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Wolff type estimates and the H^p corona problem in strictly pseudoconvex domains

Mats Andersson(1) and Hasse Carlsson

Abstract. Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$. We prove that $\bar\partial u=\varphi$, φ a $\bar\partial$-closed (0,1)-form, admits solutions in $L^p(\partial D)$, $1\le p<\infty$ and in BMO, under certain Wolff type conditions on φ. Some such results (for $1<p<\infty$) have previously been obtained by Amar in the ball, but under slightly stronger hypotheses. As a corollary we obtain a H^p-corona result for two generators.

1. Introduction

This paper deals with Wolff-type estimates for $\bar\partial_b$ in strictly pseudoconvex domains in $\mathbf{C}^n$. The origin for the study of such estimates is the corona problem in one variable, and our results lead to an H^p-variant of the corona theorem that we first discuss.

Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ and let $g_1, g_2 \in H^\infty$ satisfy

$$|g_1|^2+|g_2|^2\ge\delta^2>0.$$

Given $\phi\in H^p$ the H^p-corona problem is to find $u_1, u_2\in H^p$ such that

$$g_1u_1+g_2u_2=\phi. \tag{1.1}$$

For $n=1$ this is possible in a large class of domains for $p=\infty$ (i.e. the true corona theorem) and any number of generators g_j, and hence for all $p<\infty$. It is unknown for $p=\infty$ in strictly pseudoconvex domains in $\mathbf{C}^n$, $n>1$, but there are smooth pseudoconvex domains even in $\mathbf{C}^2$ in which it fails, see [Si] and [FSi]. However, for $0<p\le 2$ the H^p-corona problem is solvable in a large class of smooth pseudoconvex domains, including the ones from [Si] and [FSi], for any number of generators, see

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[An2] and [An3]. In [Am], Amar solves the H^p-corona problem, $1<p<\infty$, for two generators in the ball as a corollary of a certain Wolff-type estimate for $\bar\partial_b$. In [L], Lin has obtained a similar result for the polydisc, and for an arbitrary number of generators.

In this paper we generalize Amar's result in the ball to strictly pseudoconvex domains. However, even in the ball we obtain somewhat sharper estimates and we also obtain results for $p=1$ and BMO-results.

Theorem 1.1. *If D is a bounded strictly pseudoconvex domain with C^2-boundary, then the H^p-corona problem is solvable for two generators, $1\le p<\infty$, and if $p=\infty$ there is a solution in $H^\infty\cdot$BMO.*

The BMO-result above was announced already in [V1], but not elaborated in details, cf. the remark below.

The theorem is reduced to solving the equation

$$\bar\partial_b u=\varphi, \quad u\in L^p(\partial D), \tag{1.2}$$

where $\varphi=\omega\phi$,

$$\omega=(\bar g_1\bar\partial\psi_2-\bar g_2\bar\partial\psi_1)/|g|^2, \tag{1.3}$$

and ψ_j is some solution to

$$g_1\psi_1+g_2\psi_2=1,$$

since then $v=\psi\phi+ug$ solves (1.1).

When $n=1$, (1.2) is equivalent to

$$\left|\int_D g\varphi\right|\le C\left(\int_{\partial D}|g|^q\right)^{1/q}, \quad g\in H^q,\ 1/p+1/q=1. \tag{1.4}$$

This weak formulation was introduced by Hörmander, [Hö1], in his simplification of Carleson's original proof, [Ca], of the corona therorem. If $|\omega|$ is a Carleson measure, then (1.4) immediately follows from the Carleson–Hörmander inequality. However, with the simplest choice $\psi_j=\bar g_j/|g|^2$, it is not a Carleson measure in general, so one still had to use the much more refined choice of ψ_j due to Carleson.

Remark. Using such refined ψ_j in the multi-dimensional case, cf. [G] and [V1a], one can obtain Theorem 1.1 from (the non-Wolff-type) Theorem 1.3 below.

In 1979 Wolff observed, see [G], that even with the simple choice $\psi_j=\bar g_j/|g|^2$, ω in (1.3) however satisfies ($n=1$ and D the unit disc)

$$(1.5)\qquad \begin{cases} (1-|\zeta|)|\omega|^2 & \text{is a Carleson measure,} \\ (1-|\zeta|)|\partial\omega| & \text{is a Carleson measure,} \end{cases}$$

and that these two conditions together, by a comparatively simple argument, imply (1.4) for $q=1$, i.e. that (1.2) has a solution in $L^\infty(\partial D)$.

For a strictly pseudoconvex domain $D=\{\varrho<0\}$, $d\varrho\neq 0$ on ∂D, it is easily seen, using Proposition 2.1 below, that ω satisfies

$$(1.6)\qquad \begin{cases} -\varrho|\omega|^2+|\omega\wedge\bar\partial\varrho|^2 & \text{is a Carleson measure,} \\ -\varrho|\partial\omega|+\sqrt{-\varrho}(|\partial\omega\wedge\partial\varrho|+|\omega\wedge\bar\partial\varrho|)+|\partial\omega\wedge\partial\varrho\wedge\bar\partial\varrho| & \text{is a Carleson measure,} \end{cases}$$

if the Carleson measures are defined with respect to the Korányi balls, see Section 2. Thus Theorem 1.1 is a corollary of

Theorem 1.2. *Suppose that $D=\{\varrho<0\}$ is a strictly pseudoconvex domain with C^2-boundary, and ω a $\bar\partial$-closed $(0,1)$-form such that (1.6) holds. Then for any $\phi\in H^p$, $1\leq p\leq\infty$, the equation $\bar\partial u=\omega\phi$ admits a solution in $L^p(\partial D)$, $1\leq p\leq *$.*

Here and in the sequel we use the notation $L^*=\mathrm{BMO}$ and $1/*=0$. Note that we must assume that ϕ is in H^∞ to obtain a solution in BMO.

To be precise; when $p=1$ Theorem 1.2 will only be proved assuming that ∂D is C^3. However, the case with C^2-boundary can be derived from the case $p=2$ by the factorization theorem of Coifman–Rochberg–Weiss [CRW] (it is true in a strictly pseudoconvex domain with C^2-boundary).

Remark. Theorem 1.2, as well as several other statements below, are to be interpreted in the à priori sense; i.e. if ω is an appropriately smooth $\bar\partial$-closed form in D, then there is a solution to $\bar\partial u=\omega\phi$ such that $\|u\|_{L^p(\partial D)}$ is bounded by some uniform constant times the Carleson norm of

$$-\varrho|\omega|^2+|\omega\wedge\bar\partial\varrho|^2-\varrho|\partial\omega|+\sqrt{-\varrho}(|\partial\omega\wedge\partial\varrho|+|\partial\omega\wedge\bar\partial\varrho|)+|\partial\omega\wedge\partial\varrho\wedge\bar\partial\varrho|$$

times the H^p-norm of ϕ.

We will prove Theorem 1.2 as a corollary of a more general Wolff-type theorem for the $\bar\partial_b$-equation.

Let W^α be the interpolation spaces between the finite measures W^0 and the Carleson measures W^1; for exact definitions see Section 2. The following non-Wolff theorem is classic.

Theorem 1.3. *Suppose that $D=\{\varrho<0\}$ is a strictly pseudoconvex domain with C^2-boundary, and φ a $\bar\partial$-closed $(0,1)$-form such that*

$$|\varphi|+\frac{1}{\sqrt{-\varrho}}\,|\bar\partial\varrho\wedge\varphi|\in W^{1-1/p},\quad 1\le p\le *.$$

Then $\bar\partial_b u=\varphi$ has a solution in $L^p(\partial D)$.

For $p=1$ this is due to Henkin [He] and Skoda [Sk], and for $p=*$ to Varopoulos [V1].

We will prove

Theorem 1.4. *Suppose that $D=\{\varrho<0\}$ is a strictly pseudoconvex domain with C^2-boundary (C^4 if $p=1$), and φ a $\bar\partial$-closed $(0,1)$-form such that*

$$(a_p)\qquad -\varrho|\partial\varphi|+\sqrt{-\varrho}(|\partial\varphi\wedge\partial\varrho|+|\partial\varphi\wedge\bar\partial\varrho|)+|\partial\varphi\wedge\partial\varrho\wedge\bar\partial\varrho|\in W^{1-1/p}$$

and

$$(b_p)\qquad \int_D -\varrho|\varphi||\partial g|\le C_p\|g\|_{H^q},\quad 1\le p\le *,\ 1/p+1/q=1,$$

($q=*$ if $p=1$ and vice versa). Then there is a solution $u\in L^p(\partial D)$ to $\bar\partial_b u=\varphi$.

When $p\ge 2$ we can replace (b_p) by a simpler condition on $|\varphi|$.

Corollary 1.5. *Suppose that $D=\{\varrho<0\}$ is a strictly pseudoconvex domain with C^2-boundary and φ is a $\bar\partial$-closed $(0,1)$-form such that (a_p) holds and*

$$-\varrho|\varphi|^2\in W^{1-2/p},\quad 2\le p\le *.$$

Then $\bar\partial_b u=\varphi$ has a solution in $L^p(\partial D)$.

In general, the solutions in Theorem 1.4 and the corollary will not be in $L^p(D)$ (unless p=1). This is seen by taking $\varphi=\bar\partial|\zeta|^{-\alpha}$ (assuming $0\in D$). For appropriately small $\alpha>0$, φ satisfies (a_1) and (b_1) but there is no solution to $\bar\partial u=\varphi$ in $L^p(D)$ if p is large.

We will give two proofs of our main result Theorem 1.4. Since the very essence in what we call a Wolff-type estimate is that one has a size condition on $\partial\varphi$, it is natural to first solve $\partial\bar\partial u=\partial\varphi$. In our first, and we think most natural approach, we will apply the approximate solution formula for the $\partial\bar\partial$-equation from [AC3], to write the solution to $\bar\partial u=\varphi$ as

$$(1.7)\qquad u=M(\partial\varphi)+T\varphi+R\varphi,$$

where M and T are explicit integral operators and $R\varphi$ is a "nice" error term (that vanishes in the ball, see below). In fact $M\partial\varphi$ is (essentially) a solution to $\partial\bar{\partial}v=\partial\varphi$. Using the (a_p)- and the (b_p)-conditions, it is not hard to verify that u is in $L^p(\partial D)$, see Section 3. Unfortunately, we need to assume that ∂D is C^4 to obtain (1.7).

We sketch the ball case below, whereas the general case is left to Section 3.

In the ball we have $K\varphi=M\partial\varphi+T\varphi$, where $M\theta$ is the L^2-minimal solution to $\partial\bar{\partial}v=\theta$ and

$$T\varphi(z)=c\sum_{k=1}^{n+1}\int_B \frac{(1-|\zeta|^2)\varphi\wedge d(\zeta\cdot\bar{z})\wedge(\partial\bar{\partial}|\zeta|^2)^{n-1}}{(1-\zeta\cdot\bar{z})^k},$$

see [An1]. From the explicit integral representation of $M\partial\varphi$ one finds that (a_p) implies that $M\partial\varphi\in L^p(\partial D)$; see Section 3 for details.

Since $T\varphi$ is anti-holomorphic, to estimate it, we integrate it against $\psi\in H^q$ and get by Fubini's theorem,

$$\int_{\partial B} T\varphi(z)\psi(z)\,d\sigma(z)$$
$$=\int_B (1-|z|^2)\varphi(\zeta)\wedge\partial_\zeta\int_{\partial B}\sum_1^{k+1}\frac{\psi(z)\,d\sigma(z)}{(k-1)(1-\zeta\cdot\bar{z})^{k-1}}\wedge(\partial\bar{\partial}|\zeta|^2)^{n-1},$$

$(1/0x^0=\log x)$, but

$$g(\zeta)=\sum_{k=1}^{n+1}\int_{\partial B}\frac{\psi(z)\,d\sigma(z)}{(k-1)(1-\zeta\cdot\bar{z})^{k-1}}$$

is in H^q, since for $k=n+1$, the integral is nothing but the Szegő integral of ψ, and the other integrals are weakly singular. Hence by (b_p),

$$\left|\int_{\partial B} T\varphi(z)\psi(z)\,d\sigma(z)\right|\le C_p\|\psi\|_{H^q},$$

which shows that $T\varphi(z)\in H^p$, $1\le p\le *$, and hence Theorem 1.4 is proved for the ball.

In our second proof (which works for $p>1$) we will rely on previous results of solutions to the $\partial\bar{\partial}$-equation; namely that if φ satisfy the (a_p)-condition then there is a solution $u\in L^p(\partial D)$ to $\partial\bar{\partial}u=\partial\varphi$. For $p=1$ this is the Henkin–Skoda theorem, [He] and [Sk], and for $p=*$ it is due to Varopoulos [V2]. (A careful analysis shows that C^2 is enough also for $p=*$. The proof consists of two steps. First one solves a Poincaré equation. This is the hardest step and is carried out for C^2-domains in [AC1]. Then one solves a $\bar{\partial}$-equation. The solution is represented by an integral formula and admits a BMO estimate, cf. Section 6 in [AC3]).

In general, u will not solve $\bar\partial u=\varphi$, but since $d(\bar\partial u-\varphi)=0$, we can solve (disregarding for the moment possible topological obstructions) $dT=\bar\partial u-\varphi$, and T is then a unique (up to an additive constant) anti-holomorphic function. Moreover, $K=u-T$ solves $\bar\partial K=\varphi$, since $\bar\partial u-\bar\partial T=\bar\partial u-(\bar\partial u-\varphi)=\varphi$. Using that T is anti-holomorphic and the duality between H^p and H^q, one can prove that T, and hence also K, is in $L^p(\partial D)$. The breakdown for $p=1$ is due to a certain inbalance between the (a_1)- and the (b_1)-conditions, see Section 4 where this proof is carried out.

In Section 5 we prove Theorem 1.2 and Corollary 1.5. In Section 2 we have collected some preliminaries on harmonic analysis and integral representations, some proofs of which are left to Section 6.

This paper is a thoroughly revised version of [AC2].

2. Preliminaries

In this paragraph we have collected definitions and some results for strictly pseudoconvex domains which we need further on. They are all well-known, at least in the ball, but some of them we have not found in the literature in the general case, and for these we sketch proofs in Section 6.

We start with some remarks about harmonic analysis in a strictly pseudoconvex domain $D=\{\varrho<0\}$ with C^3 boundary (although C^2 is enough at several instances), where ϱ is strictly plurisubharmonic in a neighborhood of $\bar D$ and $d\varrho\neq 0$ on ∂D. A vector v at $p\in\partial D$ is complex tangential if v is a tangent vector, i.e. $d\varrho|_p v=0$, and furthermore $d^c\varrho|_p v=0$. Here d^c is the real operator $i(\bar\partial-\partial)$. A K-basis ($K=$Korányi) at $p\in\partial D$ is a basis of neighborhoods $B_t(p)\subset\partial D$, $t>0$, at p such that $B_t(p)$ has length $\sim\sqrt{t}$ in all complex tangential directions and $\sim t$ in the last one. Then clearly $|B_t(p)|\sim t^n$. Sometimes we consider neighborhoods $Q_t(p)\subset\bar D$ which have also extension $\sim t$ into D, so that $|Q_t(p)|\sim t^{n+1}$. Any two K-bases $B_t(p)$ and $B_t'(p)$ are equivalent, i.e. $B_{ct}\subset B_t'\subset B_{t/c}$, $t>0$, for some constant $c>0$. For instance, if $x_2,\dots,x_{2n}$ are local coordintes at $p\in\partial D$ such that $x(p)=0$ and $dx_2|_p$ and $d^c\varrho|_p$ are colinear, then $B_t(p)=\{x;|x_2|+\sum x_j^2<t\}$ is a K-basis at p.

If now $B_t(p)$ is any continuous choice of a K-basis at each $p\in\partial D$ one can put $\sigma(p,z)=\inf\{t;z\in B_t(p)\}$ and $d(z,\varphi)=\frac{1}{2}(\sigma(z,\varphi)+\sigma(\varphi,z))$. Then, see e.g. [AC1],

$$d(z,\varphi)+d(\varphi,\zeta)\le Cd(z,\zeta).$$

Since also

$$|B_{2t}(p)|\le C|B_t(p)|,$$

∂D is a homogeneous space, so a lot of tools of harmonic analysis are available, see [CW1].

For $p>0$ we put

$$H^p=\left\{f\in\mathcal{O}(D)\,;\sup_{\varepsilon>0}\int_{\partial D_\varepsilon}|f|^p d\sigma<\infty\right\},$$

where $D_\varepsilon=\{\varrho<-\varepsilon\}$ and $d\sigma$ is (some) surface measure. It is well-known that any $f\in H^p$ has admissible (i.e. "non-tangential" with respect to the balls $B_t(p)$) boundary values f^* a.e. $[d\sigma]$, see [St], and that f is the Poisson integral (or the Bergman–Poisson dito) of f^* if $p\geq 1$.

An $f\in L^1_{loc}(\partial D)$ is in BMO if

$$\sup_{t>0,p\in\partial D}\frac{1}{|B_t(p)|}\int_{B_t(p)}|u-u_{B_t(p)}|\,d\sigma=\|u\|_*<\infty,$$

where $u_{B_t(p)}$ is the mean value of u over $B_t(p)$. We also put BMOA=BMO$\cap\mathcal{O}(D)$.

Since ∂D is a homogeneous space there is also an atomic $\mathcal{H}^1$-space on ∂D whose dual is BMO, and singular integral operators such as G defined below, maps $\mathcal{H}^1$ boundedly into $L^1(\partial D)$, see [CW2].

A measure μ in D is a Carleson measure, $\mu\in W^1$, if

$$|\mu(Q_t(p))|\leq Ct^n,\quad t\in\partial D,\ t>0.$$

Then the Carleson–Hörmander inequality holds:

$$\int_D|g|^p d\mu\leq C_p\|g\|^p_{H^p},\quad g\in H^p,\ p>0,$$

see [Hö2]. We let W^0 denote the space of finite measures in D and let W^α, $0<\alpha<1$, be the interpolating spaces between W^0 and W^1. For our purposes it is convenient with the following operative definition, see [AmB]:

$$\begin{gathered}\mu\in W^\alpha \text{ if and only if } d\mu=kd\tau \text{ where } d\tau \text{ is a positive}\\ \text{Carleson measure and } k\in L^p(\tau), \text{where } 1/p=1-\alpha.\end{gathered}$$

In particular, note that if $\mu\in W^{1/r}$, then

$$\int|f|^p d\mu\leq\|f\|^p_{H^{rp}},\quad p>0. \tag{2.1}$$

Proposition 2.1. *If $f\in H^p$, $p<\infty$, then*

$$\int_D (-\varrho|\partial f|^2+|\partial\varrho\wedge\partial f|^2)|f|^{p-2}\lesssim\|f\|_{H^p}^p, \tag{2.2}$$

and if $f\in$ BMOA, *then*

$$-\varrho|\partial f|^2+|\partial\varrho\wedge\partial f|^2\in W^1, \tag{2.3}$$

with Carleson norm bounded by $\|f\|_*^2$.

For proofs in the ball, see [CRW] and [CW2]. We sketch proofs in the strictly pseudoconvex case in Section 6.

Let $S:L^2(\partial D)\to H^2$ be the Szegő projection, i.e. the $L^2(\partial D)$-orthogonal projection with respect to (some) surface measure $d\sigma$ on ∂D.

Proposition 2.2. *S maps* BMO *boundedly onto* BMOA.

For smooth functions f and g we put

$$(f,g)=\int_{\partial D} f\bar{g}\,d\sigma. \tag{2.4}$$

Let $H_0^1=\{f\in H^1;\ f(0)=0\}$ (0 is any point in D). We then have

Proposition 2.3. *Via the pairing* (2.4), BMOA *is the dual space of H_0^1.*

We also have to recall same facts about integral representations of holomorphic functions and solutions of $\bar\partial$ in strictly pseudoconvex domains. This is based on the formulas for the $\partial\bar\partial$-equation in [AC3] and we refer to this paper for more details.

Let $v(\zeta,z):\overline{D\times D}\to\mathbf{C}^n$ satisfy

$$2\,\mathrm{Re}\,v\geq-\varrho(\zeta)-\varrho(z)+\delta|\zeta-z|^2, \tag{2.5}$$

and

$$d_\zeta\bar{v}|_{\zeta=z}=-d_z v|_{\zeta=z}=-\partial\varrho(\zeta). \tag{2.6}$$

Then $Q_t(p)=\{\zeta\in D;\ |v(p,\zeta)|<t\}$ is a K-basis at p (take local coordinates $x_1=-\varrho$, $x_2=\mathrm{Im}\,v$ and $x_3,\dots,x_{2n}$ arbitrary) and we have the well-known estimate

$$\int_{\partial D}\frac{d\sigma(\zeta)}{|v(\zeta,z)|^{n+\alpha}}\lesssim\left(\frac{1}{-\varrho(z)}\right)^\alpha,\quad \alpha>0. \tag{2.7}$$

From [F] it follows that one can choose such a v that is also holomorphic in z for fixed $\zeta\in\bar{D}$, and in wiew of (2.5) one can then define

$$Hu(z)=c\int_{\partial D}\frac{q\wedge(\bar\partial q)^{n-1}u}{v(\zeta,z)^n},\quad z\in D, \tag{2.8}$$

where $q=\sum q_j d\zeta_j$ and $\sum q_j(z_j-\zeta_j)=v(\zeta,z)$ and $q(\zeta,z)$ is holomorphic in z. Clearly, Hu is holomorphic in D if $u\in L^1(\partial D)$ and, by the Cauchy–Fantappié formula, $Hu=u$ if u is (the boundary values of) a holomorphic function. In fact, Hu has admissible boundary values a.e. if $u\in L^p(\partial D)$, $p>1$, and this operator maps $L^p(\partial D)$ into H^p, and BMO into BMOA, see Section 6.

There are related formulas for solving $\bar\partial$. If φ is a $\bar\partial$-closed (0,1)-form on $\bar{D}$, the boundary values of a solution to $\bar\partial u=\varphi$ is given by a formula of type,

$$\mathcal{L}\varphi(z)=\int_D\frac{(-\varrho)^\alpha\varphi\wedge\mathcal{O}(1)+(-\varrho)^{\alpha-1}\varphi\wedge\bar\partial\varrho\wedge\mathcal{O}(|\zeta-z|)}{v(\zeta,z)^{n+\alpha-1}v(\zeta,z)}, \tag{2.9}$$

$\alpha>0$, $z\in\partial D$. By (2.9) and (2.7) one easily gets the Henkin–Skoda estimate,

$$\int_{\partial D}|\mathcal{L}\varphi|d\sigma\lesssim\int_D|\varphi|+\frac{1}{\sqrt{-\varrho}}|\bar\partial\varrho\wedge\varphi|,$$

if $\alpha>\frac{1}{2}$. One also gets the corresponding estimates for $L^p(\partial D)$ in Theorem 1.3. For the connection between the formulas for Hu and $\mathcal{L}\varphi$, see e.g. [AC3].

However for our purposes we need formulas with a more specific choice of $v(\zeta,z)$. Namely, one such that

$$-v(\zeta,z)=\varrho+\sum\varrho_j\eta_j+\frac{1}{2}\sum_{jk}\varrho_{jk}\eta_j\eta_k, \tag{2.10}$$

($\eta_j=z_j-\zeta_j$, $\varrho_j=\partial\varrho/\partial\zeta_j(\zeta)$ etc.) near the diagonal. The crucial property for such a $v(\zeta,z)$ is that

$$\overline{v(\zeta,z)}=v(z,\zeta)+\mathcal{O}(|\zeta-z|^3), \tag{2.11}$$

and

$$\partial_\zeta v(\zeta,z)=\mathcal{O}(|\zeta-z|^2), \tag{2.12}$$

i.e. $v(\zeta,z)$ is almost conjugate-symmmetric and anti-holomorphic in ζ. Clearly, $v(\zeta,z)$ is holomorphic in z near the diagonal but it will not be globally, so the corresponding integral Hu will not have a holomorphic kernel, and the corresponding operator $\mathcal{L}\varphi$ will just be an approximate solution for $\bar\partial$. However, one can modify the kernel for Hu by adding a smooth one and obtain a kernel $g(\zeta,z)$ and its corresponding operator

$$Gu(z)=\int_{\partial D} g(\zeta,z)u(\zeta)\,d\sigma(\zeta),\quad z\in D, \tag{2.13}$$

where $g(\zeta,z)$ is holomorphic in z and is $\sim 1/v^n$ (v from (2.10)) near the diagonal.

There is a solution operator $K\varphi$ for $\bar\partial u=\varphi$, whose principal term is the operator $\mathcal{L}\varphi$, with the special choice of $v(\zeta,z)$ discussed above. We have

Proposition 2.4. *Let $D=\{\varrho<0\}$ be a bounded strictly pseudoconvex domain with C^3-boundary and φ a $\bar\partial$-closed $(0,1)$-form. Then there is a solution $K\varphi$ to $\bar\partial u=\varphi$ such that*

$$K\varphi=M\partial\varphi+T\varphi+\widetilde{F}\varphi, \tag{2.14}$$

where

$$T\varphi(z)=\int_D -\varrho(\zeta)\varphi(\zeta)\wedge\partial_\zeta g(z,\zeta)\wedge(\partial\bar\partial\varrho(\zeta))^{n-2}, \tag{2.15}$$

and, for $z\in\partial D$,

$$|M\partial\varphi(z)|\lesssim\int_D \frac{-\varrho(-\varrho|\partial\varphi|+\sqrt{-\varrho}(|\partial\varphi\wedge\partial\varrho|+|\partial\varphi\wedge\bar\partial\varrho|)+|\partial\varphi\wedge\partial\varrho\wedge\bar\partial\varrho|)}{|v(\zeta,z)|^{n+1}}, \tag{2.16}$$

and

$$|\widetilde{F}\varphi(z)|\lesssim\int_D\frac{-\varrho|\varphi||\zeta-z|}{|v|^{n+1}}+\int_D\frac{-\varrho|\bar\partial\varrho\wedge\varphi|}{|v|^{n+1}}. \tag{2.17}$$

Remark. In general one cannot have both $T\varphi$ anti-holomorphic and $\widetilde{F}=0$, since then (2.14), applied to any d-closed (0,1)-form φ, would give a solution to $du=\varphi$ but this is possible only if $H^1(D,\mathbf{C})=0$.

The formula (2.14) will be derived from the formulas for $\partial\bar\partial$ in [AC3] and instead of recapitulating them here we only indicate the necessary modifications. We begin with formula (3.15) from [AC3] (with $\alpha=1$),

$$K\varphi=M\partial\varphi+A_{1,0,n+1}\varphi+\widetilde{F}\varphi,$$

where $\widetilde{F}$ is as in (2.17) and

$$A_{1,0,n+1}\varphi = c\int_D \frac{-\varrho\varphi\wedge d_\zeta\bar{v}\wedge(\partial\bar{\partial}\varrho)^{n-1}}{\bar{v}^{n+1}}. \tag{2.18}$$

This formula only involves third order derivatives of ϱ.

We may now replace $v(\zeta,z)$ by $v(z,\zeta)$ in (2.18) since, in view of (2.11), the occurring error term can be incorporated in $\widetilde{F}\varphi$. The kernel so obtained is

$$-\varrho\varphi\wedge\partial_\zeta\frac{1}{v(z,\zeta)^n}\wedge(\partial\bar{\partial}\varrho)^{n-1}. \tag{2.19}$$

The kernel $g(z,\zeta)$ for the holomorphic projection G above is defined so that

$$g(z,\zeta)d\sigma(z) = \frac{q(z,\zeta)\wedge(\bar{\partial}_z q(z,\zeta))^{n-1}}{v(z,\zeta)^n} + \text{smooth}.$$

Thus if we choose

$$d\sigma(z) = q(z,\zeta)\wedge(\bar{\partial}_z q(z,\zeta))^{n-1}|_{\zeta=z},$$

then

$$\partial_\zeta\left(\frac{1}{v(z,\zeta)^n} - g(z,\zeta)\right) = \partial_\zeta\frac{\mathcal{O}(|\zeta-z|)}{v(z,\zeta)^n} + \text{smooth},$$

and hence we may replace (2.19) by

$$-\varrho\varphi\wedge\partial_\zeta g(z,\zeta)\wedge(\partial\bar{\partial}\varrho)^{n-1};$$

again the occurring error term can be incorporated in $\widetilde{F}\varphi$. Hence we have obtained Proposition 2.4. $\square$

If we extend the definition of $Gu(z)$ to ∂D as the boundary values of (2.13), then, see [KSt, Theorem 3.2.1],

$$Gu(z) = \tfrac{1}{2}u(z) + G_{pv}u(z), \quad z\in\partial D,$$

where

$$G_{pv}u(z) = \lim_{\varepsilon\to 0}\int_{d(\zeta,z)>\varepsilon} g(\zeta,z)u(\zeta)\,d\sigma(\zeta), \quad z\in\partial D.$$

Thus G and G_{pv} have the same boundedness properties. By Cotlar's lemma it follows that $G_{pv}: L^2(\partial D)\to L^2(\partial D)$, and since $|g|\lesssim d^{-n}$ and $|\nabla g|\lesssim d^{-n-1}$, G_{pv} is a singular integral operator that maps $L^p(\partial D)$ into $L^p(\partial D)$, $1<p<\infty$, for details see [KSt], and also $\mathcal{H}^1$ into $L^1(\partial D)$. Thus $G: L^p(\partial D)\to H^p$, and $G:\mathcal{H}^1\to H^1$.

This argument works equally well for H in (2.8).

From (2.11) it follows that G is almost self-adjoint and it turns out that

$$S = G + SA = G + A - AS, \tag{2.20}$$

where S is the Szegő projection, and $A = G^* - G$ has the kernel $g(\zeta,z) - \overline{g(z,\zeta)}$ which by (2.11) (and (2.7)) is weakly singular. By repeated use of (2.20) it follows that $S: L^p(\partial D)\to H^p$, cf. [KSt]. In Section 6 we also show that G maps BMO onto BMOA.

3. Proof of Theorem 1.4

We will rely on Proposition 2.4. To begin with, we note that the term $M\partial\varphi$ is in $L^1(\partial D)$ if $\partial\varphi$ satisfies the condition (a_1). This immediately follows from (2.16) and the estimate (2.7). It is proved in [AC2 §6] that $M\partial\varphi$ is in $\mathrm{BMO}(\partial D)$ if (a_p) holds for $p=*$. Then it follows by interpolation that $M\partial\varphi$ is in $L^p(\partial D)$ if (a_p) holds, $1<p<*$. However, these intermediate cases can also be obtained rather easily by a direct estimation, not relying on the somewhat harder BMO estimate.

Next, we show that the term $T\varphi$ is in $L^p(\partial D)$ if the condition (b_p) holds. To this end, take $\psi\in L^q(\partial D)$, ($\in$BMO if $p=1$ and $\in\mathcal{H}^1$ if $p=*$). We then have by (2.13) and Fubini's theorem,

$$\int_{\partial D}\psi(z)T\varphi(z)\,d\sigma(z)=\int_D -\varrho\varphi\wedge\partial G\psi\wedge(\partial\bar\partial\varrho)^{n-1},$$

and hence by (b_p),

$$\left|\int_{\partial D}\psi T\varphi\,d\sigma\right|\lesssim\|G\psi\|_{H^q}\lesssim\|\psi\|_{L^q(\partial D)},$$

since $G\colon L^q(\partial D)\to H^q$, cf. Section 2. Thus $T\varphi$ is in $L^p(\partial D)$.

Since the kernel of $\widetilde F$ is less singular than that of T, the term $\widetilde F\varphi$ is in principle easier to estimate than $T\varphi$. In fact, it is straight forward to show that $\widetilde F\varphi$ is in $L^p(\partial D)$ under the conditions in Theorem 1.2 or Corollary 1.5. However, the condition (b_p) is not so easy to apply directly to the term $\widetilde F\varphi$, and we therefore rewrite (2.14) further. Unfortunately, we must then assume that ϱ is C^4.

We have $\bar\partial K=\varphi$, and by Lemma 3.5 in [AC3], the term $\widetilde F\varphi$ can be rewritten as $FK\varphi$, where F satisfies

$$|Fu_0(z)|\lesssim\int_D\frac{|u_0|}{|v|^{n+1/2}},\quad z\in\partial D. \tag{3.1}$$

Thus, $K\varphi=M\partial\varphi+T\varphi+FK\varphi$, and iterating we obtain

$$K\varphi=M\partial\varphi+FM\partial\varphi+\ldots+F^{m-1}M\partial\varphi+T\varphi+FT\varphi+\ldots+F^{m-1}T\varphi+F^mK\varphi. \tag{3.2}$$

Note that the condition (b_p) (taking $g=z_j$) implies that $-\varrho|\varphi|$ is a finite measure. Arguing as in (the proof of) Proposition 2.1 in [AC3], one obtains that

$$|F^mK\varphi(z)|\lesssim\int_D-\varrho|\varphi|<\infty,$$

if m is large enough.

The estimate (2.8) in [AC3] for $F^kM\partial\varphi$ is sharper than (2.16) for $M\partial\varphi$, and hence $F^kM\partial\varphi$ is in $L^p(\partial D)$. Note that $F^kM\partial\varphi$ is in $L^\infty(\partial D)\subset\mathrm{BMO}$ when $p{=}*$.

It remains to estimate the terms $F^kT\varphi$. If f_k is the kernel of F^k, then

$$|f_k(z,\varphi)|\lesssim\frac{1}{|v|^{n+1/2}},$$

see Lemma 3.5 in [AC3], and we have

$$F^kT\varphi(\tau)=\int_{z\in D}T\varphi(z)f(z,\lambda)=\int_{\zeta\in D}-\varrho|\varphi|\wedge(\partial\bar\partial\varrho)^{n-1}\partial_\zeta\int_{z\in D}g(z,\zeta)f_k(z,\lambda).$$

If $\psi\in L^q(\partial D)$, (b_p) implies,

$$\left|\int_{\partial D}F^kT\varphi(\tau)\psi(\tau)d\sigma(\tau)\right|\lesssim\int_{\zeta\in\partial D}-\varrho|\varphi|\left|\partial_\zeta\int_{\lambda\in\partial D}\int_{z\in D}g(z,\zeta)f_k(z,\lambda)\psi(\lambda)\right|$$
$$\lesssim\|\mathcal{K}\psi\|_{H^q},$$

where $\mathcal{K}\psi(\zeta)=\int_{\partial D}k(\zeta,\lambda)\psi(\lambda)$ is holomorphic, and

$$|k(\zeta,\lambda)|\lesssim\left|\int_D g(z,\zeta)f_k(z,\lambda)\right|\lesssim\int_D\frac{d\lambda(z)}{|v(\zeta,z)|^n|v(z,\lambda)|^{n+1/2}}\lesssim\frac{1}{|v(\zeta,z)|^{n-1/2}}.$$

For the last inequality, compare Lemma 5.2 in [AC3].

By Schur's lemma, $\|\mathcal{K}\psi\|_{H^q}\lesssim\|\psi\|_{L^q(\partial D)}$ and by duality, this implies that $F^kT\varphi\in L^p(\partial D)$ and the proof is complete. $\square$

Remark. When $p{=}1$ there is an inbalance between the (a_p)- and the (b_p)-conditions. In fact, (b_1) implies that $T\varphi$ is in $\mathcal{H}^1$, but the (a_1)-condition only yields that $M\partial\varphi$ is in $L^1(\partial D)$. It is therefore natural to look for a somewhat sharpened (a_1)-condition that would imply that also $M\partial\varphi$ is in $\mathcal{H}^1$. However, we have not found any natural such condition. When $n{=}1$ and D is the unit disc, then the kernel for M is $1-|\zeta|^2$ times the adjoint of the Poisson kernel, so in this case the condition would be that the balayage of $(1-|\zeta|)|\partial\varphi|$ is in $\mathcal{H}^1$, or equivalently, that $P\psi\in L^1((1-|\zeta|)|\partial\varphi|)$ for all Poisson integrals of BMO-functions.

4. Another proof

Let G be the Green's function for D with pole at (some) $0\in D$. Fix $\chi\in C^\infty(\overline{D})$ with $\chi{=}1$ near ∂D and $\chi{=}0$ near 0, and put $H{=}\chi G$. Then $H\lesssim-\varrho$ and (b_p) implies that

$$(c_p)\qquad\left|\int_D H\varphi\cdot\partial g\right|\lesssim\|g\|_{H^q}.$$

By taking $g(z){=}z_i$ in (b_p) we see that $-\varrho|\varphi|$ is a finite measure in D. Thus Theorem 1.4 follows from

Proposition 4.1. *Assume that* (a_p) *holds and that* $-\varrho|\varphi|$ *is a finite measure. Then, for* $1<p\leq *$ *there is a solution* $u\in L^p(\partial D)$ *to* $\bar\partial u=\varphi$ *if and only if* (c_p) *holds.*

As described in the introduction, we solve $\partial\bar\partial u_0=\partial\varphi$ with $u_0\in L^p(\partial D)$. Then we show that

$$\left|\int_D H\bar\partial u_0\cdot\partial g\right|\leq\|g\|_{H^q}. \tag{4.1}$$

Thus $\varphi_0=\varphi-\bar\partial u_0$ also satisfies (c_p), and we have reduced the problem to the simpler case where $\partial\varphi_0=0$.

If $\bar\partial\varphi_0=\partial\varphi_0=0$, we have $d\varphi_0=0$. If $\bar a$ is a solution to the Poincaré equation, $d\bar a=\varphi_0$, then for bidegree reasons $\partial\bar a=0$, so a is holomorphic, and $\bar\partial\bar a=\varphi_0$. However, even if $H^1(D,\mathbf{C})\neq0$ there is always a nice d-closed (0,1)-form α that is cohomologous to φ_0 (e.g. whose sup norm over $\bar D$ only depends on the L^1 norm of $-\varrho\varphi_0$). Then we can take a nice solution to $\bar\partial u=\alpha$ and obtain $\bar\partial(\bar a-u)=\varphi_0$.

Thus we may assume that there is a holomorphic function a with $d\bar a=\varphi_0$. To estimate the H^p-norm of a we use duality. If $g\in H^q$, Green's theorem implies (if we let $d\sigma=(\partial H)/(\partial n)dS$),

$$\int_{\partial D} g\bar a\,d\sigma=\int_D g\bar a\Delta H-H\Delta(g\bar a).$$

But by (c_p),

$$\left|\int_D H\Delta(g\bar a)\right|=\left|\int_D H\partial g\cdot\overline{\partial a}\right|=\left|\int_D H\partial g\cdot\varphi_0\right|\lesssim\|g\|_{H^q}.$$

Furthermore, as H is harmonic near ∂D, we have for some $K\subset\subset D$ that

$$\left|\int_D g\bar a\Delta H\right|\lesssim\int_K|g\bar a|\leq\|g\|_K\int_K|a|\lesssim\|g\|_{H^q},$$

so $a\in H^p\subset L^p(\partial D)$.

To complete the proof it remains to verify (4.1). If $p>1$,

$$\int_D H\bar\partial u_0\cdot\partial g=\int_D H(\Delta(u_0g)-g\Delta u_0)=\int_{\partial D}u_0g\,d\sigma+\int_D u_0g\Delta H-\int_D Hg\Delta u_0.$$

By duality,

$$\left|\int_{\partial D}u_0g\,d\sigma\right|\lesssim\|g\|_{H^q}.$$

For the second intergral we have

$$\left|\int u_0 g\Delta H\right| \lesssim \int_K |u_0 g| \leq \|g\|_{H^q} \int_D |u_0| \leq \|g\|_{H^q}\left(\int_D -\varrho|\Delta u_0| + \int_{\partial D} |u_0|\right) \lesssim \|g\|_{H^q}.$$

Finally,

$$\left|\int_D Hg\Delta u_0\right| \lesssim \int_D -\varrho|\Delta \bar{u}_0||g| \lesssim \|g\|_{H^q},$$

since $-\varrho|\partial\bar\partial u_0| = -\varrho|\partial\varphi| \in W^{1-1/p}$. □

This argument does not work for $p=1$ since to obtain (4.1), we would need $u_0 \in \mathcal{H}^1$, which however does not follow from the (a_1)-condition, cf. the remark in Section 3.

5. Proofs of Theorem 1.2 and Corollary 1.5

In this section we prove Corollary 1.5 and that Theorem 1.2 follows from Theorem 1.4 and, if $p \geq 2$, also from Corollary 1.5.

Proof of Corollary 1.5. It is enough to show that if $2 \leq p \leq *$,

$$\int -\varrho|\varphi||\partial g| \lesssim \|g\|_{H^q}, \quad \text{if } -\varrho|\varphi|^2 \in W^{1-2/p}.$$

If $p=*$,

$$\left(\int -\varrho|\varphi||\partial g|\right)^2 \leq \int -\varrho\frac{|\partial g|^2}{|g|} \int -\varrho|g||\varphi|^2 \lesssim \|g\|^2_{H^1},$$

by Proposition 2.1 and the Carleson–Hörmander inequality. If $p=2$,

$$\left(\int -\varrho|\varphi||\partial g|\right)^2 \leq \int -\varrho|\partial g|^2 \int -\varrho|\varphi|^2 \lesssim \|g\|^2_{H^2}.$$

The general case follows by interpolation. □

Proof of Theorem 1.2. We have to show that

$$-\varrho|\partial(\phi\omega)| + \sqrt{-\varrho}(|\partial(\phi\omega)\wedge\partial\varrho| + |\partial(\phi\omega)\wedge\bar\partial\varrho|) + |\partial(\phi\omega)\wedge\partial\varrho\wedge\bar\partial\varrho| \in W^{1-1/p}, \tag{5.1}$$

and

$$I = \int -\varrho|\phi\omega||\partial g| \lesssim \|\phi\|_{H^p}\|g\|_{H^q}, \tag{5.2}$$

or, if $p\geq 2$,

$$-\varrho|\phi\omega|^2\in W^{1-2/p}. \tag{5.3}$$

As $|\phi|^2\in L^{p/2}(-\varrho|\omega|^2)$ by the Carleson–Hörmander inequality, (5.3) follows.

To prove (5.2), we first assume that $2\leq p\leq\infty$. Then

$$I^2\leq\int -\varrho|g|^{q-2}|\partial g|^2\int -\varrho|\phi|^2|g|^{2-q}|\omega|^2.$$

By (2.2), the first integral is bounded by $\|g\|^q_{H^q}$. If $p=q=2$ or $p=\infty$, $q=1$, the second integral is bounded by $\|\phi\|^2_{H^2}$ and $\|\phi\|^2_{H^\infty}\|g\|_{H^1}$, respectively. If $2<p<\infty$, we note that $p/2$ and $q/(2-q)$ are dual exponents. Hence Hölder's inequality implies

$$I^2\lesssim\|g\|^q_{H^q}\left(\int -\varrho|\phi|^p|\omega|^2\right)^{2/q}\left(\int -\varrho|g|^q|\omega|^2\right)^{(2-q)/q}\lesssim\|g\|^2_{H^q}\|\phi\|^2_{H^p}$$

as desired. If $1\leq p<2$ we write $\phi\partial g=\partial(\phi g)-g\partial\phi$, and we can apply the case $p\geq 2$ of (5.2) to each of these terms.

The estimates for the different terms in (5.1) are very similar and we only give the details for one of them, namely $\sqrt{-\varrho}|\partial(\phi\omega)\wedge\bar\partial\varrho|$. Now $\partial(\phi\omega)=\phi\partial\omega+\partial\phi\wedge\omega$ and we handle the two terms separately.

To estimate the first one we note that by assumption $\mu=\sqrt{-\varrho}|\partial\omega\wedge\bar\partial\varrho|$ is a Carleson measure and thus $|\phi|\in L^p(\mu)$ and

$$\sqrt{-\varrho}|\phi\partial\omega\wedge\bar\partial\varrho|\in W^{1-1/p}.$$

The estimate for the term $\partial\phi\wedge\omega$ will be obtained by interpolation. If $p=1$,

$$\left(\int_D\sqrt{-\varrho}|\partial\phi\wedge\omega\wedge\bar\partial\varrho|\right)^2\leq\int_D -\varrho\frac{|\partial\phi|^2}{|\phi|}\int_D|\phi||\omega\wedge\bar\partial\varrho|^2\lesssim\|\phi\|^2_{H^1}.$$

If $p=*$, let Q be a Korányi-cube in D. Then

$$\left(\int_Q\sqrt{-\varrho}|\partial\phi\wedge\omega\wedge\bar\partial\varrho|\right)^2\lesssim\int_Q -\varrho|\partial\phi|^2\int_Q|\omega\wedge\bar\partial\varrho|^2,$$

and since both $-\varrho|\partial\phi|^2$ and $|\omega\wedge\bar\partial\varrho|^2$ are Carleson measures, so is $\sqrt{-\varrho}|\partial\phi\wedge\omega\wedge\bar\partial\varrho|$. The general case, $1<p<\infty$, follows by interpolation. Note that it is not enough to prove the cases $p=1$ and $p=\infty$ since it is not known whether one can interpolate between H^1 and H^∞, cf. [J]. □

6. Proofs of Propositions 2.1–2.3

Proof of Proposition 2.1, *formula* (2.2). Since $\Delta e^{\psi}=(\Delta\psi)e^{\psi}+|\nabla\psi|^2e^{\psi}$, Green's identity in $\mathbf{R}^{2n}$ yields

$$\int_D(-\varrho)\Delta\psi e^{\psi}+\int_D(-\varrho)|\nabla\psi|^2e^{\psi}+\int_D(\Delta\varrho)e^{\psi}=\int_{\partial D}e^{\psi}\frac{\partial\varrho}{\partial n},$$

where $\partial/\partial n$ is the outward normal derivative. Hence

$$\int_D(-\varrho)|\nabla\psi|^2e^{\psi}\le\int_{\partial D}e^{\psi}\frac{\partial\varrho}{\partial n}\sim\int_{\partial D}e^{\psi},$$

if ϱ and ψ are subharmonic and $d\varrho\neq 0$ on ∂D. Thus, if $\psi=(p/2)\log|f|^2$,

$$\int_D(-\varrho)|\partial f|^2|f|^{p-2}\lesssim\int_{\partial D}|f|^2.$$

Now,

$$\int_D(-\varrho)|\partial f|^2|f|^{p-2}\sim\int_D(-\varrho)\partial f\wedge\overline{\partial f}\wedge(\partial\bar\partial\varrho)^{n-1}|f|^{p-2},$$

and an integration by parts in the last integral yields

$$\begin{aligned}\int_D(-\varrho)\partial f\wedge\overline{\partial f}\wedge(\partial\bar\partial\varrho)^{n-1}|f|^{p-2}&=\int_D\partial f\wedge\overline{\partial f}\wedge\partial\varrho\wedge\overline{\partial\varrho}|f|^{p-2}(\partial\bar\partial\varrho)^{n-2}\\&\sim\int_D|\partial f\wedge\partial\varrho|^2|f|^{p-2}.\end{aligned}$$

Thus (2.2) is established. □

Proof of (2.3). The starting point is (2.8),

$$f(z)=Hf(z)=\int_{\partial D}\frac{fq\wedge(\bar\partial q)^{n-1}}{v(\zeta,z)^n}.$$

Recall from Section 2 that $H\colon L^2(\partial D)\to H^2$. Now $q\wedge(\bar\partial q)^{n-1}=r(\zeta,z)d\sigma(\zeta)$ where $r(\zeta,z)$ is holomorphic in z and $|r(\zeta,z)|\sim 1$. Thus

$$Hf(z)=\int_{\partial D}\frac{f(\zeta)}{v(\zeta,z)^n}r(\zeta,z)\,d\sigma(\zeta).$$

To estimate Hf, we will use that

$$J_h=\int_{B_h^c(p)}\frac{|f(z)-f_h|}{d(z,p)^{n+\alpha}}\,d\sigma(z)\lesssim\frac{\|f\|_*}{h^{\alpha}},\tag{6.1}$$

if $\alpha>0$ (f_h is the mean value of f over $B_h(p)$).

Let us assume (6.1) and complete the proof. Put $\mu(f)=-\varrho|\partial f|^2+|\partial f\wedge\partial\varrho|^2$. We have to show that

$$\int_{Q_h}\mu(f)d\lambda\lesssim\|f\|_*^2|B_h|, \tag{6.2}$$

where Q_h is centered at the arbitrary point $0\in\partial D$. On the boundary, we write $f=f_1+f_2+f_3$ where $f_1=f_{Ch}$, $f_2=(f-f_1)\chi$, $f_3=(f-f_1)(1-\chi)$ and χ is the characteristic function of B_{Ch} and C is a suitably large constant (depending on the constant in the triangle inequality for d). We also put $f_i(z)=Hf_i(z)$ when $z\in D$.

Since f_1 is constant, it gives no contribution to the left hand side of (6.2). To estimate $\int_{Q_h}\mu(f_2)$, we use (2.2) with $p=2$,

$$\int_{Q_h}\mu(f_2)d\lambda\lesssim\int_D\mu(f_2)d\lambda\lesssim\|Hf_2\|_{H^2}^2=\int_{B_{Ch}}|f-f_{Ch}|^2\lesssim\|f\|_*^2|B_h|,$$

where the last inequality follows from the John–Nirenberg inequality.

To estimate the contribution from f_3, we observe that

$$\partial f_3=\partial Hf_3=-n\int\left(\frac{\partial_z v(\zeta,z)r(\zeta,z)}{v(\zeta,z)^{n+1}}+\frac{\partial_z r(\zeta,z)}{v(\zeta,z)^n}\right)f_3(\zeta)\,d\sigma(\zeta).$$

Hence, since $\zeta\notin B_{Ch}$, $z\in Q_h$ implies $v(\zeta,z)\sim d(\zeta,0)$, we obtain

$$|\partial f_3(z)|\lesssim\int_{B_{Ch}^c}\frac{|f-f_{Ch}|}{d(\zeta,0)^{n+1}}\,d\sigma(\zeta)\lesssim\frac{\|f\|_*}{h}$$

by (6.1). Similarily, as $|\partial v\wedge\partial\varrho|\lesssim|z-\zeta|\lesssim d(z,\zeta)^{1/2}$,

$$|\partial f_3\wedge\partial\varrho|\lesssim\int_{B_{Ch}^c}\frac{|f-f_{Ch}|}{d(\zeta,0)^{n+1/2}}\,d\sigma(\zeta)\lesssim\frac{\|f\|_*}{\sqrt{h}}.$$

Since $-\varrho(z)\leq h$ if $z\in Q_h$, we have

$$\mu(f_3)\lesssim\frac{\|f\|_*}{h}$$

on Q_h and

$$\int_{Q_h}\mu(f_3)\,d\lambda\lesssim\frac{\|f\|_*}{h}|Q_h|\lesssim\|f\|_*^2|B_h|. \qquad\square$$

Proof of (6.1). To simplify notations, let $B_j = B_{2^j h}(p)$, $A_j = B_{j+1} \setminus B_j$, $f_j = f_{2^j h}$ and $d(z) = d(0, z)$. Then

$$J_h = \sum_{j=0}^{\infty} \int_{A_j} \frac{|f(z) - f_0|}{d(z)^{n+\alpha}}\, d\sigma(z) \lesssim \sum_{j+1}^{\infty} (2^j h)^{-(n+\alpha)} \int_{B_{j+1}} |f(z) - f_0|\, d\sigma(z). \tag{6.3}$$

But

$$\int_{B_{j-1}} |f(z) - f_j| \lesssim \int_{B_j} |f(z) - f_j| \le \|f\|_* |B_j|$$

and hence $|f_{j-1} - f_j| \le (|B_j|/|B_{j+1}|)\|f\|_*$, which implies $|f_j - f_0| \lesssim j\|f\|_*$. Thus

$$\begin{aligned}\int_{B_j} |f(z) - f_0| d\sigma(z) &\le \int_{B_j} |f(z) - f_j| d\sigma(z) + |f_j - f_0||B_j| \\ &\lesssim (1+j)|B_j|\|f\|_* \lesssim j(2^j)^n \|f\|_*.\end{aligned}$$

If we use this estimate in (6.3), we obtain

$$J_h \lesssim \sum_{j=0}^{\infty} (2^j h)^{-(n+\alpha)} j (2^j h)^n \|f\|_* \lesssim \frac{\|f\|_*}{h^\alpha}. \quad \square$$

Remark. In the proof we did not use the fact that $f \in \mathrm{BMOA}$; if $\tilde{f} \in \mathrm{BMO}(\partial D)$ and we let $f = H\tilde{f}$ or $f = G\tilde{f}$, then f satisfies (2.3). $\square$

Proof of Proposition 2.2. We first prove $H\colon \mathrm{BMO} \to \mathrm{BMOA}$. With the tools we have developed, this is accomplished in a few strokes. As above, we write $f = f_1 + f_2 + f_3$. As Hf_1 is constant, $\|Hf_1\|_* = 0$. For Hf_2 we have, as H is bounded in L^2, that

$$\begin{aligned}\left(\frac{1}{|B_h|} \int_{B_h} |Hf_2| d\sigma\right)^2 &\le \frac{1}{|B_h|} \int_{\partial D} |Hf_2|^2 d\sigma \lesssim \frac{1}{|B_h|} \int_{\partial D} |f_2|^2 d\sigma \\ &= \frac{1}{|B_h|} \int_{B_{Ch}} |f - f_{Ch}|^2 d\sigma \lesssim \|f\|_*^2.\end{aligned}$$

Hence, $\|Hf_2\|_* \lesssim \|f\|_*$. To estimate $\|Hf_3\|_*$ we need

$$|v(z,\zeta) - v(w,\zeta)| \lesssim (hd(\zeta,p))^{1/2} \quad \text{if } z, w \in B_h(p),\ \zeta \notin B_{Ch}(p).$$

This follows immediately if we write

$$v(z,\zeta) - v(w,\zeta) = v(z,w) + (q(z,w) - q(z,\zeta))(z-w) + (q(z,\zeta) - q(w,\zeta))(\zeta - w).$$

Using this, we have

$$\begin{aligned}
&\frac{1}{|B_h|}\int_{B_h}|Hf_3(z)-(Hf_3)_h|\,d\sigma(z)\\
&\quad\lesssim\frac{1}{|B_h|^2}\int_{B_h}d\sigma(z)\int_{B_h}d\sigma(w)\int_{\partial D}\left|\frac{r(\zeta,z)}{v(\zeta,z)^n}-\frac{r(\zeta,w)}{v(\zeta,w)^n}\right||f_3(\zeta)|\,d\sigma(\zeta)\\
&\quad\lesssim\int_{\partial D}\frac{\sqrt{h}\sqrt{d(\zeta,p)}}{d(\zeta,p)^{n+1}}|f_3(\zeta)|\,d\sigma(\zeta)\lesssim\sqrt{h}\int_{B^c_{Ch}}\frac{|f(\zeta)-f_{Ch}|}{d(\zeta,p)^{n+1/2}}\lesssim\|f\|_*,
\end{aligned}$$

where the last inequality follows from (6.1).

This proof also applies to G, defined by (2.13), $G\colon \mathrm{BMO}\to\mathrm{BMOA}$, as $g(\zeta,z)$ satisfies the same estimate as $r(\zeta,z)/v(\zeta,z)^n$.

By (2.20), $S-G=A-AS$ and A maps $L^p\to H^\infty$ if p is large enough, since A is weakly singular. Since S preserves L^p, we obtain $S-G\colon L^p\to H^\infty$ if p is large, and hence $S-G\colon\mathrm{BMO}\to\mathrm{BMOA}$ and so Proposition 2.2 is proved. □

Proof of Proposition 2.3. Let $d\sigma=(\partial G/\partial n)dS$, where G is the Green's function with pole in $0\in D$, and let $f\subset\mathcal{O}(\overline{D})$ and $f(0)=0$. Then, by Green's identity,

$$\int_{\partial D} f\bar{b}\,d\sigma=\int_D G\partial f\cdot\overline{\partial b},$$

and hence

$$\left|\int_{\partial D} f\bar{b}\,d\sigma\right|\lesssim\int_D-\varrho|\partial f||\partial b|+\|f\|_{H^1}\|b\|_*,$$

since $G\sim-\varrho$ near ∂D. By Schwarz' inequality, Proposition 2.1 and the Carleson–Hörmander inequality,

$$\left(\int_D-\varrho|\partial f||\partial b|\right)^2\le\int_D-\varrho\frac{|\partial f|^2}{|f|}\int_D-\varrho|\partial b|^2|f|\lesssim\|f\|^2_{H^1}\|b\|^2_*,$$

so that

$$\left|\int_{\partial D} f\bar{b}\,d\sigma\right|\lesssim\|f\|_{H^1}\|b\|_*. \tag{6.4}$$

By approximation now (6.4) follows for arbitrary $f\in H^1_0$. Thus $\mathrm{BMOA}\subset(H^1_0)^*$. On the other hand, any functional on H^1_0 is given by some $u\in L^\infty(\partial D)$, hence by $b=Su\in\mathrm{BMOA}$, where S is the Szegő projection with respect to $d\sigma$. This proves Proposition 2.3 for $d\sigma$.

Now suppose that $\psi d\sigma$ is another surface measure on ∂D (we assume $\psi\in C^1$ and $\psi\sim1$). Then by (6.4)

$$\left|\int_{\partial D} f\bar{b}\psi\,d\sigma\right|=\|f\|_{H^1_0}\|b\psi\|_*\lesssim\|f\|_{H^1_0}\|b\|_*.$$

Thus (6.4) holds also for the measure $\psi d\sigma$ and then the duality with respect to this measure follows as above. □

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Mats Andersson
Department of Mathematics
CTH & University of Göteborg
S-412 96 Göteborg
Sweden

Hasse Carlsson
Department of Mathematics
CTH & University of Göteborg
S-412 96 Göteborg
Sweden