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# Subsolutions and supersolutions in a free boundary problem

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**Abstract.** We begin by giving some results of continuity with respect to the domain for the Dirichlet problem (without any assumption of regularity on the domains). Then, following an idea of A. Beurling, a technique of subsolutions and supersolutions for the so-called quadrature surface free boundary problem is presented. This technique would apply to many free boundary problems in $\mathbf{R}^N, N \geq 2$, which have overdetermined Cauchy data on the free boundary. Some applications to concrete examples are also given.

## 1. Introduction

In a beautiful paper of 1957, [4], Arne Beurling investigated a free boundary problem for the Laplace equation with the Cauchy data prescribed on the free boundary. More precisely, this question related both to the electrochemical machining problem and to various hydrodynamics problems (see Acker [1], Alt–Caffarelli [3], Carleman [5]) is the following. Let $\Gamma$ be a given Jordan curve in the complex plane and $Q$ a positive continuous function defined on $\mathbf{C}$. Can one find an annulus $\omega$ having $\Gamma$ as one boundary component and another boundary component $\gamma$ (the free boundary) such that there exists a harmonic function $V$ in $\omega$ satisfying

$$(1.1)\qquad \begin{cases} V=0 & \text{on } \Gamma, \\ V=1 & \text{on } \gamma, \\ |\nabla V| = Q & \text{on } \gamma. \end{cases}$$

Arne Beurling in [4], (see also Tepper [21]), using the powerful tools of complex analysis presented an original way to prove existence of solutions to the prob-

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lem (1.1) by means of subsolutions and supersolutions. Here a subsolution would be a domain $\Omega$ such that the harmonic function, which solves the Dirichlet problem

$$
\begin{cases} \Delta V=0 & \text{in } \Omega, \\ V=0 & \text{on } \Gamma, \\ V=1 & \text{on } \gamma, \end{cases} \tag{1.2}
$$

has, on $\gamma$, a gradient greater than $Q$ (and similarly for a supersolution).

In the present paper we use again this idea, but for a different free boundary problem. Given are a positive distribution $\mu$ with compact support in $\mathbf{R}^N$ and a positive continuous function $g$, and we have to find a domain $\Omega$ such that there exists a potential $u$ satisfying

$$
\begin{cases} -\Delta u=\mu & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \\ |\nabla u|=g & \text{on } \partial\Omega. \end{cases} \tag{1.3}
$$

This problem is known as the quadrature surface free boundary problem and has already been investigated for instance by Shahgholian in [18] and Shapiro–Ullemar in [19]. It is related to quadrature domains and quadrature surfaces questions, see Aharonov–Shapiro [2], Gustafsson [9], [10] and Sakai [17], and also to hydrodynamics problems (like Hele-Shaw problems) or electromagnetic shaping problems (see Crouzeix–Descloux [6] and Henrot–Pierre [12], [13]).

The results we present in this paper are very similar to Beurling's results but the tools employed are completely different since we consider the problem in $\mathbf{R}^N, N\geq 2$. Furthermore, the techniques which are developed here could certainly apply to the generalization of problem (1.1) in $N$ dimensions.

For technical reasons we need in this paper some results of continuity with respect to the domain. This kind of results is important in, e.g., topics like shape optimization. Usually, to obtain such results, we need an extra assumption on the regularity of the limit domain (e.g. Lipschitz boundary). In Section 3 we choose to work with general domains but, per contra, we have to strengthen the kind of convergence.

In Section 4 we state some results of stability for the sets of domains which will be called subsolutions and supersolutions. Then, in Section 5, we give the result of existence using subsolutions and supersolutions for problem (1.3). It is easy to see afterwards that the existence can be ensured only when a subsolution is available.

In Section 6 we give some applications of these results. First we consider combination of Dirac measures with an example in two dimensions which shows that uniqueness of a solution is not true in general. Then we give a constructive way to obtain existence which is applied to the case of the distribution $m\mu$ when $m$ describes $\mathbf{R}_+$.

To conclude, this idea of Beurling seems to me applicable to many free boundary problems with overdetermined Cauchy data on the free boundary even in $N$-dimensions.

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## 2. Statement of the problem

Let $\mu$ be a given positive distribution in the Sobolev space $H^{-1}(\mathbf{R}^N)$ (which can be identified to the topological dual of the Sobolev space $H^1(\mathbf{R}^N)$, see e.g. [7]), $N \geq 2$ with compact support $K$ in $\mathbf{R}^N$ and $g$ a positive continuous function given in $K^c$ (the complement of $K$). We look for a bounded domain $\Omega$ in $\mathbf{R}^N$, containing $K$, such that there exists a solution to the following overdetermined Cauchy problem,

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega, \\ |\nabla u| = g & \text{on } \Gamma. \end{cases} \tag{2.1}$$

We do not assume any regularity property of $\Omega$, so the solution is to be taken in the weak sense, i.e.

$$u \in H_0^1(\Omega) \quad \text{and, for each } v \in H_0^1(\Omega), \ \int_\Omega \nabla u(x) \nabla v(x)\, dx = \langle \mu, v \rangle, \tag{2.2}$$

the meaning of the somewhat ambiguous third condition being specified below (see (2.6)).

Let us denote by $C$ the set of all bounded (open) domains containing $K$. For a given $\omega \in C$, we denote by $V_\omega$ the solution of the Dirichlet problem

$$\begin{cases} -\Delta V_\omega = \mu & \text{in } \omega, \\ V_\omega = 0 & \text{on } \partial\omega, \end{cases} \tag{2.3}$$

and we assign to $\omega$ the following quantities which are defined on $\partial\omega$,

$$\begin{aligned} a(\omega, g) &= a(\omega) = \liminf (g^{-1} |\nabla V_\omega|) \\ b(\omega, g) &= b(\omega) = \limsup (g^{-1} |\nabla V_\omega|), \end{aligned} \tag{2.4}$$

where the limits are taken for $x \in \omega$, tending to the boundary of $\omega$. By means of these limits we define the following three sets of domains,

$$
\begin{aligned}
A(g) &= A = \{\omega \in C; a(\omega) \geq 1\}, \\
B(g) &= B = \{\omega \in C; a(\omega) \leq 1\}, \\
A_0(g) &= A_0 = \{\omega \in C; a(\omega) > 1\}.
\end{aligned} \tag{2.5}
$$

We are now able to give a precise sense to the third condition in (2.1).

(2.6) $\Omega \in C$ is a solution of the free boundary problem if $\Omega \in A \cap B$.

We introduce some notation which will be useful in the following. We will denote by $\hat{\mu}$ the fundamental solution of $-\Delta\hat{\mu} = \mu$ in $\mathbf{R}^N$ which is given by

$$
\hat{\mu}(x) = \int_K E_N(x-y)\, d\mu(y), \tag{2.7}
$$

where $E_N$ is the fundamental solution of $-\Delta$ in $\mathbf{R}^N$ given by

$$
\begin{cases}
E_N(x) = -\dfrac{1}{2\pi} \log|x| & \text{for } N=2 \text{ (we will set } k_2 = -2\pi), \\[2mm]
E_N(x) = \dfrac{1}{k_N} \dfrac{1}{|x|^{N-2}} & \text{for } N \geq 3,
\end{cases} \tag{2.8}
$$

where $k_N$ is a negative constant whose exact value is $k_N = (2(2-N)\pi^{N/2})/\Gamma(N/2)$.

## 3. Some results of continuity with respect to the domain

In free boundary problems like in shape optimization the question of continuity with respect to the domain is very important. It is related both to the kind of convergence of the domains $\omega_n$ to the limit domain $\omega$ and to the regularity of the domain $\omega$. In this section we choose a strong kind of convergence (see (3.2) and (3.9)) but we do not assume any regularity property for $\omega$. In [16], Pironneau considers a weaker convergence of the domains (Hausdorff convergence) but he needs some extra assumptions to obtain his convergence result. In [15], Osipov and Suetov give a sufficient condition of convergence for the solutions of the Dirichlet problem in term of Mosco convergence of $H_0^1(\omega_n)$ to $H_0^1(\omega)$. See also [20] where Sverak obtains interesting results in dimension 2, using the orthogonal projections on $H_0^1(\omega_n)$ and $H_0^1(\omega)$.

Let $\omega$ be a bounded open set in $\mathbf{R}^N$. For each $\varepsilon > 0$ we will denote by $\mathcal{V}_\varepsilon$ the following domain,

$$
\mathcal{V}_\varepsilon = \{x \in \mathbf{R}^N, d(x, \partial\omega) < \varepsilon\}, \tag{3.1}
$$

where $d(x,L)$ is defined for every compact subset $L$ of $\mathbf{R}^N$ by

$$d(x,L)=\inf_{y\in L}|x-y|.$$

**Proposition 3.1.** *Let $\omega_n$ be a sequence of domains converging to a domain $\omega$ belonging to $C$ in the following sense,*

$$\forall\varepsilon>0, \exists n_0\in\mathbf{N} \quad \textit{such that } n\geq n_0 \Rightarrow \omega_n\subset\mathcal{V}_\varepsilon\cup\omega \textit{ and } \omega_n^c\subset\mathcal{V}_\varepsilon\cup\omega^c. \tag{3.2}$$

*Assume moreover that $\nabla V_{\omega_n}$ is uniformly bounded outside $K$, i.e.*
(3.3) *there exists a neighbourhood $W$ of $K$, with $\overline{W}\subset\omega$ and a positive constant $M$ such that $\forall n,\forall x\in\omega_n\setminus W,\quad|\nabla V_{\omega_n}(x)|\leq M$.*

*Then $V_{\omega_n}$ converges to $V_\omega$ in $H^1(\mathbf{R}^N)$ (as usual we extend $V_{\omega_n}\in H_0^1(\omega_n)$ by zero outside $\omega_n$).*

*Proof.* Let $D$ denote a ball containing $\omega$ and all the $\omega_n$. We have

$$\int_{\omega_n}\nabla V_{\omega_n}(x)\nabla w(x)\,dx=\langle\mu,w\rangle \quad \text{for each } w\in H_0^1(\omega_n).$$

Extending all these functions by 0 in $D$ yields ($\widetilde{\ }$ denotes such an extension)

$$\int_D\nabla\widetilde{V}_{\omega_n}(x)\nabla\widetilde{w}(x)\,dx=\langle\mu,\widetilde{w}\rangle \quad \text{for each } \widetilde{w}\in H_0^1(D) \text{ such that } \widetilde{w}/\omega_n\in H_0^1(\omega_n).$$

Applying this relation to $\widetilde{w}=\widetilde{V}_{\omega_n}$ yields $\int|\nabla\widetilde{V}_{\omega_n}(x)|^2dx\leq\|\mu\|\,\|V_{\omega_n}\|_{H_0^1}$, where $\|\mu\|$ is the norm of $\mu$ as a continuous linear form on $H_0^1(D)$ and the Poincaré inequality shows that $\widetilde{V}_{\omega_n}$ is bounded in $H_0^1(D)$. Therefore, we can extract a subsequence which converges, in the weak topology of $H^1$, to $V\in H^1(D)$.

Let $w\in\mathcal{D}(\omega)$, since $d(\operatorname{supp}w,\partial\omega)$ is strictly positive, by assumption (3.2) there exists $n_0$ such that $n\geq n_0\Rightarrow w\in\mathcal{D}(\omega_n)$, and then

$$\int_D\nabla\widetilde{V}_{\omega_n}(x)\nabla\widetilde{w}(x)\,dx=\int_\omega\nabla V_{\omega_n}(x)\nabla w(x)\,dx=\langle\mu,w\rangle,$$

and taking the weak limit,

$$\int_\omega\nabla V(x)\nabla w(x)\,dx=\langle\mu,w\rangle. \tag{3.4}$$

This relation (3.4) is also true for every $w\in H_0^1(\omega)$ by density.

Now, it remains to prove that $V\in H_0^1(\omega)$ to identify $V$ and $V_\omega$. We use for this a characterization of $H_0^1(\omega)$ that we can find in Mikhailov [14] in the regular case and in Osipov and Suetov [15] for the general case.

**Lemma 3.2.** *Let $V$ be in $H^1(\omega)$. If there exists a constant $C$ such that for each $\varepsilon>0$ small enough, we have*

$$\int_{\mathcal{V}_\varepsilon\cap\omega} V^2(x)\,dx \le C\varepsilon^2, \tag{3.5}$$

*then $V$ belongs to $H_0^1(\omega)$ (see proof in* [15]*).*

So, let $\varepsilon>0$ be fixed. According to (3.2) there exists $n_0$ such that $n\ge n_0$ yields

$$\partial\omega_n\subset\mathcal{V}_\varepsilon. \tag{3.6}$$

Let us fix $n$ greater than $n_0$ and consider a sequence $(\varphi_q)_{q\ge1}$ of functions in $\mathcal{D}(\omega_n)$ converging in $H^1(\omega_n)$ to $V_{\omega_n}$. Since $\nabla\varphi_q\to\nabla V_{\omega_n}$ a.e., we have $|\nabla\varphi_q|\le M+1$ a.e. (and then everywhere since $\nabla\varphi_q$ is continuous) in $\omega_n\setminus W$ using (3.3).

For each $x$ in $\mathcal{V}_\varepsilon\cap\omega$, there exists $y$ in $D$, $|x-y|\le 2\varepsilon$ and $\varphi_q(y)=0$. Then

$$\begin{aligned}|\varphi_q(x)| &= \left|\int_0^1 \nabla\varphi_q\big(y+t(x-y)\big)\cdot(x-y)\,dt\right| \le 2\varepsilon(M+1)\\ &\Rightarrow \int_{\mathcal{V}_\varepsilon\cap\omega}\varphi_q^2(x)\,dx \le 4\varepsilon^2(M+1)^2\,\mathrm{mes}(\mathcal{V}_\varepsilon\cap\omega)\le C\varepsilon^2,\end{aligned}$$

and then

$$\int_{\mathcal{V}_\varepsilon\cap\omega} V_{\omega_n}^2(x)\,dx \le C\varepsilon^2,$$

since $\varphi_q$ converges to $V_{\omega_n}$ in $L^2(D)$.

This property being true for each $n$ (with $C$ independent of $n$), we have

$$\int_{\mathcal{V}_\varepsilon\cap\omega} V^2(x)\,dx \le C\varepsilon^2, \tag{3.7}$$

because we can extract from the sequence $V_{\omega_n}$, a subsequence converging to $V$ in $L^2(D)$ by Rellich's theorem. Using Lemma 3.2, we have proved that $V$ belongs to $H_0^1(\omega)$.

Now applying (3.4) with $w=V$ yields:

$$\int_\omega |\nabla V(x)|^2\,dx = \langle\mu,V\rangle = \lim_{n\to+\infty}\int_{\omega_n}|\nabla V_{\omega_n}(x)|^2\,dx,$$

and the convergence of $V_{\omega_n}$ to $V$ is therefore strong in $H^1(D)$, which finishes the proof.

*Remark* 3.1. When we investigate the previous proof, we observe that the convergence of $\omega_n$ to $\omega$ occurs twice. First when we claim that $w\in\mathcal{D}(\omega)$ implies $w\in\mathcal{D}(\omega_n)$ for $n$ large enough, and later when we claim that for every $x$ belonging to $\mathcal{V}_\varepsilon\cap\omega$, we can find $y$ in $D\backslash\omega_n$ with $|x-y|<2\varepsilon$. Consequently it is possible to weaken assumption (3.2). For example, when $\omega_n$ is a decreasing sequence of open domains, the first point is automatically fulfilled and we can state the following proposition.

**Proposition 3.3.** *Let $\omega_n$ be a decreasing sequence of domains which belong to $C$ converging to $\omega$ in the following sense:*

$$\forall\varepsilon>0, \exists n_0 \quad \textit{such that}\ \forall x\in\mathcal{V}_\varepsilon\cap\omega,\ B(x,4\varepsilon)\cap\omega_{n_0}^c\neq\emptyset. \tag{3.8}$$

*Assume moreover that* (3.3) *is satisfied. Then $V_{\omega_n}$ converges to $V_\omega$ in $H^1(\mathbf{R}^N)$.*

*Proof.* Obviously, we have then $B(x,4\varepsilon)\cap\omega_n^c\neq\emptyset$ for all $n\geq n_0$ and the proof is the same as in the previous proposition (with $2\varepsilon$ replaced by $4\varepsilon$).

In the case of a decreasing sequence of domains $\omega_n$ converging to $\omega$ in the sense of (3.8), we can use the property of the functions $V_{\omega_n}$ and $V_\omega$ to be harmonic on $\omega\backslash K$ to specify the kind of convergence. We recall that a sequence of harmonic functions $V_{\omega_n}$ converges to a harmonic function $V_\omega$ in $\mathcal{H}(\omega\backslash K)$ (the space of harmonic functions on $\omega\backslash K$) if $V_{\omega_n}$ uniformly converges to $V_\omega$ as well as all its derivatives on every compact subset of $\omega\backslash K$.

**Proposition 3.4.** *Let $\omega_n$ be a decreasing sequence of domains converging to a domain $\omega$ belonging to $C$ in the sense* (3.8). *Assume moreover that* (3.3) *is satisfied. Then $V_{\omega_n}$ converges to $V_\omega$ in $\mathcal{H}(\omega\backslash K)$.*

*Proof.* By the maximum principle $V_{\omega_n}$ is a decreasing sequence of harmonic functions on $\omega\backslash K$. Let $x_0$ be fixed in $\omega\backslash K$ near the boundary of $\omega$. By assumption (3.3) and the mean value inequality, the sequence $V_{\omega_n}(x_0)$ is bounded in $\mathbf{R}$ and then the Harnack theorem (see e.g. [7], vol. 1, Corollary 9, p. 300) gives the announced result.

Now, we give a similar result of convergence when $\omega$ is perturbed only in a neighbourhood of a point of its boundary. This result may allow, for instance, to replace any open domain by a domain locally more regular.

**Proposition 3.5.** *Let $\omega\in C$, $x\in\partial\omega$ and $\omega_n$ be a sequence of domains such that*

$$\omega_n\Delta\omega \ \textit{is included in the ball centered at}\ x\ \textit{of radius}\ 1/n, \tag{3.9}$$

*where the symmetric difference of* $\omega_n$ *and* $\omega$ *is defined by*

$$\omega_n \Delta \omega = (\omega_n \setminus \omega) \cup (\omega \setminus \omega_n).$$

*Then* $V_{\omega_n}$ *converges to* $V_\omega$ *in* $H^1(\mathbf{R}^N)$.

*Proof.* The beginning of the proof is the same as in Proposition 3.1 since it is clear that for any compact $K_1 \subset \omega$ we have $K_1 \subset \omega_n$ for $n$ large enough. Again, it remains to prove that $V$ (= weak-limit of $V_{\omega_n}$) belongs to $H_0^1(\omega)$. For this purpose, we are going to use the characterization of $H_0^1(\omega)$ related to spectral synthesis that we can find in Hedberg [11] or Deny [8] (for initial reference),

$$V \in H_0^1(\omega) \quad \text{if} \quad V|_{\omega^c} = 0 \text{ quasi-everywhere.}$$

(We define, as usual, $V|_{\omega^c}$ as the restriction to $\omega^c$ of a $(2,1)$ quasi-continuous representative of $V$.)

Let $F$ be any compact subset of $\omega^c$ which does not contain $x$. Since $d(x, F) > 0$, there exists $N_0$ such that $F \subset \omega_n^c$ for every $n \geq N_0$, and then $V_{\omega_n}|_F \equiv 0$. Now since $V_{\omega_n}$, $n \geq N_0$ converges weakly to $V$, there exists a linear convex combination of the $V_{\omega_n}$, $n \geq N_0$ which converges strongly in $H^1$ to $V$. Such a convex combination vanishes on $F$ and converges quasi-everywhere to $V$, so $V \equiv 0$ q.e. on $F$. This result is true for each $F$ not containing $x$ and the $(2,1)$ capacity of $x$ (see e.g. [11] for more details) being zero (in dimension $N \geq 2$), we have proved that $V \equiv 0$ q.e. on $\omega^c$ which finishes the proof.

## 4. Preliminary results concerning the sets $A$ and $B$

Let us now claim some stability results for the set of domains $B$.

**Proposition 4.1.** *Let* $\omega_n$ *be a decreasing sequence of domains in* $B$ *converging to a domain* $\omega$, *belonging to* $C$, *in the sense of* (3.8). *Then* $\omega$ *belongs to* $B$.

*Proof.* We first want to apply proposition (3.4), so we have to prove uniform boundedness of $|\nabla V_{\omega_n}|$ (assumption (3.3)).

The function $V_{\omega_n} - \hat{\mu}$ and all the functions $\partial(V_{\omega_n} - \hat{\mu})/\partial x_i$ are harmonic on $\omega_n$. By the maximum principle applied to each function $\partial(V_{\omega_n} - \hat{\mu})/\partial x_i$, it follows that on $\omega_n$

$$|\nabla(V_{\omega_n} - \hat{\mu})|^2 = \sum_{i=1}^{N} \left( \frac{(\partial(V_{\omega_n} - \hat{\mu})}{\partial x_i} \right)^2 \leq N \sup_{\partial \omega_n} \left|\nabla(V_{\omega_n} - \hat{\mu})\right|^2. \tag{4.1}$$

Now the assumption that $\omega_n$ belongs to $B$ implies that, in a neighbourhood of $\partial\omega_n$, we have

$$(4.2)\qquad |\nabla V_{\omega_n}| \le \|g\|_{L^\infty(D)}.$$

Let $W$ be any neighbourhood of $K$ strictly contained in $\omega$. Using (4.1) and (4.2) we obtain for each $x$ in $\omega_n \backslash \overline{W}$

$$|\nabla(V_{\omega_n} - \hat{\mu})(x)| \le \sqrt{N}\Big(\|g\|_{L^\infty(D)} + \sup_{D\backslash\overline{W}} |\nabla\hat{\mu}|\Big),$$

and so we have obtained that for each $x$ in $\omega_n \backslash \overline{W}$, we have

$$\big|\nabla V_{\omega_n}(x)\big| \le \sqrt{N}\|g\|_{L^\infty(D)} + \Big(\sqrt{N}+1 \sup_{D\backslash\overline{W}} |\nabla\hat{\mu}|\Big),$$

the right-hand side being a constant independent of $x$ and $n$.

Then, according to Proposition 3.4, $V_{\omega_n}$ converges to $V_\omega$ in $\mathcal{H}(\omega\backslash K)$. Now an elementary calculation proves that $|\nabla V_{\omega_n}|^2$ is subharmonic where $V_{\omega_n}$ is harmonic, i.e. on $\omega_n\backslash K$. Let us denote by $\Gamma$ the boundary of $W$ defined above. Since $\nabla V_{\omega_n}$ converges uniformly to $\nabla V_\omega$ on $\Gamma$, we have $|\nabla V_{\omega_n}|^2 \le |\nabla V_\omega|^2+1$ on $\Gamma$ for $n$ large enough.

Let us introduce a sequence of harmonic functions $\phi_n$ as solutions of the Dirichlet problem

$$\begin{cases} \Delta\phi_n = 0 & \text{in } \omega_n\backslash\overline{W}, \\ \phi_n = g^2 & \text{on } \partial\omega_n, \\ \phi_n = |\nabla V_\omega|^2+1 & \text{on } \Gamma. \end{cases}$$

Since $|\nabla V_{\omega_n}|^2$ is subharmonic on $\omega_n\backslash\overline{W}$ and $|\nabla V_{\omega_n}|^2 \le g^2$ on $\partial\omega_n$, we have

$$(4.3)\qquad |\nabla V_{\omega_n}|^2 \le \phi_n \quad \text{on } \omega_n\backslash\overline{W}.$$

We want to prove that $\omega$ belongs to $B$. Assume, for a contradiction, that there exists $x \in \partial\omega$ such that

$$(4.4)\qquad \limsup_{\substack{z\to x\\ z\in\omega}} |\nabla V_\omega(z)|^2 \ge g^2(x)+\alpha \quad \text{with } \alpha>0.$$

If we can prove that, for $n$ large enough,

$$(4.5)\qquad \phi_n(x) \le g^2(x)+\alpha/2$$

the result will follow immediately from uniform convergence of $|\nabla V\omega_{\omega_n}|^2$ to $|\nabla V_\omega|^2$ and from inequality (4.3).

Now let us introduce the harmonic function $\phi$ which solves the Dirichlet problem

$$\begin{cases} \Delta\phi=0 & \text{in } \omega\backslash\overline{W}, \\ \phi=g^2 & \text{on } \partial\omega, \\ \phi=|\nabla V_\omega|^2+1 & \text{on } \Gamma. \end{cases}$$

It follows from Harnack's theorem and monotony of the sequence $\phi_n$ that $\phi_n$ converges to $\phi$ in $\mathcal{H}(\omega\backslash K)$ (see the proof of Proposition 3.4) and inequality (4.5) follows immediately which finishes the proof.

**Lemma 4.2.** *If $\omega_1, \omega_2$ belong to $B$, so does their intersection.*

*Proof.* Let us set $V=V_{\omega_1\cap\omega_2}$, $V_1=V_{\omega_1}$ and $V_2=V_{\omega_2}$. Since $\omega_1\cap\omega_2\subset\omega_1$ and $V_1-V$ is harmonic on $\omega_1\cap\omega_2$, we have by the maximum principle $V_1-V\geq 0$ on $\omega_1\cap\omega_2$ and moreover for each $x$ belonging to $\partial\omega_1\cap\partial(\omega_1\cap\omega_2)$ (where both $V_1$ and $V$ vanishes) we have

$$\limsup|\nabla V_1(x_n)|\geq\limsup|\nabla V(x_n)| \tag{4.6}$$

for every sequence $x_n$ converging to $x$ in $\omega_1\cap\omega_2$ (we have obviously the same property for $V_2$ and $V$). Since every point of $\partial(\omega_1\cap\omega_2)$ belongs either to $\partial\omega_1$ or $\partial\omega_2$, we obtain immediately from (4.6)

$$\limsup\frac{|\nabla V(x_n)|}{g(x_n)}\leq 1 \quad \text{for } x_n \text{ converging to } x \text{ in } \omega_1\cap\omega_2$$

and then $\omega_1\cap\omega_2\in B$.

**Lemma 4.3.** *Assume $\omega_1\subset\omega_2$, $\omega_1\in A_0$ and $\omega_2\in B$. Then $\omega_2$ contains the boundary of $\omega_1$.*

Same proof as in [4], Lemma II, since $V_{\omega_1}-V_{\omega_2}$ is harmonic in $\omega_1$.

## 5. The existence result

We are now able to prove the main result of this paper which is a result of existence when a subsolution (i.e. a domain in $A_0$) and a supersolution (a domain in $B$) are available.

**Theorem 5.1.** *Assume* $\Omega_0 \subset \omega_0$, $\omega_0 \in B$, $\Omega_0 \in A_0$. *Then there exists a solution* $\Omega$ *for the problem* (2.1) *such that* $\Omega_0 \subset \Omega \subset \omega_0$.

*Proof.* We consider the set $S = \{\omega \in B : \Omega_0 \subset \omega \subset \omega_0\}$, which is not empty since it contains $\omega_0$. We are going to construct a minimal element of $S$ which will be the desired solution.

For this purpose, we form $I$, the intersection of all $\omega \in S$, and we set $\Omega = \mathring{I}$. The domain $\Omega$ is not empty since $\Omega_0$ is included in $I$.

First we have to prove that $\Omega$ belongs to $B$.

We can choose a sequence $(D_n)_{n=1,\dots,\infty}$ of domains belonging to $S$ such that

$$\bigcap_{n=1}^{\infty} D_n = \bigcap_{\omega \in S} \omega = I.$$

We define a decreasing sequence $\omega_n$ by $\omega_1 = D_1$ and $\omega_{n+1} = \omega_n \cap D_{n+1}$.

By Lemma 4.2, all the $\omega_n$ belong to $B$. To prove that $\Omega$ belongs to $B$ it remains to prove, according to Proposition 4.1, that $\omega_n$ converges to $\Omega$ in the sense of (3.8). Let $\varepsilon > 0$ be given, we assume for an indirect proof that

$$\text{for each } n, \text{there exists } x_n \in \mathcal{V}_\varepsilon \cap \Omega \text{ such that } B(x_n, 4\varepsilon) \subset \omega_n. \tag{5.1}$$

We extract by compactness a subsequence $x_{n_k}$ converging to $x$ in $\overline{\mathcal{V}_\varepsilon \cap \Omega} \subset \bar{\mathcal{V}}_\varepsilon \cap \bar{\Omega}$. By the triangle inequality, (5.1) implies that

$$\text{there exists } k_0 \text{ such that } k \geq k_0 \Rightarrow B(x, 2\varepsilon) \subset \omega_{n_k}. \tag{5.2}$$

But it follows that $B(x, 2\varepsilon) \subset I$ and then $B(x, 2\varepsilon) \subset \Omega$ which is in contradiction with $x \in \bar{\mathcal{V}}_\varepsilon$ and we have proved that $\Omega$ belongs to $B$.

Moreover, according to Lemma 4.3, $\Omega$ contains the boundary of $\Omega_0$. This domain $\Omega$ is not only minimal in $S$ (by construction), it is also locally minimal in $B$ in the sense that there exists a neighbourhood $N$ of its boundary such that $\Omega$ does not contain any $\omega \in B$, $\omega \neq \Omega$ and $\partial\omega \subset N$.

We are going to prove that $\Omega$ is a solution of the free boundary problem.

Assume that this is not true, then there exists a point $x \in \partial\Omega$ such that

$$\liminf_{x_n \to x,\, x_n \in \Omega} \frac{|\nabla V_\Omega|}{g} = 1 - \alpha < 1. \tag{5.3}$$

Let us introduce $\mathcal{O} = \{z \in \Omega$ such that $|\nabla V_\Omega(z)|/g(z) < 1 - \alpha/2\}$ which is an open subset of $\Omega$. Assumption (5.3) means that for every $n \in \mathbf{N}$, $B(x, n^{-1}) \cap \mathcal{O}$ is non empty.

Let us choose for each $n$ a closed subset $F_n$ of positive capacity in $B(x,n^{-1})\cap \mathcal{O}$ and let us set

$$\widetilde{\Omega}_n = \Omega - F_n. \tag{5.4}$$

We are going to prove that $\widetilde{\Omega}_n$, which is strictly contained in $\Omega$, belongs to $S$ for $n$ large enough, which leads to a contradiction.

Since $\Omega$ contains $\widetilde{\Omega}_n$, by the maximum principle we have

$$\widetilde{V}_n := V_{\widetilde{\Omega}_n} \leq V_\Omega \quad \text{on } \Omega. \tag{5.5}$$

Now, the boundary of $\widetilde{\Omega}_n$ is composed on the one hand of $\partial\Omega$, but for every point $y$ in this part we have (since $\widetilde{V}_n(y)=V_\Omega(y)=0$), by (5.5)

$$\limsup \frac{|\nabla \widetilde{V}_n(y)|}{g(y)} \leq \limsup \frac{|\nabla V_\Omega(y)|}{g(y)} \leq 1,$$

and of the boundary of $F_n$. But on $F_n$, $|\nabla V_\Omega|/g<1-\alpha/2$ and since $\widetilde{\Omega}_n$ converges to $\Omega$ in the sense (3.9), we have, according to Proposition 3.5,

$$\widetilde{V}_n - V_\Omega \to 0 \quad \text{in } H^1.$$

Then the $L^2$-convergence of $\nabla V_n$ to $\nabla V_\Omega$ proves that, up to a subsequence, and for $n$ large enough,

$$\frac{|\nabla \widetilde{V}_n|}{g} \leq 1 - \frac{\alpha}{4} \quad \text{on } \partial F_n \tag{5.6}$$

and for such $n$, we have therefore

$$\limsup \frac{|\nabla \widetilde{V}_n|}{g} \leq 1 \quad \text{on } \partial\widetilde{\Omega}_n$$

which finishes the proof.

The interest of Theorem 5.1 is that to prove the existence of a solution for the problem (2.1), we can compute solutions of the Dirichlet problem (2.3) for particular domains. For a ball, for instance, we possess an explicit integral representation of the solution of (2.3) that we are going to use now.

In all the following, we are going to make an extra assumption on the growth of the function $g$ at infinity, we assume from now on that

$$\frac{1}{g(x)} = o\big(|x|^{N-1}\big) \quad \text{when } |x| \to +\infty. \tag{5.7}$$

**Theorem 5.2.** *There exists a solution to the problem* (2.1) *if* $A_0$ *is non empty.*

*Proof.* To prove Theorem 5.2 it is sufficient, according to Theorem 5.1, to prove that for $R$ large enough the ball $B_R=B(0,R)$ belongs to $B$.

Now it is well known that the solution of problem (2.3) for the ball $B_R$ is given by (see for instance [7])

$$V_R(x)=\int_K E_N(x-y)-E_N\left(\frac{|y|x}{R}-\frac{Ry}{|y|}\right)d\mu(y) \tag{5.8}$$

and then on $\Gamma_R=\partial B_R$; $|\nabla V_R(x)|=\nabla V_R(x)\cdot n$ is given by

$$|\nabla V_R(x)|=C_N\int_K \frac{R^2-|y|^2}{R|x-y|^N}\,d\mu(y) \quad \text{for } x\in\Gamma_R, \tag{5.9}$$

where $C_N$ is a positive constant depending on $N$. We obtain immediately

$$|\nabla V_R(x)|\le C_N\frac{R}{(R-\delta)^N}|\mu| \quad \text{for } x\in\Gamma_R,$$

where $\delta=\max_{y\in K}|y|$, and $|\mu|$ is the total variation of $\mu$, and then the convergence of $|\nabla V_R(x)|/g(x)$ to zero on $\Gamma_R$ follows from (5.7).

For a given distribution $\mu$, the proof of existence of a solution for the problem (2.1) amounts to looking for a domain in the class $A_0$. We give in the following section a constructive way to obtain such a domain and some examples of application.

## 6. Application and examples

### (a) First example: combination of Dirac measures

Let $x_1,x_2,\ldots,x_m$ be $m$ points in $\mathbf{R}^N$ and $\alpha_1,\alpha_2,\ldots,\alpha_m$ positive coefficients. We consider the following combination of Dirac measures,

$$\mu=\sum_{i=1}^{m}\alpha_i\delta_{x_i}. \tag{6.1}$$

Although $\mu$ does not belong to $H^{-1}(\mathbf{R}^N)$, for $N\ge 2$, the theory that we have developed in the previous section works in this situation. Indeed, the two main points we need in order to apply the theory are the following.

There exists a solution $V_\omega$ for every bounded domain $\omega$ for the problem (2.3). This solution is given using the Green function $G_\omega$ of $\omega$, by

$$V_\omega(x) = -\sum_{i=1}^{m} \alpha_i G_\omega(x, x_i). \tag{6.2}$$

We are able to prove Propositions 3.1, 3.4 and 3.5 (continuity with respect to the domain), which can be done either using formula (6.2) and continuity of Green functions with respect to the domain or using variational formulation of problem (2.3) (see [7]), let $\theta \in ]0,1[$ then $V_\omega$ is characterized by

$$V_\omega \in W_0^{1,p}(\Omega) \quad \text{with } p = \frac{N-\theta}{N-1} \text{ and}$$
$$\int_\Omega \nabla V_\omega \nabla \xi = \langle \mu, \xi \rangle \quad \text{for every } \xi \in W_0^{1,q}(\Omega) \quad \text{with } q = \frac{N-\theta}{1-\theta}$$

and following the proof of Proposition 3.1.

Now we are able to prove the following proposition.

**Proposition 6.1.** *Let $\mu$ be given by* (6.1), *then for every $g$ satisfying* (5.7) *there exists a solution for the problem* (2.1).

*Proof.* According to Theorem 5.2, we have to find a domain in $A_0$. Let us denote by

$$\Omega_\varepsilon = \bigcup_{i=1}^{m} B(x_i, \varepsilon)$$

with $\varepsilon$ small enough so that all the balls are disjoint. We obtain immediately

$$V_\varepsilon(x) := V_{\Omega_\varepsilon}(x) = \alpha_i \big(E_N(x - x_i) - E_N(\varepsilon)\big) \quad \text{for } x \in B(x_i, \varepsilon) \tag{6.3}$$

and on $\partial\Omega_\varepsilon$

$$|\nabla V_\varepsilon| = c_N \alpha_i \frac{1}{|x - x_i|^{N-1}} = \frac{c_N \alpha_i}{\varepsilon^{N-1}} \quad \text{for } x \in \partial B(x_i, \varepsilon),$$

where $c_N$ is a positive constant. So it is clear that for $\varepsilon$ small enough, $\Omega_\varepsilon$ belongs to $A_0$.

When $N=2$, we can use conformal mappings to compute the solution(s) of problem (2.1), see [12], [13] and also [19]. These computations show that, in general, we have not a unique solution to this problem for a given $\mu$. Another interesting situation is to let the total mass $\sum_{i=1}^{m} \alpha_i$ increase. We observe various topologies

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | (1 convex) | | (convex) |
| connected solutions | | 0 | | 2 | | 2 | | 1 |
| $\alpha/l$ | 0 | | 2.300 | | 2.827 | | $\pi$ | |
| disconnected solutions | | 1 | | 1 | | 1 | | 0 |

Figure 1.

for the solutions $\Omega$ and also a sort of bifurcation situation happens. We are going to present here a simple but typical situation, the case of two Dirac measures with the same coefficient,

$$\mu = \alpha\delta_{-x} + \alpha\delta_x \tag{6.4}$$

with $\alpha > 0$ and $x$ and $-x$ on the real axis of the complex plane ($x > 0$). We denote by $l(=2x)$ the distance between the two points. We choose for function $g$ the constant function 1.

We can state the following proposition.

**Proposition 6.2.** *Let $\mu$ be given by* (6.4), *$l=2x$ and $g\equiv 1$. We look for the regular domains $\Omega$ (i.e. of class $C^2$) which are solutions of the problem* (2.1).

(1) *For $\alpha < \pi l$, there exists a (unique) disconnected solution, it has two connected components and it is given by*

$$\Omega_\alpha = B\left(-x, \frac{\alpha}{2\pi}\right) \cup B\left(x, \frac{\alpha}{2\pi}\right);$$

*for $\alpha \geq \pi l$ there is no disconnected solution.*

(2) *For $2.300 \ldots l < \alpha < \pi l$, there exist two connected solutions, one contained in the other.*

(3) *For $\pi l \leq \alpha$, there exists a unique solution (which is connected). Moreover, for $2.827 \ldots l < \alpha$ there exists a convex solution.*

We summarize the results in Figure 1.

*Sketch of the proof.* (1) Each connected component of a solution $\Omega$ must meet the support of $\mu$ (otherwise, we would have $u=0$ on this component and then $|\nabla u|=0$), therefore a solution has at most two connected components. If there are exactly two, on each one we have to solve the problem

$$\begin{cases} -\Delta u = \alpha\delta_z & \text{in } \omega \text{ with } z = x \text{ or } -x, \\ u = 0 & \text{on } \partial\omega, \\ |\nabla u| = 1 & \text{on } \partial\omega, \end{cases}$$

whose unique (regular) solution is a disk of center $z$ and radius $2\pi/\alpha$, see [12] or [18], so the result follows since these two disks must be disjoint.

(2) and (3) Looking for the connected solutions $\Omega$ we introduce a conformal map $\Phi$ transforming the unit disk $\Omega_0$ into $\Omega$. We can assume that the inverse image of $x$ and $-x$ are $a$ and $-a$ on the real axis, with $0<a<1$.

If we set $v(x,y)=\log|\Phi'(x+iy)|$ we have proved in [13], see also [12], that $v$ is the solution of the Dirichlet problem

$$\text{(6.5)} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega_0, \\ v = \log\left(\dfrac{2\alpha(1-a^4)}{|a^2-e^{2i\theta}|^2}\right) & \text{on } \partial\Omega_0 = \Gamma_0 \end{cases}$$

We are able to solve explicitly (6.5) using a Fourier expansion of the function appearing in the boundary condition (see [12]), and we obtain

$$\Phi'(z) = \frac{\alpha}{\pi}\,\frac{(1-a^4)}{a^4}\,\frac{1}{(z^2-1/a^2)^2}$$

and by integration

$$\Phi(z) = \frac{\alpha(1-a^4)}{4\pi a^2}\left[\frac{-2z}{z^2-1/a^2} + a\log\frac{1/a+z}{1/a-z}\right].$$

These maps are univalent for every $a$, $0<a<1$. It remains to solve the equation

$$\Phi(a) = x = l/2$$

to obtain effectively a convenient conformal map. The study of the function $a \to \Phi(a)$ gives the announced result (for the convexity, we observe that $\Gamma = \Phi(\Gamma_0)$ is convex if and only if $a \le 1/\sqrt{3}$).

## (b) Use of level sets

If $\gamma_c$ is any level surface of $\hat{\mu}$, the function defined in (2.7), which includes $K$,

$$\text{(6.6)} \qquad \gamma_c = \left\{x \in \mathbf{R}^N ; \hat{\mu}(x) = c \in \mathbf{R}\right\},$$

we denote by $\omega_c = \{x \in \mathbf{R}^N ; \hat{\mu}(x) > c\}$ the "interior" of $\gamma_c$ and the potential associated to $\omega_c$ is given by

$$V_c := V_{\omega_c} = \hat{\mu} - c.$$

Since $|\nabla V_c|=|\nabla\hat{\mu}|$ which is a known function, a simple way to prove existence of a solution for the free boundary problem (2.1) is the following,

$$(6.7)\qquad \begin{cases} \text{if there exists a level surface of } \hat{\mu}, \text{ including } K, \\ \text{on which } |\nabla\hat{\mu}|/g \geq 1, \text{ then there is a solution to problem (2.1).} \end{cases}$$

This criterion (6.7) is indeed only a sufficient criterion of existence but it is rather simple to use. For instance, we can determine the region outside $K$ where $|\nabla\hat{\mu}|/g\geq 1$, and then look for a level surface of $\hat{\mu}$ in this region to obtain existence. Another application is given in the following result.

**Proposition 6.3.** *Let $\mu$ be given. There exists a number $m_0\in\mathbf{R}+$ such that*
(i) if $m<m_0$ *there is no solution to the free boundary problem associated to the measure $m\mu$;*
(ii) if $m>m_0$ *there is at least one solution to the free boundary problem associated to the measure $m\mu$.*

*Proof.* We set

$$I=\{m\in\mathbf{R}+ \text{ such that there exists a solution for the measure } m\mu\}.$$

We have to prove that $I$ is an interval like $[m_0,+\infty[$ or $]m_0,+\infty[$. For this purpose we prove
(1) if $m_1\in I$ then $m>m_1\Rightarrow m\in I$,
(2) $I\neq\emptyset$.
The first point. Let $(\Omega_1,U_1)$ be the solution associated to the measure $m_1\mu$.
For $m>m_1$, we set $V=m/m_1U_1$ and we have

$$\begin{cases} -\Delta V=\dfrac{m}{m_1}(-\Delta U_1)=m\mu & \text{on } \Omega_1, \\ V=0 & \text{on } \partial\Omega_1, \end{cases}$$

and

$$\frac{|\nabla V|}{g}=\frac{m}{m_1}\frac{|\nabla U_1|}{g}>1,$$

so $\Omega_1$ belongs to $A_0(m\mu)$ which proves existence of a solution for $m\mu$.

For the second point we are going to use criterion (6.7), the fundamental solution associated to $m\mu$ is here $m\hat{\mu}$, so we begin by proving;

(a) There is always a level surface of $\hat{\mu}$ including $K$, say $\gamma$.
Since $\gamma$ is also a level surface for the measure $m\mu$, it will remain to prove
(b) when $m$ goes to infinity, $\gamma$ is contained in the set

$$\left\{x\in\mathbf{R}^N : m\frac{|\nabla\hat{\mu}(x)|}{g(x)}>1\right\}.$$

Let

$$\delta > \sup_{y \in K} |y|.$$

We can easily prove that there is a level surface of $\hat{\mu}$ in the ring

$$\mathcal{C} = \left\{ x \in \mathbf{R}^N : \sqrt{N}\delta \leq |x| \leq (\sqrt{N}+2)\delta \right\}$$

which includes $K$. Indeed, for every $\xi \in \sum^{N-1}$, the unit sphere of $\mathbf{R}^N$, we have

$$\hat{\mu}(\sqrt{N}\delta\xi) = \int_K E_N(\sqrt{N}\delta\xi - y)\, d\mu(y) \geq \int_K E_N(\sqrt{N}\delta + |y|)\, d\mu(y) \geq |\mu| E_N(\sqrt{N}\delta + \delta)$$

while

$$\hat{\mu}\big((\sqrt{N}+2)\delta\xi\big) \leq \int_K E_N\big((\sqrt{N}+2)\delta\xi - |y|\big)\, d\mu(y) \leq |\mu| E_N(\sqrt{N}\delta + \delta)$$

and then there is at least one point on the segment $[\sqrt{N}\delta\xi, (\sqrt{N}+2)\delta\xi]$ where $\hat{\mu}$ takes the value $c = |\mu| E_N(\sqrt{N}\delta + \delta)$ and then $\gamma_c \subset \mathcal{C}$.

Now to prove (b), we have to verify that on $\gamma_c$,

$$\frac{|\nabla \hat{\mu}|}{g} \geq \alpha \geq 0 \tag{6.8}$$

(since for $m$ large enough, we then have $|\nabla(m\hat{\mu})|/g > 1$ on $\gamma_c$).

Now,

$$\frac{\partial \hat{\mu}}{\partial x_i} = \frac{1}{k_N} \int_K \frac{x_i - y_i}{|x-y|^N}\, d\mu(y),$$

but in the ring $\mathcal{C}$, we have $|x| \geq \sqrt{N}\delta$ and then, for each $x$ in $\mathcal{C}$, there always exists at least one index $i_0 \in \{1, ..., N\}$ such that $x_{i_0} > y_{i_0}$ for every $y$ in $K$ (because the hypercube circumscribed to the ball of center 0 and radius $\delta$ is contained in the ball of center 0 and radius $\sqrt{N}\delta$). It follows that $\partial\hat{\mu}/\partial x_{i_0} > 0$ for this $x$, that is to say $|\nabla\hat{\mu}| > 0$ on $\gamma_c$, which proves (6.8) and Proposition 6.3.

## 7. Conclusion

Following Beurling, we are able to distinguish three different types of solutions for the free boundary problem (2.1) defined as follows.

If $(\Omega, u)$ is a solution of

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = g & \text{on } \partial\Omega, \end{cases} \tag{7.1}$$

we denote by $(\Omega_\lambda, u_\lambda)$ an (eventual) solution of

$$(7.2)\qquad \begin{cases} -\Delta u_\lambda = \frac{\mu}{\lambda} & \text{in } \Omega_\lambda, \\ u_\lambda = 0 & \text{on } \partial\Omega_\lambda, \\ |\nabla \mu_\lambda| = g & \text{on } \partial\Omega_\lambda \end{cases}$$

with $1-\varepsilon<\lambda<1+\varepsilon$.

If $\Omega_\lambda$ is shrinking as $\lambda$ increases, the solution $\Omega$ will be called elliptic. If $\Omega_\lambda$ is monotonic increasing with $\lambda$, $\Omega$ will be called hyperbolic. Finally, if $\Omega_\lambda$ does not exist for $\lambda>1$ (or for $\lambda<1$) but two solutions $\Omega_\lambda$ and $\Omega'_\lambda$ do exist for $1-\varepsilon<\lambda<1$ (or for $1<\lambda<1+\varepsilon$) such that $\Omega_\lambda\subset\Omega\subset\Omega'_\lambda$, where both regions tend to $\Omega$ as $\lambda\to 1$, then $\Omega$ will be called a parabolic solution.

The procedure developed in the present paper allows us to obtain only elliptic and parabolic solutions. This is due to the fact that each elliptic solution is locally minimal in the set $B$ while hyperbolic solutions are locally maximal in the same set. The difference between these cases is that each locally minimal region in $B$ is a solution while maximal regions in general are not solutions.

Nevertheless, using conformal mappings like in 6(a) we are able to obtain all types of solutions. For instance in the case of two Dirac measures like in (6.4) we can classify the solutions with the value of $a=\Phi^{-1}(l/2)$ (see the proof of Proposition 6.2),

| | |
|---|---|
| for $0<a<0.832$ | we obtain an elliptic solution, |
| for $a=0.832$ | it is a parabolic one, and |
| for $0.832<a<1$ | we have a hyperbolic solution. |

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