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Zak's theorem on superadditivity

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0. Introduction

This paper is concerned with an important theorem due to *Fyodor L. Zak*, which appeared in [Z1]: Let $X \subset \mathbf{P}^N$ be a (reduced and irreducible) subvariety. A *k-secant space* to X is a k-dimensional linear subspace of $\mathbf{P}^N$ which is spanned by $k+1$ points from X, the *k-secant variety* of X in $\mathbf{P}^N$ is the (closure of the) union of all the k-secant spaces of X. Zak denotes this space by $S^k(X)$, we shall also use a different terminology: Whenever X and Y are subvarieties of $\mathbf{P}^N$, we define their *join* XY in $\mathbf{P}^N$ as the closure of the union of all *lines* in $\mathbf{P}^N$ spanned by a point from X and a point from Y. This defines a commutative and associative operation on the set of subvarieties of $\mathbf{P}^N$, making it into a *commutative monoid*, see [Å] for details. We have $S^k(X)=X^{k+1}$.

Zak considers a *relative secant defect*, defined as

$$\delta_k(X) = \dim(X^k) + \dim(X) + 1 - \dim(X^{k+1}).$$

We shall always assume that the ground field is algebraically closed of characteristic zero. Under this assumption, Zak states the following

Theorem (Zak's Theorem on Superadditivity). *Let $X \subset \mathbf{P}^N$ be a nonsingular projective variety, such that $\delta_1(X)>0$. Let p and q be integers such that $X^{p+q} \neq \mathbf{P}^N$. Then $\delta_{p+q}(X) \geq \delta_p(X) + \delta_q(X)$.*

The assumption that $\delta_1(X)>0$ was not explicitly stated in the formulation of this theorem in Zak's paper referred to above, but it is quite clear from the introduction that only this case is considered. In fact, there are counterexamples to the asserted inequality (for $p=q=2$) if $\delta_1(X)=0$, and also for singular varieties, see §2 below.

In the applications of this theorem in his paper [Z1], Zak uses the theorem above only in the case $q=1$.

The present paper grew out of an effort to understand Zak's proof of superadditivity. The proof is not easy to follow, in particular it is not easy to see precisely how the assumption $\delta_1(X)>0$ is used for $p,q\geq 2$. In fact, if $q=1$ the claim is true also when $\delta_1(X)=0$, it is then an immediate consequence of the string of inequalities

$$\delta_1 \leq \delta_2 \leq \delta_3 \leq \ldots \leq \delta_q$$

which holds as long as $X^q \neq \mathbf{P}^N$. See Proposition 1.2.

When $q=1$, we give in §4 an alternative proof of the theorem, which is of an *infinitesimal* nature: One reasons with *embedded tangent spaces* and their projections where Zak works with the actual varieties. In §3 we show some results about embedded tangent spaces which are used in the alternative proof. In the case $q=1$ we also fully understand Zak's proof, of which we will give an exposition and fill in some details in the following two sections, §1 and §2. Our contribution to this picture essentially is Proposition 2.4.

Zak's proof is truly impressive, blending as it does a mastery of intricate technical algebraic reasoning with profound intuition from *"synthetic geometry"* in a modern setting. Unfortunately the case of p and $q \geq 2$ remains a mystery to us, as we have not been able to fill in the details in this case. But we also know of no counterexamples. In §2 we analyze in some detail what seems to be needed to make Zak's proof work in the case $q\geq 2$. To sum up, the conclusion of the superadditivity theorem does not hold in the case $p,q\geq 2$ if the hypothesis $\delta_1>0$ is dropped.

This subject has also been treated by *Barbara Fantechi* in [Fa], independently of our work. Her approach is to verify that Zak's proof works if the secant variety $S^{q-1}(X)$ satisfies a *regularity condition*—which she introduces, called *almost smoothness*, see §2 for the definition. Since smoothness implies almost smoothness, her result then yields a proof of superadditivity in the case when $q=1$, along the lines of Zak's proof. But examples show that almost smoothness for $S^{q-1}(X)$ is a stronger property than what is really needed, see our Remark 2.11.

Our approach to the general problem has not been to look for new, global conditions on the geometry of the secant varieties, but rather to study the local structure along the various entry point loci, see §1 for the definition and §2 for the discussion.

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1. Generalities

In general, let $V_0, \dots, V_r \subset \mathbf{P}^N$ be subvarieties. We use the notation of joins $V_0 \cdots V_r$ from the introduction, and if $V_i = \{x_i\}$, then we denote the join by $x_0 \cdots x_r$. Put

$$S^0_{V_0,\dots,V_r} = S^0_V = \{(v_0, \dots, v_r, u) = (v, u) \mid v_i \in V_i, i = 0, \dots, r, \\ \dim(v_0 \cdots v_r) = r, \quad u \in v_0 \cdots v_r\}$$

in $V_0 \times V_1 \times \dots \times V_r \times \mathbf{P}^N$. V denotes the tuple $(V_0, \dots, V_r)$ and v denotes $(v_0, \dots, v_r)$. Let

$$S_V = \overline{S^0_V}.$$

We introduce the notation $\varphi^V = pr_{\mathbf{P}^N} \colon S_V \longrightarrow \mathbf{P}^N$, $p_i^V = pr_{V_i} \colon S_V \longrightarrow V_i$. The join of $V_0, \dots, V_r$ is $V_0 \cdots V_r = \varphi^V(S_V)$ and as stated in §0 we write $S^k(X) = X^{k+1}$. When $V_0 = \dots = V_r = X$ we put $S_V = S^k_X$ and $\varphi^V = \varphi^k$, $p_i^V = p_i^k$. Another frequent case is when $V_i = S^{a_i}(X) = X^{a_i+1}$, where $a_0, \dots, a_r$ are integers such that $a_0 + a_1 + \dots + a_r = k - r$. (Zak makes the additional assumption that $a_0 \le a_1 \le \dots \le a_r$. This is not needed other than as a normalizing convention, and since it tends to destroy the symmetry, we will omit this assumption here.) Again we let a denote the tuple $(a_0, \dots, a_r)$, and put $\varphi^V = \varphi^a = \varphi^{a_0,\dots,a_r}$, $p_i^V = p_i^a = p_i^{a_0,\dots,a_r}$ and $S_V = S_{S^a(X)} = S_{S^{a_0}(X),\dots,S^{a_r}(X)}$ where $S^a(X)$ denotes the tuple $(S^{a_0}(X), \dots, S^{a_r}(X))$. Clearly we have $\varphi^a(S_{S^a(X)}) = \varphi^k(S^k_X) = S^k(X)$.

Letting $v_k \in S^k(X)$, we define

$$Y_{v_k} = p_0^{0,k-1}((\varphi^{0,k-1})^{-1}(v_k)) \subseteq X \tag{1}$$

as the *variety of entry points* for v_k in X. For a general point $v_k \in S^k(X)$ it is the closure of the subset

$$Y^0_{v_k} = \{x \in X \mid \exists v_{k-1} \in S^{k-1}(X) \text{ such that } v_k \in x v_{k-1} \text{ and } v_{k-1} \ne x\}.$$

More generally, let $a_0 + a_1 = k - 1$ as above. Then we have the morphisms

$$\varphi^{a_0,a_1} \colon S_{S^{a_0}(X),S^{a_1}(X)} \longrightarrow S^k(X),$$
$$p_i^{a_0,a_1} \colon S_{S^{a_0}(X),S^{a_1}(X)} \longrightarrow S^{a_0}(X).$$

We define, for $v_k \in S^k(X)$, $Y^{a_0}_{v_k} = p_0^{a_0,a_1}((\varphi^{a_0,a_1})^{-1}(v_k))$, $Y^{a_1}_{v_k} = p_1^{a_0,a_1}((\varphi^{a_0,a_1})^{-1}(v_k))$.

In general we put

$$s_k(X) = \dim(S^k(X)) \quad \text{and} \quad \delta_k(X) = \dim(Y_{v_k})$$

where v_k is a general point of $S^k(X)$. With $n = \dim(X)$, we have the result below, which shows that this definition of $\delta_k(X)$ is equivalent to the one given in the introduction. When no confusion is possible, we shall write s_k instead of $s_k(X)$ and δ_k instead of $\delta_k(X)$:

Proposition 1.1. *The following equality holds for all k such that $S^k(X)\neq\mathbf{P}^N$:*

$$s_k=s_{k-1}+n+1-\delta_k.$$

The *proof* is straightforward. The following forms part of Proposition 3 in [Z1]:

Proposition 1.2. *We have the inequalities*

$$\delta_1\leq\delta_2\leq\cdots\leq\delta_{k_0}\leq n$$

where k_0 is minimal among the numbers k such that $S^k(X)=\mathbf{P}^N$.

Next, we recall the following basic result. The simplest proof we know is due to *B. Ådlandsvik*, [Å, Corollary 1.10].

Theorem 1.3 (Terracini's Lemma). *Let X and Y be subvarieties of $\mathbf{P}_k^N$, where k is an algebraically closed field, not necessarily of characteristic* 0. *Let $x\in X$ and $y\in Y$, and let u be a point on the line xy. Then we have an inclusion of embedded tangent spaces $t_{XY,u}\supseteq t_{X,x}t_{Y,y}$. If k is of characteristic* 0, *then this inclusion is an equality for all u in some open dense subset of the join XY.*

Let $Z\subset\mathbf{P}^N$ be any projective scheme, and let $Y\subset Z$ be a projective subscheme. Then we denote by $T^*(Y,Z)$ the union of all *relative tangent stars* $t^*_{Y,Z,y}$ as $y\in Y$. $t^*_{Y,Z,y}$ is the union of all lines through y which are limiting positions of lines $y'z$, with $y'\in Y$ and $z\in Z$. The (usual) tangent star at a point $z\in Z$ is the union of all lines through z which are limiting positions of secant lines to Z, thus $t^*_{Z,z}=t^*_{Z,Z,z}$. We have the diagram

$$\begin{tikzcd}
\mathbf{T}_Y(Z) \arrow[r, hook] \arrow[d, "f_{Y,Z}"] & \widetilde{Y\times Z} \arrow[r, hook] \arrow[d, "\pi_{\Delta(Y)}"] & \widetilde{\mathbf{P}^N\times\mathbf{P}^N} \arrow[r, "\lambda"] \arrow[d, "\pi_\Delta"] & \mathbf{P}(\Omega^1_{\mathbf{P}^N}) \arrow[d, "f"] \\
Y \arrow[r, hook] & Y\times X \arrow[r, hook] & \mathbf{P}^N\times\mathbf{P}^N \arrow[r, "pr_2"'] & \mathbf{P}^N
\end{tikzcd}$$

Here π_Δ denotes the blowing up of $\mathbf{P}^N\times\mathbf{P}^N$ with center in the diagonal Δ, $\pi_{\Delta(Y)}$ the blowing up of $Y\times Z$ with center in the diagonal $\Delta(Y)$ in $Y\times Y$, identified with the canonical subscheme of $Y\times Z$, $\mathbf{T}_Y(Z)$ is the exceptional divisor. λ induces the identity on the exceptional divisor of the blowing up π_Δ. λ is a $\mathbf{P}^1$-bundle. We have that

$$YZ=pr_1(\pi_\Delta(\lambda^{-1}(\lambda(\widetilde{Y\times Z})))).$$

In fact, the fiber $f^{-1}(y)$ parametrizes the directions of all lines through the point $y\in\mathbf{P}^N$. Thus if $t\in\mathbf{P}(\Omega^1_{\mathbf{P}^N})$, then the fiber $\lambda^{-1}(t)$ consists of a line with a selected point on it—namely, the point $y=f(t)$ and the line through it given by the direction which corresponds to t. So for $y \in Y$, the limiting position of secant lines $y'z$, as $y' \in Y$ and $z \in Z$ approach $y \in Y$, are parametrized by $\mathbf{T}_Y(Z)_y$. In particular, if $Y = Z$ then this fiber is the projectivized (usual) tangent star, and if y is a smooth point then it is the projectivized tangent space. In general it is the projectivized *relative tangent star.* Thus

$$t^*_{Y,Z,y} = pr_1(\pi_\Delta(\lambda^{-1}(\lambda(\mathbf{T}_Y(Z)_y))))$$

and

$$T^*(Y,Z) = pr_1(\pi_\Delta(\lambda^{-1}(\lambda(\mathbf{T}_Y(Z))))).$$

We now have the following result, which is due to Zak:

Theorem 1.4. *If $T^*(Y,Z) \neq YZ$, then*

$$\dim(YZ) = \dim(Y) + \dim(Z) + 1.$$

Proof. Assume that

$$m = \dim(YZ) \leq \dim(Y) + \dim(Z)$$

but that $T^*(Y,X) \neq YZ$. Choose a linear subspace $L \subset \mathbf{P}^N$ of codimension m, so $L \cap YZ \neq \emptyset$, and such that $L \cap T^*(Y,Z) = \emptyset$ and $L \cap Z = \emptyset$. Let

$$p = p_L : Z \longrightarrow \mathbf{P}^{m-1}$$

be induced by the projection with center L. Let

$$\varphi = p \times p\,|_{Y \times Z} : Y \times Z \longrightarrow \mathbf{P}^{m-1} \times \mathbf{P}^{m-1}$$

be the induced morphism. Then φ is a finite morphism, and hence $\dim \varphi(Y \times Z) = \dim(Y) + \dim(Z) \geq m$. We recall the *Connectedness Theorem of Fulton and Hansen* (see Theorem 3.1 in [FL]):

Theorem 1.5. *Let X be a complete variety, and let $f : X \longrightarrow \mathbf{P}^m \times \mathbf{P}^m$ be a morphism with* $\dim(f(X)) > m$. *If Δ denotes the diagonal, then $f^{-1}(\Delta)$ is connected.*

Thus in our situation we get that $\varphi^{-1}(\Delta_{\mathbf{P}^{m-1}})$ is *connected.* Moreover, there exist $y \in Y$ and $z \in Z$ such that $y \neq z$ and $L \cap yz \neq \emptyset$. But $\varphi^{-1}(\Delta_{\mathbf{P}^{m-1}}) = \Delta_Y \cup \overline{D}$, where

$$D = \{\, (y,z) \in Y \times Z \mid y \neq z \text{ and } p(y) = p(z) \,\}.$$

By connectedness we therefore have $\overline{D} \cap \Delta_Y \neq \emptyset$. Let (y,y) be a point in this intersection. Then there exists a line $l \subset \mathbf{P}^N$, which is a limiting position of lines of the form $y'z$ meeting L, where $y' \in Y$ and $z \in Z$ when y', $z \longrightarrow y$. Then $l \cap L \neq \emptyset$. As $l \subset T^*(Y,Z)$ and $L \cap T^*(Y,Z) = \emptyset$, this is a contradiction, and the proof is complete.

Following Zak we now prove the corollary below.

Corollary 1.6. *If $S^k(X) \neq \mathbf{P}^N$, then $\delta_k(X) \leq n - \delta_1(X)$.*

Indeed, let u be a general point of $S^k(X)$. Then by Terracini's Lemma we get $T^*(Y_u, X) \subset t_{S^k(X),u}$. Hence we must have $Y_u X \neq T^*(Y_u, X)$, so $\dim(Y_u X) = \delta_k(X) + n + 1$. Since $S^1(X) \supset Y_u X$ and $\dim(S^1(X)) = 2n + 1 - \delta_1(X)$, the claim follows.

2. Comparison of the cases $q=1$ and $q>1$

Let p, q be integers such that $p, q \geq 0$ and $p+q=k$. We then have the following diagram, where the dotted arrows are only rational maps.

$$\begin{tikzcd}
 & S_{X,S^{p-1}(X),S^{q-1}(X)} \arrow[dl, dotted, "\lambda"'] \arrow[dr, dotted, "\mu"] \arrow[dd, "\varphi^{0,p-1,q-1}"] & \\
S_{S^p(X),S^{q-1}(X)} \arrow[dr, "\varphi^{p,q-1}"'] & & S_{S^p(X),S^{q-1}(X)} \arrow[dl, "\varphi^{p-1,q}"] \\
 & S^k(X) &
\end{tikzcd} \tag{1}$$

On the dense open subset where it is defined, λ is given by $\lambda(x, v_{p-1}, v_{q-1}, u) = (v_p, v_{q-1}, u)$ where $\{v_p\} = xv_{p-1} \cap v_{q-1}u$. Similarly $\mu(x, v_{p-1}, v_{q-1}, u) = (v_{p-1}, v_q, u)$ where $\{v_q\} = xv_{q-1} \cap v_{p-1}u$. The situation is shown in the Diagram (2):

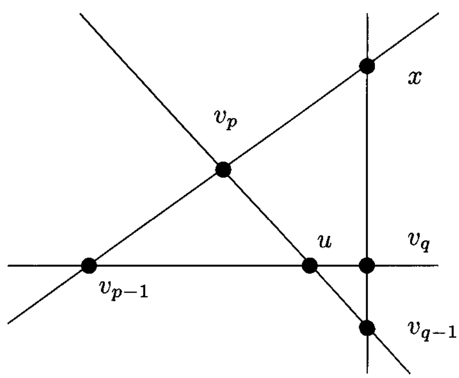

Diagram (2)

We will now show that

$$\varphi^{p,q-1}(\lambda(\mu^{-1}(v_{p-1}, v_q, u))) = u. \tag{3}$$

Indeed, let $\Gamma(\mu)\subset S_{X,S^{p-1}(X),S^{q-1}(X)}\times S_{S^{p-1}(X),S^q(X)}$ denote the closed graph of the correspondence μ. Then $\mu^{-1}(v_{p-1},v_q,u)=\Gamma(\mu)_{(v_{p-1},v_q,u)}$ which is contained in $\Gamma(\mu)_u$. Thus equality (3) follows. (3) implies similarly that

$$\lambda^{-1}(\lambda(\mu^{-1}(v_{p-1},v_q,u)))\subseteq(\varphi^{0,p-1,q-1})^{-1}(u). \tag{4}$$

Thus

$$p_0^{0,p-1,q-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{p-1},v_q,u)))]\subseteq p_0^{0,p-1,q-1}((\varphi^{0,p-1,q-1})^{-1}(u)). \tag{5}$$

In fact

$$\begin{aligned} Y_u &= p_0^{0,a_1,\dots,a_r}((\varphi^{0,a_1,\dots,a_r})^{-1}(u)) \\ &= p_0^{0,a_1,\dots,a_r}((S_{X,S^{a_1}(X),\dots S^{a_r}(X)})_u) \end{aligned} \tag{6}$$

when $a_1+\dots+a_r=k-r$. Hence for a general point $u\in S^k(X)$,

$$\delta_k=\dim(Y_u)\geq\dim p_0^{0,p-1,q-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{p-1},v_q,u)))]. \tag{7}$$

We next show that

$$\dim\lambda(\mu^{-1}(v_{p-1},v_q,u))=\dim\mu^{-1}(v_{p-1},v_q,u)=\delta_q. \tag{8}$$

We note first that λ and μ obviously are dominating. Next we compute the dimension of the fiber of μ and its image under λ. Since we are considering a general point u, we may restrict our attention to open dense subschemes of $S_{X,S^{p-1}(X),S^{q-1}(X)}$, $S_{S^{p-1}(X),S^q(X)}$ and $S_{X,S^p(X),S^{q-1}(X)}$. Thus *we may assume that* λ *and* μ *are morphisms*: Namely, we have the situation

$$\begin{tikzcd} & \Gamma(\mu) \arrow[r, "\mu"] & S_{S^{p-1}(X),S^q(X)} \\ S_{X,S^{p-1}(X),S^{q-1}(X)} \supset & D(\lambda,\mu) \arrow[u, hook] \arrow[d, hook] \arrow[ur, "\bar{\mu}"'] \arrow[dr, "\bar{\lambda}"] & \\ & \Gamma(\lambda) \arrow[r, "\lambda"'] & S_{S^p(X),S^{q-1}(X)} \end{tikzcd} \tag{9}$$

where $D(\mu,\lambda)$ is (isomorphic to) a dense open subset of $\Gamma(\mu)$ and $\Gamma(\lambda)$ and is such that both μ and λ are defined on it. Then it follows that for a general point $\sigma\in S_{S^{p-1}(X),S^q(X)}$

$$\begin{aligned} \dim\mu^{-1}(\sigma) &= \dim\Gamma(\mu)-\dim S_{S^{p-1}(X),S^q(X)} \\ &= \dim D(\mu,\lambda)-\dim S_{S^{p-1}(X),S^q(X)} \\ &= \dim\bar{\mu}^{-1}(\sigma). \end{aligned}$$

Now assume that

$$\dim \bar{\mu}^{-1}(\sigma) = \dim \bar{\lambda}(\bar{\mu}^{-1}(\sigma)).$$

To show is that

$$\dim \mu^{-1}(\sigma) = \dim \bar{\lambda}(\mu^{-1}(\sigma)).$$

But this is clear since $\geq$ holds, and moreover $=$ holds if we replace the fiber $\mu^{-1}(\sigma)$ by the open (not necessarily dense but of the same dimension) subset $\bar{\mu}^{-1}(\sigma)$. This proves the emphasized assertion above.

To prove (8), note that when we observed that λ is dominating, then the set of points x satisfying the condition of the argument

$$\exists v_{p-1} \text{ such that } v_p \in x v_{p-1} \text{ and } (x, v_{p-1}, v_{q-1}, u) \text{ maps to } (v_p, v_{q-1}, u)$$

constitute a dense subset of the variety Y_{v_p} of entry points for v_p in X. Thus

$$\dim \lambda^{-1}(v_p, v_{q-1}, u) = \delta_p \tag{10}$$

and similarly

$$\dim \mu^{-1}(v_{p-1}, v_q, u) = \delta_q \tag{11}$$

which proves the last equality in (8). To prove the first equality, it now suffices to show that $\bar{\lambda}$ induces a generically finite morphism

$$\nu \colon \bar{\mu}^{-1}(v_{p-1}, v_q, u) \longrightarrow S_{S^p(X), S^{q-1}(X)}.$$

To see this, let $(v_p, v_{q-1}, u) \in \operatorname{im}(\nu)$. Then the set of points in $\bar{\mu}^{-1}(v_{p-1}, v_q, u)$ which map to (v_p, v_{q-1}, u) consist of those points (x, v_{p-1}, v_{q-1}, u) such that the configuration in Diagram (2) holds. In this picture v_p, v_{p-1}, v_q, v_{q-1} and u are *given*. This determines x uniquely, unless $v_p = v_{p-1}$, and (hence) $v_q = v_{q-1}$. But then $u \in S^{k-1}(X)$, contradicting the assumption that u be a general point of $S^k(X)$. Thus we have shown that ν is generically 1-1. Hence the remaining part of (8) follows.

It is an immediate consequence of (8) and (10) that

$$\dim \lambda^{-1}[\lambda(\mu^{-1}(v_{p-1}, v_q, u))] = \delta_p + \delta_q. \tag{12}$$

In view of (7) and (12) the proof of the general version of *Zak's Theorem on Superadditivity* would be complete if we could show the following:

Lemma 2.1. (First Conjectured Lemma) *The restriction of the morphism*

$$p_0^{0,p-1,q-1}\colon S_{X,S^{p-1}(X),S^{q-1}(X)} \longrightarrow X$$

to the closed subscheme $\lambda^{-1}[\lambda(\mu^{-1}(v_{p-1},v_q,u))]$ *is finite-to-one at a generic point of that subscheme.*

Analysis of proof. Let

$$y \in p_0^{0,p-1,q-1}[\lambda^{-1}(\lambda(\mu^{-1}(v_{p-1},v_q,u)))]$$

be a general point. Then $(p_0^{0,p-1,q-1})^{-1}(y)$ consits of quadruples $(y, v'_{p-1}, v_{q-1}, u)$ where

$$v_p \in yv'_{p-1}, \quad \{v_p\} = xv_{p-1} \cap v_{q-1}u, \quad v_q \in xv_{q-1}.$$

To show is that there only exist a finite number of such triples. The configuration is shown in Diagram (13):

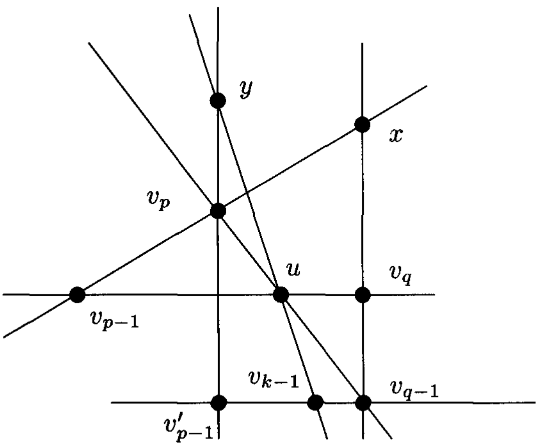

Diagram (13)

Choose v_{k-1} as indicated, $\{v_{k-1}\} = v'_{p-1}v_{q-1} \cap yu$. Then

$$\begin{aligned} Y^{q-1}_{v_{k-1}} &= p_1^{p-1,q-1}[(\varphi^{p-1,q-1})^{-1}(v_{k-1})] \\ &\supseteq \{w \in S^{q-1}(X) \mid \exists v'_p \in S^{p-1}(X) \text{ such that } w \in v_{k-1}v'_{p-1} \text{ and } v'_{p-1} \neq w\}, \end{aligned}$$

where the subset on the second line is dense in $Y^{q-1}_{v_{k-1}}$. Similarly, we also have

$$\begin{aligned} Y^{q-1}_{v_q} &= p_1^{0,q-1}[(\varphi^{0,q-1})^{-1}(v_q)] \\ &\supseteq \{w \in S^{q-1}(X) \mid \exists x \in X \text{ such that } w \in v_q x \text{ and } x \neq w\}, \end{aligned}$$

where again the subset is dense.

Now the configuration in Diagram (13) holds if and only if v_{q-1} satisfies the two conditions for w, i.e. $v_{q-1}\in Y^{q-1}_{v_{k-1}}\cap Y^{q-1}_{v_q}$. Thus the assertion of Lemma 2.1 would follow if we could show the

Lemma 2.2 (Second Conjectured Lemma). *If y is a general point as above, and v_{k-1} is an entry point for uy in $S^{k-1}(X)$, then $Y^{q-1}_{v_{k-1}}\cap Y^{q-1}_{v_q}$ is a finite set.*

This, in turn, would follow from the

Lemma 2.3. (Third Conjectured Lemma) *Suppose that $S^{k-1}(X)\neq\mathbf{P}^N$. Let $v_{k-1}\in S^{k-1}(X)$ be a general point, and $Y^{q-1}_{v_{k-1}}$ be the locus of entry points for v_{k-1} in $S^{q-1}(X)$, as above. Then* $\dim(Y^{q-1}_{v_{k-1}}X)=\dim(Y^{q-1}_{v_{k-1}})+n+1$.

We show that Lemma 2.3 implies Lemma 2.2: Namely, assume the negation. Then for fixed, general v and y, $v=(v_{p-1},v_q,u)$, $y\in p_0^{0,p-1,q-1}[\lambda^{-1}(\lambda(\mu^{-1}(v)))]$ there would exist an infinite number of points (y,v'_{p-1},v_{q-1},u) satisfying the configuration in Diagram (13). Thus the locus of points $x\in X$ which can appear in Diagram (13) is infinite. This is easily seen to contradict the hypothesis that $Y^{q-1}_{v_{k-1}}$ be of maximal dimension. Thus the implication is proven.

The proof of superadditivity is completed for $q=1$ by the following

Proposition 2.4. *The conjectured Lemma 2.3 is true when $q=1$.*

Proof. We have $S^{q-1}(X)=X$, put $Y^{q-1}_{v_{k-1}}=Y\subset X$. It suffices to show

$$T^*(Y,X)\neq YX. \tag{14}$$

To see this, let $L=t_{S^{k-1}(X),v_{k-1}}$. v_{k-1} is a general point of $S^{k-1}(X)$, hence smooth. Thus $\dim L=s_{k-1}$, in particular $L\neq\mathbf{P}^N$. Since $X\not\subseteq L$, we have $YX\not\subseteq L$, so (14) will follow if we can show that $T^*(Y,X)\subseteq L$. X is smooth, so for all $y\in Y$

$$t^*_{Y,X,y}=t_{X,y}\subset L, \tag{15}$$

and the claim follows.

In order to make this work for $q>1$, one needs a generalization of (14):

$$T^*(Y^{q-1}_{v_{k-1}},S^{q-1}(X))\neq Y^{q-1}_{v_{k-1}}S^{q-1}(X)=Y^{q-1}_{v_{k-1}}X^q. \tag{16}$$

This will imply the claim in Lemma 2.3: By Theorem 1.4, (16) implies that

$$\dim(Y^{q-1}_{v_{k-1}}S^{q-1}(X))=\dim(Y^{q-1}_{v_{k-1}})+s_{q-1}+1.$$

But since $q \geq 2$, we can write $Y^{q-1}_{v_{k-1}} S^{q-1}(X) = Y^{q-1}_{v_{k-1}} X S^{q-2}(X)$, and hence obtain the diagram

$$
\begin{tikzcd}
S_{Y^{q-1}_{v_{k-1}},X} \arrow[r, hook] \arrow[d, "\beta"] & S_{Y^{q-1}_{v_{k-1}},X,S^{q-2}(X)} \arrow[r, "\varphi"] \arrow[d, "\beta'"] & S_{Y^{q-1}_{v_{k-1}},S^{q-1}(X)} \arrow[dl, "\alpha"'] \\
Y^{q-1}_{v_{k-1}} X \arrow[r, hook] & Y^{q-1}_{v_{k-1}} S^{q-1} &
\end{tikzcd}
\tag{17}
$$

Let $u \in Y^{q-1}_{v_{k-1}} S^{q-1}(X)$ be a general point. Then there exists a point $v \in Y^{q-1}_{v_{k-1}} X$ such that $u \in vw$ for some $w \in S^{q-2}(X)$. For the purpose of computing fiber dimensions we may replace S by S^0 in the top level of the diagram. Then every point in $\beta^{-1}(v)$ gives a point in $\beta'^{-1}(u)$ by adding the coordinate w. Now $\varphi(\beta'^{-1}(u)) = \alpha^{-1}(u)$. We need to conclude $\dim(\alpha^{-1}(u)) \geq \dim(\beta^{-1}(u))$. This follows if we have φ finite-to-one over $\alpha^{-1}(u)$. If this were not so, there would exist a point $(y, v_{q-1}, u) \in S_{Y^{q-1}_{v_{k-1}},S^{q-1}(X)}$ with $\varphi^{-1}(y, v_{q-1}, u)$ infinite. Thus the line yv_{q-1} would meet $S^{q-2}(X)$ in infinitely many points and hence be contained in $S^{q-2}(X)$. So $u \in S^{q-2}(X)$, i.e., not a general point. Thus it suffices to prove (16), which in the present approach amounts to proving the following assertion:

$$\text{For all } y \in Y^{q-1}_{v_{k-1}} \text{ we have } t^*_{Y_{v_{k-1}},S^{q-1}(X),y} \subseteq t_{S^{k-1}(X),v_{k-1}}. \tag{18}$$

Remark 2.5. This presents a non-trivial difficulty. The assertion of (18) is *false* for some counter examples to superadditivity for which $\delta_1 = 0$, see below. Thus this is where the assumption of $\delta_1 > 0$ would have to be used: In fact, it was not needed in the case of $q = 1$, as we have noted above. If X is a cone (thus singular), then (18) fails completely, as is easily seen.

Let Y be a nonsingular rational curve of degree d in $\mathbf{P}^{N-2} \subset \mathbf{P}^N$, and let L be a line such that $L \cap \mathbf{P}^{N-2} = \emptyset$, so that $C_L(Y)$ is a 3-dimensional cone in $\mathbf{P}^N$.

In [Da], M. Dale constructed families of smooth surfaces $X_{d,b} \subset C_L(Y)$. The subscript b is the degree of the curve which is the intersection of the surface $X = X_{d,b}$ with a generating plane of the cone $C_L(Y)$. In the case where $b = 1$ and Y is a rational normal curve, this type of surface is a rational normal scroll. Let B be the blowup of $\mathbf{P}^N$ along L. It is well known that there is a commutative diagram:

$$
\begin{tikzcd}
& B \arrow[dl, "\pi"'] \arrow[d, hook] \arrow[dr, "\lambda"] & \\
\mathbf{P}^N & \mathbf{P}^N \times \mathbf{P}^{N-2} \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & \mathbf{P}^{N-2}
\end{tikzcd}
\tag{19}
$$

where π is the structural map of the blowing up, and λ is a $\mathbf{P}^2$-bundle map. In fact, B is the graph of projection from L, and the exceptional divisor is isomorphic to $L\times\mathbf{P}^{N-2}$. It is also well known that $B\cong\mathbf{P}(\mathcal{O}^{\oplus 2}_{\mathbf{P}^{N-2}}\oplus\mathcal{O}_{\mathbf{P}^{N-2}}(1))$ as schemes over $\mathbf{P}^{N-2}$.

Now, we consider a nonsingular rational curve $Y\subset\mathbf{P}^{N-2}\subset\mathbf{P}^N$. It will always be assumed that Y spans $\mathbf{P}^{N-2}$. The blowing up $\widetilde{C_L(Y)}$ along L of the cone $C_L(Y)$ fits into a commutative diagram:

(20)
$$\begin{tikzcd} & \widetilde{C_L(Y)} \arrow[dl, "\pi"'] \arrow[d, hook] \arrow[dr, "\lambda"] & \\ C_L(Y) & C_L(Y)\times Y \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & Y \end{tikzcd}$$

and $\widetilde{C_L(Y)}$ is the closure of the graph of the projection from L of $C_L(Y)$ onto Y. Alternatively, $\widetilde{C_L(Y)}\cong\mathbf{P}(\mathcal{O}_Y\oplus\mathcal{O}_Y(1))$. Since $Y\cong\mathbf{P}^1$ is embedded as a curve of degree d, we have an isomorphism $\widetilde{C_L(Y)}\cong\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}\oplus\mathcal{O}_{\mathbf{P}^1}(d))$ of schemes over $\mathbf{P}^1$. Let E be the exceptional divisor of the blowing up π. It follows by the observation above that λ is a $\mathbf{P}^2$-bundle and that $E\cong L\times Y\cong\mathbf{P}^1\times\mathbf{P}^1$.

For any $b\geq 1$, the invertible sheaf $\mathcal{L}(b)=\mathcal{O}_{\mathbf{P}(\mathcal{F})}(b)\otimes\lambda^*(\mathcal{O}_{\mathbf{P}^1}(1))$ is very ample. Following Dale, we define $Z\subset C_L(Y)$ to be the scheme of zeros of a sufficiently general section of $\mathcal{L}(b)$, so that Z is a nonsingular surface and $Z\cap E$ is a nonsingular curve on $E\cong\mathbf{P}^1\times\mathbf{P}^1$. We define $X_{d,b}=\pi(Z)$.

Proposition 2.6. *Let $X=X_{d,b}$ be defined as above. Then X is a nonsingular surface.*

Proof. Because of the isomorphism $E\cong\mathbf{P}^1\times\mathbf{P}^1$ and our hypotheses about Z, it follows that $Z\cap E$ is a smooth curve of type $(b,1)$ on $\mathbf{P}^1\times\mathbf{P}^1$. Hence Z intersects each fiber of the projection $\pi\colon C_L(Y)\longrightarrow C_L(Y)$ transversally at a single point. Thus π maps Z bijectively onto its image, and $\pi|_Z$ is unramified. Therefore $\pi|_Z$ is proper and quasi-finite, and hence finite. Since $\pi|_Z$ is also bijective and unramified it follows that Z is mapped isomorphically onto its image.

We are now ready to state and prove our main results about these surfaces.

Theorem 2.7. *Let Y be a nonsingular rational curve of degree d in $\mathbf{P}^{N-2}$, and let $X=X_{d,b}\subset C_L(Y)$ be defined as above. Then:*

(i) *$S^k(X)=C_L(S^k(Y))$ for all $k\geq 1$. Equivalently, $X^k=Y^kL$ for all $k\geq 2$.*

(ii) *If $N\geq 5$, then $\delta_1(X)=0$.*

(iii) $\dim(S^k(X))=2k+3$ *for all* k *such that* $2k+3\leq N$.

(iv) $\delta_k(X)=1$ *for all* $k>1$ *such that* $2k+3\leq N$.

Corollary 2.8. *If* $N\geq 11$, *then* $\delta_4(X)<2\delta_2(X)$.

Proof. Without loss of generality, we may assume that $N\geq 5$. Since Y is a curve which spans $\mathbf{P}^{N-2}$, we have $\dim(S^k(Y))=2k+1$ for all k such that $2k+1\leq N-2$ (see [Å, Corollary 1.5]).

Clearly $S^k(X)\subset C_L(S^k(Y))$ for all k. In particular, $\dim(C_L(S(Y)))=5$ and $\dim(S(X))=5$ because X is not the Veronese surface. Thus $S(X)=C_L(S(Y))$ and $\dim(S^k(X))\leq\dim(C_L(S^k(Y)))=2k+3$. Let k_0 be the first subscript k such that $\dim(S^k(X))<2k+3$. Then $\dim(S^{k_0-1}(X))=2k_0+1$, and $\dim(S^{k_0}(X))\leq 2k_0+2$. Now Proposition 1.3 of [Å] asserts that if X and Y are subvarieties of $\mathbf{P}^N$ such that $\dim(XY)=\dim(X)+1$, then X is a cone whose vertex contains Y. This implies that $S^{k_0}(X)=XS^{k_0-1}(X)$ is a cone whose vertex contains X, but as the vertex is a linear subspace, $S^{k_0}(X)=\mathbf{P}^N$. Hence $S^k(X)=C_L(S^k(Y))$ for all $k<k_0$. This implies (i). (iii) and (iv) are easy consequences of (i).

We shall see how the proof of superadditivity fails if applied to $X=X_{d,b}$. We investigate whether or not Lemmas 2.1, 2.2 and 2.3 are valid in this case. In particular suppose that $k=4$ and $p=q=2$. Since $S^{q-1}(X)=S(X)$ is a cone with vertex L and $S^{k-1}(X)=S^3(X)=S(S(X))$, it follows that for a general point $v_{k-1}\in S^{k-1}(X)$, the entry point set $Y^{q-1}_{v_{k-1}}$ is a union of planes which contain L. The entry point set $Y^{q-1}_{v_q}$ has dimension $\delta_2=1$. It is not hard to see that $Y^{q-1}_{v_q}$ is a union of curves which are contained in planes which contain L. Since a general point of $S^{q-1}(X)=S(X)$ is contained in a unique plane which contains L, the intersection $Y^{q-1}_{v_{k-1}}\cap Y^{q-1}_{v_q}$ contains at least one of these curves. Thus, the intersection is not finite. In other words, the conclusion of Lemma 2.3 does not hold. The conclusion of Lemma 2.3 also does not hold. The point is that $Y^{q-1}_{v_{k-1}}$ is a cone with vertex L. Since X intersects each generating plane of the cone $C_L(Y)$ in a curve, we have $t_{X,x}\cap L\neq\emptyset$ for every $x\in X$. Therefore, it follows from Terracini's Lemma that $\dim(Y^{q-1}_{v_{k-1}}X)<\dim(Y^{q-1}_{v_{k-1}})+\dim(X)+1$.

Returning to general theory, note that clearly (18) means that the singularities of $S^{q-1}(X)$ are not "too bad" along $Y^{q-1}_{v_{k-1}}$: The restricted tangent stars are not too much bigger than the tangent cones there.

The approach taken by *Barbara Fantechi* in [Fa] is to introduce a condition on the singularities which the higher secant variety $S^{q-1}(X)$ can have: In general she defines a point z of the embedded variety $Z\subset\mathbf{P}^n$ to be *almost smooth* if the tangent star of Z at z is contained in the join zZ. With the above considerations it is then proved that the conjectured lemmas hold if one assumes that $S^{q-1}(X)$ is almost smooth, see her Theorem 2.5.

While this certainly settles the question for $q=1$, it is not clear how to verify the condition of almost smoothness for $S^{q-1}(X)$ when $q>1$. Also, the condition is rather special since one can have superadditivity without this condition being satisfied, as we can see from the following class of examples.

Let $2\leq d\leq n/2$, and let $V=\mathbf{P}^d\times\mathbf{P}^{n-d}\subset\mathbf{P}^N$, embedded by the Segre embedding. We consider $\mathbf{P}^N\subset\mathbf{P}^{N+1}$, and set $Z=C_P(V)$, where $P\in\mathbf{P}^{N+1}-\mathbf{P}^N$. Define $X=Z\cap H$, where H is a general hypersurface of degree ≥ 2 in $\mathbf{P}^{N+1}$ such that $P\notin H$. Thus X is a nonsingular variety of dimension n.

Proposition 2.9. *In the situation above we have*
(i) $S^k(X)=C_P(S^k(V))$ *for all* $k\geq 1$.
(ii) $\delta_1(X)=1$.
(iii) $\delta_k(X)=2k$ *for* $k=2,\dots,d$, *and* $S^d(X)=\mathbf{P}^{N+1}$.

Corollary 2.10. *If* $d\geq 2$, *then* $\delta_2(X)>2\delta_1(X)$.

Proof. We show first that $\delta_k(V)=2k$ for all $k\leq d$ and $S^d(V)=\mathbf{P}^N$. Indeed, $V\subset\mathbf{P}^N$ can be described as the zero-set of the 2×2 minors of a $(d+1)\times(n-d+1)$ matrix of homogeneous indeterminates and $S^k(V)$ is then the zero-set of the $(k+2)\times(k+2)$ minors of this matrix. Hence, the codimension of $S^k(V)$ in $\mathbf{P}^N$ is $(d-k)(n-d-k)$ for $k\leq d$, and $S^d(V)=\mathbf{P}^N$. It follows that

$$\dim(S^k(V))-\dim(S^{k-1}(V))=\operatorname{codim}(S^{k-1}(V))-\operatorname{codim}(S^k(V))=n+1-2k,$$

for $k=1,\dots,d$. This implies that $\delta_k(V)=2k$ for $k\leq d$.

In particular, we have $\dim(S(V))=2n-1$. Since $Z=C_P(V)$, it follows that we have $\dim(S(Z))=\dim(S(V))+1=2n$. By Terracini's Lemma Theorem 1.3, this implies that $\dim(t_{Z,x}\cap t_{Z,y})=2$ for general points $x,y\in X\subset Z$. The tangent spaces $t_{X,x}$ and $t_{X,y}$ are the intersections of $t_{Z,x}$ and $t_{Z,y}$ with the hyperplanes $t_{H,x}$ and $t_{H,y}$ respectively. It follows that if H is sufficiently general, then

$$\dim(t_{X,x}\cap t_{X,y})=\dim(t_{Z,x}\cap t_{Z,y})-2=0.$$

By Terracini's Lemma, this implies that $\dim(S(X))=2n$. Therefore $S(X)=S(Z)$.

It is clear that $S^k(X)\subset S^k(Z)=C_P(S^k(V))$. This implies that $\dim(S^k(X))\leq\dim(S^k(V))+1$. Since $S(X)=S(Z)=C_P(S(V))$, we have $P\in t_{S(X),v}$ for general $v\in S(X)$. Equivalently, $P\in t_{X,x}t_{X,y}$ for general $x,y\in X$. It follows that $P\in t_{S^k(X),u}$ for every $k\geq 1$ and general $u\in S^k(X)$. For general $x\in X$, projection from P maps $t_{X,x}$ onto $t_{V,x'}$, where $x'\in V$ is the image of x. Since $t_{S^k(X),u}$ is a span of tangent spaces $t_{X,x}$, projection from P must also map $t_{S^k(X),u}$ onto $t_{S^k(V),u'}$, where $u'\in S^k(V)$ is the image of u. Since $P\in t_{S^k(X),u}$, it follows that $\dim(t_{S^k(X),u})=\dim(t_{S^k(V),u'})+1$.

Therefore, $\dim(S^k(X))=\dim(S^k(V))+1$, and $S^k(X)=C_P(S^k(V))$. This proves (i). Conclusion (ii) is already proved, and (iii) follows because

$$\dim(S^k(X))-\dim(S^{k-1}(X))=\dim(S^k(V))-\dim(S^{k-1}(V)).$$

Remark 2.11. This class of varieties shows that *Fantechi's* notion of *almost smoothness* as a condition on $S^{q-1}(X)$ in order to have the superadditivity result for q is not sufficiently general: Indeed, by (i) in Proposition 2.9 we find that this condition does not hold in this case for $q\geq 1$: Recall that almost smoothness for the embedded variety $Z\subset\mathbf{P}^N$ at z means that the tangent star of Z at z is contained in the join zZ. When Z is a cone with z in the vertex, this will be true only when Z is a linear subspace of $\mathbf{P}^N$.

We end this section by some details concerning the local geometric structure of the secant variety $S^{q-1}(X)$. We put $Y=Y^{q-1}_{v_{k-1}}$. In the case where $S^{q-1}(X)$ is smooth at y, the inclusion $t^*_{Y,S^{q-1}(X),y}\subseteq t_{S^{k-1}(X),v_{k-1}}$ follows in the same way as in the proof of Proposition 2.4 above. In the case where $S^{q-1}(X)$ is singular at y, the validity of this inclusion seems more doubtful, for reasons which we will explain in the next paragraph.

It is easy to check the inclusion for any point $y\in Y$ such that v_{k-1} lies on a line of the form yz, where $z\in S^p(X)$ and $z\neq y$. Indeed, Terracini's lemma implies the stronger inclusion $t_{S^{q-1}(X),y}\subseteq t_{S^{k-1}(X),v_{k-1}}$. On the other hand, this inclusion is not obvious in the case when $y\in Y$ merely lies on some line l which is a limiting position of lines of the form $y'z'$, where $y'\in Y$, $z'\in S^p(X)$, and $v_{k-1}\in y'z'$. If $S^{q-1}(X)$ is smooth at y, then the inclusion follows as before: Indeed, the linear space $t_{S^{k-1}(X),v_{k-1}}$ contains certain irreducible components of the closed set $T=\bigcup_{y\in Y} t_{S^{q-1}(X),y}$ because $t_{S^{q-1}(X),y}\subseteq t_{S^{k-1}(X),v_{k-1}}$ for every y in some dense open subset of Y. If $S^{q-1}(X)$ is singular at y, it is not clear how to verify the corresponding inclusion. The point is that there does not seem to be any way to conclude that $t_{S^{k-1}(X),v_{k-1}}$ contains the particular irreducible component of T which contains $t_{S^{q-1}(X),y}$.

3. Entry point loci and their tangent spaces

We consider closed subvarieties $X\subset\mathbf{P}^N$ and $Y\subset\mathbf{P}^N$, and define XY to be their *join* as defined in the introduction and in §1. We will be especially interested in the case where $X\subset Y$, but this hypothesis will not actually be used in this section. In §4, however, we shall apply our results to the case where $Y=X^k$ for some $k\geq 2$.

Let $S\subset X\times Y\times\mathbf{P}^N$ be the *join correspondence*. Recall that S is the closure of

$$S_0=\{\,(x,y,z)\,|\,x\in X, y\in Y, x\neq y,\ \text{and}\ z\in xy\,\},$$

where as before xy denotes the line joining the points x and y, i.e., the join of $\{x\}$ and $\{y\}$. We have the maps $p_1: S \longrightarrow X$, $p_2: S \longrightarrow Y$, and $p_3: S \longrightarrow \mathbf{P}^N$ induced by the projections, and similarly the morphism $p_{1,2}: S \longrightarrow X \times Y$. If $(x,y) \in X \times Y - \Delta$, then $p_{1,2}^{-1}(x,y) \cong \mathbf{P}^1$. Thus, $\dim(S) = \dim(X) + \dim(Y) + 1$. Moreover $p_3(S) = XY$; it follows that $\dim(XY) \leq \dim(X) + \dim(Y) + 1$. Hence an immediate generalization of the secant defects considered earlier is the *join defect* defined by $\delta(X,Y) = \dim(X) + \dim(Y) + 1 - \dim(XY)$.

It follows from standard facts that $\dim(p_3^{-1}(z)) = \delta(X,Y)$ for general $z \in XY$. Our first goal is to describe the tangent spaces of the fibers $p_3^{-1}(z)$, where $z \in XY$, and $z \notin X \cup Y$. In an obvious way, we can define the embedded tangent space $t_{S,(x,y,z)}$ as a subspace of $\mathbf{P}^N \times \mathbf{P}^N \times \mathbf{P}^N$, or more precisely as a subspace of $t_{X,x} \times t_{Y,y} \times \mathbf{P}^N$, where $t_{X,x}$ and $t_{Y,y}$ are the enbedded tangent spaces of X and Y at x and y, respectively. Clearly, the tangent space at (x,y,z) to the fiber $p_3^{-1}(z)$ is contained in $t_{X,x} \times t_{Y,y} \times z$, and the linear tangent space map $(dp_3)_{(x,y,z)}$ is induced by the projection of $t_{X,x} \times t_{Y,y} \times \mathbf{P}^N$ onto the third factor. It follows from standard facts that the tangent space to the fiber is actually equal to $\ker(dp_3)_{(x,y,z)}$.

Proposition 3.1. *Let X and Y be as above; let $x \in X$ and $y \in Y$ be points such that $x \notin t_{Y,y}$ and that $y \notin t_{X,x}$, and let $z \in xy$ with $z \neq x, y$. Let $(\xi, \eta) \in t_{X,x} \times t_{Y,y}$. Then:*

(a) If (ξ, η, z) is contained in the tangent space of the fiber, then $\xi \in t_{X,x} \cap x t_{Y,y}$ and similarly $\eta \in t_{Y,y} \cap y t_{X,x}$.

(b) If $\xi \in t_{X,x} \cap x t_{Y,y}$, then there is a unique point $\eta \in t_{Y,y} \cap y t_{X,x}$ such that (ξ, η, z) is contained in the tangent space of the fiber.

Proof. As in [FR, §2], we begin by working with with affine open sets. We can choose an affine open piece $\mathbf{A}^N \subset \mathbf{P}^N$ with $x, y, z \in \mathbf{A}^N$. Let $X_0 = X \cap \mathbf{A}^N$ and $Y_0 = Y \cap \mathbf{A}^N$; assume that $x = (x_1, ..., x_N) \in X_0$ and $y = (y_1, ..., y_N) \in Y_0$. By renumbering variables we may assume that $y_1 \neq x_1$. Then the defining equations of S in some neighborhood of (x,y,z) are:

$$(z_i - x_i)(y_1 - x_1) = (z_1 - x_1)(y_i - x_i), \quad i = 2, ..., N$$

together with the defining equations of X_0 and Y_0 in the appropriate sets of variables. Therefore, the condition for $(x, h, z) \in \mathbf{A}^{3N}$ to be in the embedded tangent space of S at (x,y,z) is:

$$(1) \qquad (\zeta_i - \xi_i)(y_1 - x_1) + (\eta_1 - \xi_1)(z_i - x_i) = (\eta_i - \xi_i)(z_1 - x_1) + (\zeta_1 - \xi_1)(y_i - x_i)$$

for $i = 2, ..., N$, together with the defining equations of $t_{X,x}$ in the ξ-variables and the defining equations of $t_{Y,y}$ in the η-variables.

As noted in [FR], we may assume that the coordinate system is such that $x_i=y_i=z_i=0$ for $i=2,\dots,N$. Under this assumption, (1) simplifies to:

$$(\zeta_i-\xi_i)(y_1-x_1)=(\eta_i-\xi_i)(z_1-x_1), \quad i=2,\dots,N,$$

or equivalently:

$$(y_1-x_1)\zeta_i=(z_1-x_1)\eta_i-(z_1-y_1)\xi_i, \quad i=2,\dots,N. \tag{2}$$

Note that there is no condition on ζ_1, because the line $x_2=\dots=x_N=0$ lies in the join correspondence S. (More precisely, $x\times y\times xy$ lies in S.)

Suppose that $(\xi,\eta,\zeta)\in\ker(dp_3)_{(x,y,z)}$. The subspace $\ker(dp_3)_{(x,y,z)}\subset t_{S,(x,y,z)}$ is defined by the equations $\zeta_1-z_1=\zeta_2=\dots=\zeta_N=0$. If we substitute these equations into (2), we obtain:

$$\xi_i=\frac{z_1-x_1}{z_1-y_1}\eta_i \quad \text{for } i=2,\dots,N. \tag{3}$$

Conversely, it is easy to see that if (3) is satisfied, then $(\xi,\eta,z)\in\ker(dp_3)_{(x,y,z)}$.

To prove (a) it is enough to show that if (3) holds, then $(1,\xi_1,\dots,\xi_N)$ is a linear combination of $(1,x_1,0,\dots,0),(1,\eta_1,\dots,\eta_N)$, and $(1,y_1,0,\dots,0)$. Let $\varrho=(z_1-x_1)/(z_1-y_1)$. Then we can use (3) to show that

$$(1,\xi_1,\dots,\xi_N)-\varrho(1,\eta_1,\dots,\eta_N)=(1-\varrho,\xi_1-\varrho\eta_1,0,\dots,0).$$

Since $y_1\neq x_1$, the expression on the right side of this last equation is a linear combination of $(1,x_1,0,\dots,0)$ and $(1,y_1,0,\dots,0)$. This implies that $\xi\in t_{X,x}\cap xt_{Y,y}$, which proves (a).

To prove (b), suppose that $\xi\in t_{X,x}\cap xt_{Y,y}$. Then

$$(\xi_1,\dots,\xi_N)=\sigma(\eta_1',\dots,\eta_N')+(1-\sigma)(x_1,0,\dots,0)$$

for some $\sigma\in k$ and $(\eta_1',\dots,\eta_N')\in t_{Y,y}$. We define $(\eta_1,\dots,\eta_N)\in t_{Y,y}$ by the formulas

$$\eta_1-y_1=\frac{z_1-y_1}{z_1-x_1}\sigma(\eta_1'-y_1)=\frac{\sigma}{\varrho}(\eta_1'-y_1)$$
$$\eta_i=\frac{z_1-y_1}{z_1-x_1}\sigma\eta_i'=\frac{\sigma}{\varrho}\eta_i' \quad \text{for } i=2,\dots,N,$$

where ϱ is defined as above. With this choice of η it follows that (3) is satisfied, so that $(\xi,\eta,z)\in\ker(dp_3)_{(x,y,z)}$. It is clear from (3) that $\eta_2,\dots,\eta_N$ are uniquely determined by ξ. Since $x\notin t_{Y,y}$, any line parallel to xy meets $t_{Y,y}$ in at most one point. Therefore η_1 is also uniquely determined. This completes the proof of (b).

Let X and Y be closed subvarieties of $\mathbf{P}^N$, and let $u\in XY$. As in §1 we denote the variety of entry points for u in X by X_u. Recall that it is the closure of

$$\{\,x\mid x\in X \text{ and } u\in xy \text{ for some } y\in Y,\ y\neq x\,\}.$$

The entry point set Y_u in Y is given in a similar way. The following observation will be useful in proving the main result of this section.

Lemma 3.2. *Let X and Y be closed subvarieties of $\mathbf{P}^N$. If S is the join correspondence, and $p_3: S \longrightarrow XY$ is defined as above, then $p_3^{-1}(u) \subseteq X_u \times Y_u \times \{u\}$ for general $u \in XY$.*

Proof. We have

$$p_3^{-1}(u) \cap S_0 \subseteq X_u \times Y_u \times \{u\}$$

because S_0 is closed in $(X \times Y - \Delta) \times \mathbf{P}^N$. If we have $p_3(S - S_0) \neq XY$, then $p_3^{-1}(u) = p_3^{-1}(u) \cap S_0$ for every $u \in XY - p_3(S - S_0)$, so that the conclusion is immediate in this case. In general, we have $\dim p_3^{-1}(u) = \dim p_3^{-1}(u) \cap S_0 = \delta(X, Y)$ for every u in some dense open subset $U_0 \subset XY$. If $p_3(S - S_0) = XY$, then there is a dense open subset $U_1 \subset U_0$ such that $\dim p_3^{-1}(u) \cap (S - S_0) < \delta$ for every $u \in U_1$. In this case, $p_3^{-1}(u)$ is the closure of $p_3^{-1}(u) \cap S_0$ for every $u \in U_1$, so that the conclusion of the lemma follows.

It would be interesting to know whether the inclusion $p_3^{-1}(u) \subseteq X_u \times Y_u \times \{u\}$ holds for all $u \in XY$, or at least to have a more precise version of the lemma.

We can now state and prove the main result of this section.

Proposition 3.3. *Let X and Y be closed subvarieties of $\mathbf{P}_k^N$, where the field k is of characteristic 0, as before. Let u be a general point of $XY - (X \cup Y)$ and let x and y be general points of the entry point varieties X_u and Y_u, respectively, such that $u \in xy$. In particular, assume that $x \notin t_{Y,y}$ and $y \notin t_{X,x}$. Then $t_{X_u,x} = t_{X,x} \cap x t_{Y,y}$ and $t_{Y_u,y} = t_{Y,y} \cap y t_{X,x}$.*

Proof. Let S be the join correspondence, and let $p_3: S \longrightarrow XY$ be defined as before. Then $p_3^{-1}(u) \cap (S - \mathrm{Sing}(S))$ is smooth of dimension $\delta(X, Y)$ for general $u \in XY - (X \cup Y)$. Moreover, $p_3^{-1}(u) \subseteq X_u \times Y_u \times \{u\}$ for general u, and each general $u \in XY$ lies on at least one line xy such that $x \notin t_{Y,y}$ and that $y \notin t_{X,x}$. Since $u \notin X \cup Y$, each line through u contains at most finitely many points of $X \cup Y$. This implies that $\dim(X_u) = \dim(Y_u) = \delta(X, Y)$ and that $p_1: S \longrightarrow X$ and $p_2: S \longrightarrow Y$ induce generically finite morphisms of $p_3^{-1}(u)$ onto X_u and Y_u respectively. By generic smoothness, the corresponding tangent space maps are surjections of the tangent space $t_{p_3^{-1}(u),(x,y,u)}$ onto $t_{X_u,x}$ and $t_{Y_u,y}$, respectively. Therefore, the conclusion follows from Proposition 3.1.

4. An alternative proof

In this section we will present an alternative proof of the case $q = 1$ in Zak's theorem. Specifically we prove the following, where the notation is as in §1:

Theorem 4.1. *Let $X \subset \mathbf{P}^N$ be a nonsingular projective variety which spans $\mathbf{P}^N$. Then $\delta_{k+1} \geq \delta_k + \delta_1$ for all k such that $X^{k+1} = S^k(X) \neq \mathbf{P}^N$.*

If $\delta_1 = 0$ then the claim is proved in Proposition 1.2. So assume that $\delta_1 \geq 1$. The proof uses our description of the tangent spaces of the variety of entry points, Terracini's Lemma, and the Fulton–Hansen Connectedness Theorem. We will discuss the case $k=1$ before presenting the general case. This is done only to exhibit the main ideas separately from the notational complexities of the general case.

The case $k=1$. In the first part of the proof, we will show that three general embedded tangent spaces of X have empty intersection. In doing this, we will use a method of applying the connectedness theorem which is due to Zak, along with our description of the tangent space of an entry point set.

Let u be a general point of $X^2 = S(X)$, and let X_u be the corresponding entry point locus in X. Let $T(X_u, X) = \bigcup \{ t_{X,x} \,|\, x \in X_u \}$. Then $T(X_u, X) \subset t_{S(X),u}$ by Terracini's Lemma, Theorem 1.3. Since $S(X) \neq \mathbf{P}^N$, the tangent space $t_{S(X),u}$ is a proper subspace of $\mathbf{P}^N$, so that $X_u X \not\subseteq t_{S(X),u}$. Hence, $T(X_u, X) \neq X_u X$. Exactly as in the proof of Proposition 2.4, it follows from Theorem 1.4 that $\dim(X_u X) = \dim(X_u) + \dim(X) + 1$. By Theorem 1.3 we conclude that

$$t_{X_u,x} \cap t_{X,z} = \emptyset \quad \text{for general } x \in X_u,\ z \in X.$$

If x, y is a general pair of points such that $u \in xy$, then

$$t_{X_u,x} = t_{X,x} \cap x t_{X,y},$$

by Proposition 3.3. On the other hand, if x and y are general points of X and u is a general point of the line xy, then u is a general point of $S(X) = X^2$. Therefore

$$t_{X,x} \cap x t_{X,y} \cap t_{X,z} = \emptyset \quad \text{for general } x, y, z \in X.$$

It follows *a fortiori* that

$$t_{X,x} \cap t_{Y,y} \cap t_{X,z} = \emptyset \quad \text{for general } x, y, z \in X. \tag{1}$$

Remark. In the case $\delta_1 = 1$, Theorem 1.3 implies that the pairwise intersections $t_{X,x} \cap t_{Y,y}$, etc. are points, so that the relation (1) is an immediate consequence of the fact that X is not a cone. But this identity is not as obvious in the case $\delta_1 \geq 2$, since the pairwise intersections then have dimensions ≥ 1. There does not seem to be any elementary way to rule out the possibility that the triple intersection could be a variable point.

To finish the proof in this case, we study incidence properties of certain subspaces of $\mathbf{P}^N$. Thus, let x, y, z be general points of X; let $M_1 = t_{X,x}$, $M_2 = t_{X,y}$, and $\mathcal{N} = t_{X,x}t_{X,y}$. Let $\Lambda_i = t_{X,z} \cap M_i$ for $i = 1, 2$. Now, $\mathcal{N} = t_{S(X),u}$, where u is a general point of xy. We have $\dim(\Lambda_i) = \delta_1 - 1$ by Theorem 1.3, and $\dim(t_{X,z} \cap \mathcal{N}) = \delta_2 - 1$ similarly. The relation (1) implies that $\Lambda_1 \cap \Lambda_2 = \emptyset$, therefore $\dim \Lambda_1\Lambda_2 = 2\delta_1 - 1$. Since $\Lambda_1 \cap \Lambda_2 \subset t_{X,z} \cap \mathcal{N}$, it follows that $\delta_2 \geq 2\delta_1$, as claimed.

The proof in general. To simplify the notation, we will prove the equivalent statement that $\delta_k \geq \delta_{k-1} + \delta_1$ provided that $X^k \neq \mathbf{P}^N$. Let u be a general point of X^k, and let X_u be the corresponding entry point set on X. Define $T(X_u, X)$ as before, it is contained in $t_{X^k,u}$ by the same argument as in the special case. Because $X^k \neq \mathbf{P}^N$ the tangent space $t_{X^k,u}$ is a proper subspace of $\mathbf{P}^N$, thus $X_u X \not\subseteq t_{X^k,u}$. As in the special case it then follows that $\dim(X_u X) = \dim(X_u) + \dim(X) + 1$, which implies as before that

$$t_{X_u,x} \cap t_{X,z} = \emptyset \quad \text{for general } x \in X_u,\ z \in X.$$

By Proposition 3.3 we have $t_{X_u,x} = t_{X,x} \cap x t_{X^{k-1},v}$, where $(x, v) \in X \times X^{k-1}$ is a general pair such that $u \in xv$. As in the special case, we must check that the hypotheses of Proposition 3.3 are satisfied, namely that $u \notin X^{k-1}$, $v \notin t_{X,x}$, and $x \notin t_{X^{k-1},v}$. For this we first take $x \in X$ and $v \in X^{k-1}$ to be general points and then take u to be a general point of the line xy. It is not hard to see that u can be moved to any point of some dense open subset of X^k by moving x and v through appropriate dense open subsets of X and X^{k-1} respectively. The equality $t_{X_u,x} \cap t_{X,z} = \emptyset$ leads immediately to:

$$t_{X,x} \cap x t_{X^{k-1},v} \cap t_{X,z} = \emptyset \quad \text{for general } x, z \in X \text{ and } v \in X^{k-1}.$$

As before it follows that

$$t_{X,x} \cap t_{X^{k-1},v} \cap t_{X,z} = \emptyset \quad \text{for general } x, z \in X \text{ and } v \in X^{k-1}. \tag{2}$$

Also as in the special case, we finish the proof by studying the incidence properties of certain linear subspaces of $\mathbf{P}^N$. Thus, let $x \in X$ and $v \in X^{k-1}$ be general points, and let u be a general point of xv. Then u is a general point of X^k. Let $M_1 = t_{X,x}$, $M_2 = t_{X^{k-1},v}$, and $\mathcal{N} = M_1 M_2$, so that $\mathcal{N} = t_{X^k,-u}$ by Theorem 1.3. Consider another general point $z \in X$ and set $\Lambda_1 = t_{X,z} \cap M_1$ and $\Lambda_2 = t_{X,z} \cap M_2$. Thus, $\dim(\Lambda_1) = \delta_1 - 1$ and $\dim(\Lambda_2) = \delta_{k-1} - 1$ by Theorem 1.3, similarly $\dim(t_{X,z} \cap \mathcal{N}) = \delta_k - 1$. The relation shown in Diagram (2) implies that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Therefore, we have $\dim \Lambda_1\Lambda_2 = \delta_1 + \delta_{k-1} - 1$. Since $\Lambda_1\Lambda_2 \subseteq t_{X,z} \cap \mathcal{N}$, it follows that we have $\delta_k \geq \delta_1 + \delta_{k-1}$, as claimed.

To use similar methods to prove the general inequality $\delta_{p+q} \geq \delta_p + \delta_q$, one might consider general points $v \in X^p$ and $w \in X^q$, and a general point $u \in vw$. Let $M_1 = t_{X^p,v}$ and $M_2 = t_{X^q,w}$. As above, define $\mathcal{N} = M_1 M_2 = t_{X^{p+q},u}$. Let z be a general point of X, put $\Lambda_1 = t_{X,z} \cap M_1$ and $\Lambda_2 = t_{X,z} \cap M_2$. As above we see that $\dim(\Lambda_1) = \delta_p - 1$ and $\dim(\Lambda_2) = \delta_q - 1$, while $\dim(t_{X,z} \cap \mathcal{N}) = \delta_{p+q} - 1$. Since $\Lambda_1 \Lambda_2 \subseteq tX, z \cap \mathcal{N}$, it suffices to prove that $\Lambda_1 \cap \Lambda_2 = \emptyset$, or equivalently that $t_{X^p,v} \cap t_{X^q,w} \cap t_{X,z} = \emptyset$. This is an analogue of Diagram (2), and one might plausibly seek to prove this by studying the tangent spaces of entry point sets on one of the higher secant varieties. But since the higher secant varieties X^k have singular points, the variety $T(X_u, X)$ must be replaced by an appropriate variety of *relative tangent stars*. For the type of proof proposed here, the main difficulty is that we do not know much about the size of the relative tangent star at a point of some subvariety of X^k which is a singular point of X^k. In particular, we don't know whether there is a useful replacement for the inclusion $T(X_u, X) \subset t_{X^k,u}$.

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