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Equivalent norms for the Sobolev space $W_0^{m,p}(\Omega)$

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Abstract. A new proof is given for a theorem by V. G. Maz'ya. It gives a necessary and sufficient condition on the open set Ω in $\mathbf{R}^N$ for the functions in $W_0^{m,p}(\Omega)$ to have the ordinary norm equivalent to the norm obtained when including only the highest order derivatives in the definition. The proof is based on a kind of polynomial capacities, Maz'ya capacities.

Introduction

Let Ω be an open subset of $\mathbf{R}^N$. The Sobolev norm is given by

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\le m} \left(\int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}.$$

Here α is a multiindex. Define $W_0^{m,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm $\| \ \|_{W^{m,p}(\Omega)}$. There is a seminorm, $| \ |_{m,p,\Omega}$, defined as

$$|u|_{m,p,\Omega} = \sum_{|\alpha| = m} \left(\int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}.$$

A question solved by V. G. Maz'ya, (see the book [4, 11.4.2] and the reference given there), is the following: What is the necessary and sufficient condition on Ω for

$$|u|_{m,p,\Omega} \sim \|u\|_{W^{m,p}(\Omega)},$$

when $u \in W_0^{m,p}(\Omega)$? Here "$\sim$" as usual denotes that the quotients between the quantities are bounded by positive constants. Here the equivalence is independent of u.

R. A. Adams has given a sufficient, non-necessary condition in his book, [2, Th. 6.28], unaware of the result by V. G. Maz'ya. The aim of this article is to give a different proof of the theorem by V. G. Maz'ya based on a kind of capacities here called Maz'ya capacities, see [3] or [4, 10.3.3], instead of capacities related to Bessel capacities. This application gives an indication of the usefulness of the concept of Maz'ya capacities. The proof is divided into two parts. The first one consists of a proof that a necessary and sufficient condition can be given in terms of Maz'ya polynomial capacities. The second one is a proof that this condition can be translated to corresponding capacities related to Bessel capacities. Generally these capacitities can not be translated to each other like this, but here it is possible due to the geometric formulation of the condition. The major part of the first part of the proof will be used elsewhere as a lemma in the proof of a certain Hardy inequality for domains in $\mathbf{R}^N$. If $p=N$ then this lemma makes it possible to give a necessary and sufficient condition on the domain for the Hardy inequality to hold. The lemma is used in the proof of the necessity. It can be added that the sufficiency in fact holds for general parameters. (Observe that the condition is formulated in a way that admits testing on actual domains—a property not properly emphazised in this area.) As mentioned above we in the present article actually give two different formulations of the necessity and sufficiency condition, but since the capacity formulation related to Bessel capacities is so superior in practice we do not emphasize this and here instead see the Maz'ya capacities only as a tool in the proof procedure.

V. G. Maz'ya actually proves a stronger statement, but this is a corollary of the theorem we indicated above. This statement has as an application a result on the solvability of a Dirichlet problem. Let α and β be multiindices, $|\alpha|, |\beta| \leq m$. Let $a_{\alpha\beta}(x)$ be complex, bounded and measurable functions on the open set Ω in $\mathbf{R}^N$, such that $a_{\alpha\beta}=\bar{a}_{\beta\alpha}$ and

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\zeta_\alpha\bar{\zeta}_\beta \geq \delta \sum_{|\beta|=m} |\zeta_\beta|^2$$

for $\delta=\text{const.}>0$. Now define the operator

$$Bu=(-1)^m \sum_{|\alpha|=|\beta|=m} D^\beta(a_{\alpha\beta}D^\alpha u).$$

From the theorem by V. G. Maz'ya it follows that the above necessary and sufficient condition also is a necessary and sufficient condition for the equation $BT=f$, with f in $L^r(\Omega)$, $r/(r-1)\geq 2$, $2\leq r<2N/(N-2m)$ for $N\geq 2m$ and $2\leq r\leq\infty$ for $N<2m$, to have a distribution solution in the space $L_0^{m,2}(\Omega)$, defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$\left(\sum_{|\alpha|=m}\int_\Omega |D^\alpha u|^2\,dx\right)^{1/2}.$$

For this see [4, 11.7.2]. We will use the notation that A is a generic positive constant, which may change even within the same string of inequalities.

Results

First we give a definition of a condenser capacity equivalent to the corresponding Bessel capacity when $p>1$. Let Q denote an open cube in $\mathbf{R}^N$, λQ denote the cube with the same centre and orientation, but dilated a factor λ. Let m denote a positive integer.

Definition 1. Let $p\geq 1$, let K be a closed set in $\overline{Q}$, let

$$S_K=\{\,u\in C_0^\infty(2Q):u\,|_K\geq 1\,\}$$

and define

$$C_{m,p}(K,2Q)=\inf_{u\in S_K}\sum_{|\alpha|=m}\|D^\alpha u\|_{L^p}.$$

According to the results by D. R. Adams and J. C. Polking in [1] and also by V. G. Maz'ya, see [4, 9.3.2], we have that this capacity is equivalent for $p>1$ to the capacity $C^\sharp_{m,p}(K,2Q)$ defined as follows.

Definition 2. Let $p\geq 1$, let K be a closed set in $\overline{Q}$ and let

$$S^\sharp_K=\{\,u\in C_0^\infty(2Q):u=1 \text{ in a neighbourhood of } K\,\}$$

and define

$$C^\sharp_{m,p}(K,2Q)=\inf_{u\in S^\sharp_K}\sum_{|\alpha|=m}\|D^\alpha u\|_{L^p}.$$

We shall use Maz'ya capacities, which are a generalization of the capacities above.

Definition 3. Let K be a closed set in $\overline{Q}$. Let $\mathcal{P}_k$ denote the set of polynomials of degree less or equal to k with $\int_Q |P|^p\,dx=(\text{side } Q)^N$. Furthermore let $\mathcal{S}_{P,K}$ denote the set of functions in $C_0^\infty(2Q)$, which coincides with P in a neighbourhood of K. Then define the Maz'ya capacity $\gamma_{m,k,p}$ by

$$\gamma_{m,k,p}(K,2Q)=\inf_{P\in\mathcal{P}_k}\inf_{u\in\mathcal{S}_{P,K}}\sum_{|\alpha|=m}\int_{2Q}|D^\alpha u|^p\,dx.$$

In these kind of situations it has no significance if we take this definition or e.g. the one where the pth root of the quantity is taken.

Relative versions of the capacities above are defined as the value of the capacity of the image of the set K after a dilation mapping Q onto a unit cube. They will be denoted by a * on the corresponding capacity.

Much of the usefulness of the Maz'ya capacities lies in the following result by V. G. Maz'ya, see [3] or [4, 10.3.3]. For $u\in C_0^\infty(K^c)$, $1\le p$ and Q with side h we have that

$$\sup_{u\in C_0^\infty(K^c)} \frac{\int_Q |u|^p\,dx}{\sum_{|\alpha|=m}\int_Q |D^\alpha u|^p\,dx} \sim \frac{h^{mp}}{\gamma^*_{m,m-1,p}(K,2Q)}, \tag{1}$$

when $\gamma^*_{m,m-1,p}(K,2Q)$ is small and non-zero, i.e. there is a positive constant c_0 independent of K, such that this capacity is less than c_0.

On the other hand LHS$\le A\cdot$RHS for all closed sets K.

Now we turn to the main theorem, proved by V. G. Maz'ya in [4, 11.4.2].

Theorem. *Let $p\ge 1$. The following statements are equivalent for the open set Ω in $\mathbf{R}^N$:*

(i) $\mathbf{R}^N$ *can be tesselated into equally sized cubes $\{\overline{Q}_i\}$ such that*

$$\inf_i \; C^\sharp_{m,p}(\Omega^c\cap\overline{Q}_i, 2Q_i) > 0;$$

(ii) $|u|_{m,p}\sim\|u\|_{W^{m,p}}$ *if* $u\in W_0^{m,p}(\Omega)$.

Proof. Obviously it does not matter if we prove the result for relative capacities instead.

The structure of the proof is that first we prove the theorem with a relative Maz'ya capacity instead of the capacity $C^\sharp_{m,p}$, and then we end with an estimate that shows that with the geometrical situation at hand the otherwise inequivalent capacities turn out to fill the same purpose.

Accordingly we begin with a proof that (ii) is equivalent to

(iii) $\mathbf{R}^N$ can be tesselated into equally sized cubes $\{\overline{Q}_i\}$ such that

$$\inf_i \gamma^*_{m,m-1,p}(\Omega^c\cap\overline{Q}_i, 2Q_i) > 0.$$

First we prove that (iii) implies (ii). The truth or falsehood of either statement is not changed by a dilation. Without loss of generality we may hence assume that the cubes have side 1. Assume (iii) is true and let $u\in W_0^{m,p}(\Omega)$. We use interpolation, see [2, Th. 4.13], and the result (1) by V. G. Maz'ya and get for $|\beta|\le m$ that

$$\int_{Q_i} |D^\beta u|^p\,dx \le A\cdot\int_{Q_i} |u|^p\,dx + A\cdot\sum_{|\alpha|=m}\int_{Q_i} |D^\alpha u|^p\,dx$$
$$\le A\cdot((\gamma^*_{m,m-1,p}(\Omega^c\cap\overline{Q}_i, 2Q_i))^{-1}+1)\cdot\sum_{|\alpha|=m}\int_{Q_i}|D^\alpha u|^p\,dx.$$

We sum the inequality over β and i and get

$$\|u\|_{W^{m,p}} \le A\cdot((\inf_i \gamma^*_{m,m-1,p}(\Omega^c\cap\overline{Q}_i, 2Q_i))^{-1}+1)\cdot|u|_{m,p,\Omega}.$$

Thus we have proved that (iii) implies (ii).

Conversely to prove that (ii) implies (iii) we assume that (iii) is false for Ω and prove that then also (ii) is false.

Let the sidelength of the cubes be h, which later will be chosen arbitrarily large. Choose $Q_{i_0}=Q$ such that $\gamma^*_{m,m-1,p}(\Omega^c\cap\overline{Q}, 2Q)<\varepsilon$, where ε is small and to be chosen later. There is a $u\in C_0^\infty(\Omega)$, with

$$h^{mp}\cdot\sum_{|\alpha|=m}\int_Q |D^\alpha u|^p\,dx < A\varepsilon\cdot\int_Q |u|^p\,dx \tag{2}$$

according to the result (1) by V. G. Maz'ya.

Now let $\phi\in C_0^\infty(Q_0)$ with $\phi\equiv 1$ on $\frac{1}{2}Q_0$, where Q_0 is a unit cube. Put $\phi_Q=\phi(h^{-1}(x-x_Q))$, where x_Q is the center of Q. We want to compare $\|\phi_Q u\|_{W^{m,p}(\Omega)}$ and $|\phi_Q u|_{m,p,\Omega}$. Note that

$$\|\phi_Q u\|^p_{W^{m,p}(\Omega)} \ge \int_{(1/2)Q} |u|^p\,dx.$$

On the other hand by Leibnitz' formula, interpolation, [2, Th. 4.13], and (2) we have that

$$|\phi_Q u|^p_{m,p,\Omega} \le A\cdot\sum_{|\alpha|\le m} h^{-(m-|\alpha|)p}\cdot\int_Q |D^\alpha u|^p\,dx$$
$$\le Ah^{-mp}\cdot\int_Q |u|^p\,dx + A\cdot\sum_{|\alpha|=m}\int_Q |D^\alpha u|^p\,dx$$
$$\le Ah^{-mp}(1+\varepsilon)\int_Q |u|^p dx.$$

If we can prove that

$$A\int_{(1/2)Q}|u|^p\,dx\geq\int_Q|u|^p\,dx$$

we are done, since then

$$A\cdot\|\phi_Q u\|_{W^{m,p}(\Omega)}\geq h^{mp}\cdot|\phi_Q u|_{m,p,\Omega}$$

and now h can be made arbitrarily large.

We have a weak Poincaré inequality, see [4, 1.1.11. Lemma], that says that there is a polynomial P of degree less than or equal to $m-1$, such that

$$\int_Q|u-P|^p\,dx\leq Ah^{mp}\cdot\sum_{|a|=m}\int_Q|D^\alpha u|^p\,dx.$$

(Actually the formulation given there covers only $h=1$, but it is easy to derive the formulation above from this. Start with a unit cube. Then dilate the inequality with a factor h. Then h^{mp} appears as a dilation factor. Furthermore, the polynomial is changed by the dilation, but this does not matter, since it is only the existence of a polynomial that matters.)

We may by compactness of the set of polynomials in question assume that P makes the LHS a minimum. From this and (2) we get

$$\int_Q|u-P|^p\,dx\leq A\varepsilon\cdot\int_Q|u|^p\,dx. \tag{3}$$

Thus by the triangle inequality and by (3) we have that

$$\begin{aligned}\left(\int_{(1/2)Q}|u|^p\,dx\right)^{1/p}&\geq\left(\int_{(1/2)Q}|P|^p\,dx\right)^{1/p}-\left(\int_{(1/2)Q}|u-P|^p\,dx\right)^{1/p}\\&\geq\left(\int_{(1/2)Q}|P|^p\,dx\right)^{1/p}-A\varepsilon^{1/p}\left(\int_Q|u|^p\,dx\right)^{1/p}.\end{aligned}\tag{4}$$

But if we use a fixed cube Q, then since norms in finite dimensions are equivalent, we get for any P of degree less than or equal to $m-1$ that

$$A\left(\int_{(1/2)Q}|P|^p\,dx\right)^{1/p}\geq\left(\int_Q|P|^p\,dx\right)^{1/p}.$$

Now by dilation this holds for any cube. The RHS above is greater than or equal to

$$\left(\int_Q|u|^p\,dx\right)^{1/p}-\left(\int_Q|u-P|^p\,dx\right)^{1/p}$$

by the triangle inequality. This quantity is by (3) greater than or equal to

$$\left(\int_Q |u|^p\,dx\right)^{1/p} - A\varepsilon^{1/p}\left(\int_Q |u|^p\,dx\right)^{1/p}.$$

Now by (4) and the estimates above we obtain

$$A\int_{(1/2)Q} |u|^p\,dx \geq \int_Q |u|^p\,dx$$

if ε is small enough.

This proves that (ii) implies (iii). Hence we have that (iii) and (ii) are equivalent.

In the second step we show that we can substitute $C^{\sharp*}_{m,p}$ for $\gamma^*_{m,m-1,p}$ in condition (iii) and obtain condition (i). We assume that (iii) holds. First we note that

$$\gamma^*_{m,m-1,p}(K,2Q)^{1/p} \leq A\cdot C^{\sharp*}_{m,p}(K,2Q).$$

This follows if we take P=const. in the definition of the Maz'ya capacity. Now it remains to estimate the Maz'ya capacity under the condition (i).

Let $\{Q_i\}$ be the cubes of (i). Let k be a large integer. Form a tesselation of $\mathbf{R}^N$ with a subset of $\{kQ_i\}$. We denote this subset $\{Q'_n\}$. The statements are invariant under dilations. Without loss of generality we may assume that the sides of $\{Q'_n\}$ equal 1. Let ψ_{Q_i} be a partition of unity with respect to the cubes $\{Q_i\}$, such that the functions are the translates of each other, with respect to any translation mapping the respective cubes on each other. Let ψ_{Q_i} be a C^∞ function. Furthermore let $\psi_{Q_i}|_{Q_i}$=const.$\neq 0$ and $\psi_{Q_i}|_{(2Q_i)^c}=0$. Pick an arbitrary cube Q' in $\{Q'_n\}$. Assume that u and P are nearly optimal in the definition of $\gamma_{m,m-1,p}(\Omega^c\cap\overline{Q}', 2Q')$. By Leibnitz' rule, by a trivial estimate, by a Poincaré inequality for functions with compact support and finally by the definition of $\gamma_{m,m-1,p}$, we have that

$$\begin{aligned}
\sum_{\{i:Q_i\subset Q'\}}\sum_{|\alpha|=m}\int |D^\alpha(\psi_{Q_i}u)|^p\,dx &\leq A\sum_{\{i:Q_i\subset Q'\}}\sum_{|\alpha|\leq m}\int |D^\alpha u|^p\,dx\\
&\leq A\sum_{|\alpha|\leq m}\int |D^\alpha u|^p\,dx\\
&\leq A\sum_{|\alpha|=m}\int |D^\alpha u|^p\,dx\\
&\approx A\cdot\gamma_{m,m-1,p}(\Omega^c\cap\overline{Q}', 2Q').
\end{aligned} \tag{5}$$

Hence it suffices to prove that

$$\sum_{|\alpha|=m} \int |D^\alpha(\psi_{Q_{i_0}} u)|^p \, dx \geq \text{const.} > 0$$

for some i_0, $Q_{i_0} \subset Q'$. We claim that if k is sufficiently large it is possible to, to any polynomial P in $\mathcal{P}_{m-1}$ to choose a cube Q_{i_0} such that $\|P\|_{Q_{i_0}} \geq$const.>0. This follows by the following argument: Since norms in finite dimensional space are equivalent we have that

$$1 = \|P\|_{L^p(Q')} \sim \|P\|_{L^\infty(Q')}.$$

Furthermore we have by the equivalence of norms in finite dimensional space that

$$\|\nabla P\|_{L^\infty(Q')} \leq A \|P\|_{L^\infty(Q')},$$

which implies that we can choose k such that there is a Q_{i_0} covering a maximum of $|P|$ in Q' and such that

$$|P| \geq \tfrac{1}{2} \max_{x \in Q'} |P|$$

in Q_{i_0}.

Since $\psi_{Q_{i_0}} u/P$ can be taken as test function for $C^\sharp_{m,p}(\Omega^c \cap \overline{Q}_{i_0}, 2Q_{i_0})$, since $|P| \sim 1$ in $2Q_{i_0}$, by Leibnitz rule, by the triangle inequality, by a Poincaré inequality and by (5) we get that

$$\begin{aligned}
C^\sharp_{m,p}(\Omega^c \cap \overline{Q}_{i_0}, 2Q_{i_0}) &\leq A \biggl(\sum_{|\alpha|=m} \int \left| D^\alpha \left(\frac{\psi_{Q_{i_0}} u}{P} \right) \right|^p \cdot |P|^p \, dx \biggr)^{1/p} \\
&\leq A \biggl(\sum_{|\alpha|=m} \int |D^\alpha(\psi_{Q_{i_0}} u)|^p \, dx \biggr)^{1/p} \\
&\quad + A \sum_{\substack{|\beta|+|\gamma|=m \\ |\beta|<m}} \biggl(\int |D^\beta(\psi_{Q_{i_0}} u)|^p \cdot \left| D^\gamma \left(\frac{1}{P} \right) \cdot P \right|^p dx \biggr)^{1/p} \\
&\leq A \biggl(\sum_{|\alpha|=m} \int |D^\alpha(\psi_{Q_{i_0}} u)|^p \, dx \biggr)^{1/p} \\
&\leq A \gamma_{m,m-1,p}(\Omega^c \cap \overline{Q}', 2Q').
\end{aligned}$$

End of proof.

The following stronger formulation is a consequence. It is how V. G. Maz'ya formulated his theorem.

Corollary. *Let $p\geq 1$. The following statements are equivalent for an open set Ω in $\mathbf{R}^N$.*

(i) $\mathbf{R}^N$ *can be tesselated into equally sized closed cubes $\{Q_i\}$ such that*

$$\inf_i C^{\sharp}_{m,p}(\Omega^c\cap\bar{Q}_i, 2Q_i)>0.$$

(ii) *Let $u\in W_0^{m,p}(\Omega)$. Then*

$$\left(\int |u|^q\,dx\right)^{1/q}\leq A\left(\sum_{|\alpha|=m}\int |D^\alpha u|^p\,dx\right)^{1/p},$$

where

(1) $p\leq q<Np/(N-mp)$ *when* $p\geq 1$ *and* $mp<N$;
(2) $p\leq q<\infty$ *when* $p\geq 1$ *and* $mp=N$;
(3) $p\leq q\leq\infty$ *when* $mp>N$ *and* $p\geq 1$.

Proof. By the Sobolev imbedding theorem and the proof of the previous theorem we have that

$$\left(\int_{Q_i} |u|^q\,dx\right)^{1/q}\leq A\left(\int_{Q_i} |u|^p\,dx\right)^{1/p}+A\left(\sum_{|\alpha|=m}\int_{Q_i} |D^\alpha u|^p\,dx\right)^{1/p}$$
$$\leq A\left(\sum_{|\alpha|=m}\int_{Q_i} |D^\alpha u|^p\,dx\right)^{1/p}.$$

Now raise to the pth power and sum over i.

$$\sum_i\left(\int_{Q_i} |u|^q\,dx\right)^{p/q}\leq A\sum_{|\alpha|=m}\int |D^\alpha u|^p\,dx,$$

but by an elementary inequality for sums we have that the LHS is greater or equal to

$$\left(\sum_i\int_{Q_i} |u|^q\,dx\right)^{p/q},$$

since $p/q\leq 1$. End of proof.

References

1. ADAMS, D. R. and POLKING, J. C., The equivalence of two definitions of capacity, *Proc. Amer. Math. Soc.* **37** (1973), 529–533.
2. ADAMS, R. A., *Sobolev Spaces*, Academic Press, New York, 1975.
3. MAZ'YA, V. G., On (p, l)-capacity, imbedding theorems, and the spectrum of a self-adjoint elliptic operator, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 356–385 (Russian). English transl.: *Math. USSR-Izv.* **7** (1973), 357–387.
4. MAZ'YA, V. G., *Sobolev Spaces*, Springer-Verlag, Berlin–New York, 1985.

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