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Weierstrass points and gap sequences for families of curves

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Abstract. The theory of Weierstrass points and gap sequences for linear series on smooth curves is generalized to smooth families of curves with geometrically irreducible fibers, and over an arbitrary base scheme.

0. Introduction

We present a theory of Weierstrass points and gap sequences for linear series on smooth families of curves with geometrically irreducible fibers; the parameter space is an arbitrary base scheme. The theory provides a flexible framework for studying the behavior of Weierstrass points under deformation of curves, generalizing the classical theory of Weierstrass points on a single Riemann surface.

The work is a continuation and extension of the results in [25] where we studied enumerative formulas for multiple points of linear systems on smooth families of curves. That such an extension is natural and interesting is indicated by the relations between multiple points and gaps of linear systems on smooth curves (see [36] and [24]). Compared to [25] the present work has several new features:

First of all, we introduce and develop a theory of gaps and Weierstrass points. We work, as in [25], in great generality with Wronski systems on a noetherian scheme. For such systems we define gap sequences and Weierstrass points and show that they satisfy properties analogous to those of gaps and Weierstrass points on smooth curves. The gaps associated with a linear system on a smooth curve have the following two key properties:

(1) The number of gaps at each point is equal to the number of gaps at the generic point.

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(2) The number of gaps at the generic point is equal to the dimension of the given linear system.
Moreover, a Weierstrass point is characterized as a point where the gap sequence is different from the generic gap sequence. The main result (Corollary (3.3)) is that the Weierstrass points are the points of the zero scheme of the wronskians associated to the Wronski system. In particular, the Weierstrass points that we define have the fundamental property of being the zeros of sections of locally free sheaves on the total space.

When the scheme is the total space of a family of smooth curves and the Wronski system is obtained from the sheaves of principal parts our theory generalizes that for smooth curves (see [34], [35], [30], [23], [24]). We even show (Note (5.2)) that, when the family is proper, we can interpret gaps in terms of the existence of poles of meromorphic functions.

Secondly, we extend, in Section 4, the theory of higher derivations in abstract algebra developed in [25], and we obtain further generalizations of the works [21], [34], [35] and [29]. In particular, we find in Theorem (4.11) a sufficient condition for the gaps associated to an iterative derivation to have the second key property above. Our results make it possible to give the exact relations between the local theory of Weierstrass points, based upon higher derivations and most commonly used in the literature, and the global approach of [23], [24], [25] and of the present work. In fact, the present work started as an attempt to clarify these relations.

A third new feature of the present work is the refinement of Theorem (4.6) of [25] and the adaptation of that result for use in our theory of gaps and Weierstrass points. For the Wronski systems obtained from linear systems on a smooth family of curves with geometrically irreducible fibers we establish in Theorem (5.5) both key properties of the gaps. We establish the key properties without using the Riemann–Roch Theorem for curves! The Riemann–Roch Theorem is an essential ingredient in all other treatments, limiting them to *complete* curves (see [18] for example).

The final new feature of this work is its applicability to families with possibly singular fibers. In this direction we present in the last section for a Gorenstein curve the Wronski systems that form the natural generalization of the systems of principal parts for a smooth curve. Our results generalize those of [27] to arbitrary characteristics and globalize the wronskian determinant used in their work.

The main applications of our theory for smooth curves are to the universal families of curves over various moduli schemes. Such families contain interesting subsets defined by conditions of Weierstrass type. This is evidenced by a large number of works (see for instance [2], [3], [4], [6], [7], [8], [9], [22], [26], [32], [33]). In these works, the subsets are defined fiberwise, that is, as subsets of the individual members of the family. Also in the series of articles [10], [11], [12], [13], [14], [15], containing many important results from the theory of curves, the central definition of limit series on curves is based upon a Plücker formula (also called the Brill–Segre formula) on the individual members of the family. In this work we show how to define such sets globally as relative schemes over the moduli scheme, and even as zero schemes of sections of locally free sheaves. For families of hyperelliptic curves such a global construction of Weierstrass schemes was given in [22].

We will express our thanks to S. L. Kleiman for his remarks that led to the present formulation of Theorem (5.5).

1. Preliminaries

We collect, in this section, some of the fundamental properties of ranks of maps between sheaves that are essential for the study of Wronski systems and their wronskians in the sequel.

(1.1) Setup. We shall work with modules over a fixed (noetherian) scheme X. If $\mathcal{F}$ is a module and x is a point of X, we denote by $\boldsymbol{\kappa}(x)$ the residue field at x and by $\mathcal{F}(x){:=}\mathcal{F}\otimes\boldsymbol{\kappa}(x)$ the fiber of $\mathcal{F}$ at x. If $v{:}\,\mathcal{G}\to\mathcal{F}$ is a map of modules, we denote by $v(x){:}\,\mathcal{G}(x)\to\mathcal{F}(x)$ the $\boldsymbol{\kappa}(x)$-linear map induced on the fibers. Unless the contrary is stated explicitly, a module is assumed to be coherent and a locally free module is assumed to be of finite rank.

(1.2) *Remark.* Let $v{:}\,\mathcal{W}\to\mathcal{P}$ be a map of quasi coherent $\mathcal{O}_X$-modules. The following two results are well known, see for instance [25, Lemma (4.2), p. 144]:

(1) If $\mathcal{W}$ is flat and v is injective, then the map $v(\xi)$ is injective for all associated points ξ of X.

(2) Assume that $\mathcal{W}$ and $\mathcal{P}$ are locally free (not necessarily of finite rank). Then v is injective if and only if the map $v(\xi)$ is injective for all associated points ξ of X.

Assume that $\mathcal{W}$ and $\mathcal{P}$ are locally free. Then it follows from (2) that if v is injective, then every exterior power $\wedge^i v$ is injective. The latter result can be shown to hold without the noetherian hypothesis, see [5, AIII.88], or [38, Lemma (1.4), p. 265].

(1.3) *Definition.* By definition, the *rank* of a map $v{:}\,\mathcal{W}\to\mathcal{Q}$ is the largest integer r such that $\wedge^r v\neq 0$.

When the target $\mathcal{Q}$ is locally free, the map v will be said to be of *generically constant rank* if for every associated point ξ of X, the rank of v is equal to the rank of the map $v(\xi)$. Note that if X is integral, then every map into a locally free sheaf is of generically constant rank.

(1.4) Lemma. *Let $v\colon \mathcal{W}\to\mathcal{Q}$ be a map of modules, and let x be a point in X. Assume that $\mathcal{Q}$ is locally free. Then the following inequality holds:*

$$\operatorname{rk} v \geq \operatorname{rk} v(x). \tag{1.4.1}$$

Moreover, the following two conditions are equivalent:

(i) *Equality holds in the above inequality.*

(ii) *In a neighborhood of x, the cokernel of v is free and the image of v is free of rank equal to* $\operatorname{rk} v$.

Proof. The inequality (1.4.1) holds trivially. The equivalence of the conditions is well known. To prove the equivalence, assume first that (i) holds. Let r be the rank of v. To prove that the cokernel of v is locally free, we may work locally, and assume that $\mathcal{Q}$ is free. Moreover, covering $\mathcal{W}$ with a free module, we may assume that $\mathcal{W}$ is free. Then v is represented by a matrix. By condition (i), there exists an $r\times r$ minor Δ of v, such that $\Delta(x)\neq 0$. Restricting the neighborhood of x we may assume that Δ is invertible. Denote by v_0 the submatrix consisting of the first r columns of v. Without loss of generality we may assume that Δ is the determinant of the submatrix consisting of the first r rows in v_0. Then, since Δ is invertible and $\wedge^{r+1}v=0$, it follows from Cramer's rule that every column in v is a linear combination of the first r columns. In other words, there exists in $\mathcal{W}$ a free rank-r submodule $\mathcal{W}_0$ which is mapped by v_0 onto all of $\operatorname{Im} v$. Thus the cokernel of v is equal to the cokernel of v_0. Again, since Δ is invertible, it follows by Cramer's rule that the map v_0 from $\mathcal{W}_0$ into $\mathcal{Q}$ has a retraction. Hence the cokernel of v_0 is locally free and the image of v is free of rank r. Thus (ii) holds. Conversely, it is clear that (ii) implies (i).

Hence the Lemma has been proved.

Note. It follows easily from Lemma (1.4) that a map into a locally free sheaf is of generically constant rank if and only if the image of the map is locally free of constant rank on some open subset containing all associated points of X.

(1.5) Lemma. *Consider a commutative diagram,*

$$\begin{tikzcd} \mathcal{W} \arrow[d, "v"'] \arrow[dr, "u"] & \\ \mathcal{Q} \arrow[r, "q"] & \mathcal{P} \end{tikzcd}$$

where q is a surjection of locally free sheaves. Assume that the kernel of q is an invertible module $\mathcal{K}$. Let x be a point of X. Then the following inequalities hold:

$$\operatorname{rk} v \geq \operatorname{rk} u \geq \operatorname{rk} v - 1, \tag{1.5.1}$$

$$\operatorname{rk} v(x) \geq \operatorname{rk} u(x) \geq \operatorname{rk} v(x) - 1. \tag{1.5.2}$$

Moreover, the following three conditions are equivalent:

(i) *The first inequality of* (1.5.2) *is strict.*

(ii) *The map of fibers at x of the cokernels,* $\operatorname{Coker} v(x) \to \operatorname{Coker} u(x)$, *induced by the diagram, is an isomorphism.*

(iii) *Equality holds in the second inequality of* (1.5.2).

Proof. In the inequalities of (1.5.1) and of (1.5.2), the first inequality holds trivially, and the second follows from [25, Lemma (1.4), p. 134]. To prove the equivalence of the three conditions, consider the surjection $\operatorname{Im} v(x) \to \operatorname{Im} u(x)$ induced by $q(x)$. Denote by J the kernel of the surjection. Then $J \neq 0$ if and only if the first inequality of (1.5.2) is strict. Clearly, J fits into an exact sequence,

$$0 \longrightarrow J \longrightarrow \mathcal{K}(x) \longrightarrow \operatorname{Coker} v(x) \longrightarrow \operatorname{Coker} u(x) \longrightarrow 0.$$

The term $\mathcal{K}(x)$ is one-dimensional over $\kappa(x)$. Hence $J \neq 0$ if and only if the map of (ii) is an isomorphism. Therefore conditions (i) and (ii) are equivalent. The equivalence of conditions (i) and (iii) is immediate from the inequalities (1.5.2).

We have thus proved the Lemma.

2. Gaps and Weierstrass points of Wronski systems

The central notion of this work is that of a Wronski system. In this section we associate to any Wronski system the corresponding gap sequences and Weierstrass points and give their main properties. We note in (2.10) that these properties are analogous to the corresponding properties for meromorphic functions on Riemann surfaces.

(2.1) Setup. Fix a *Wronski system* as defined in [25], that is, a sequence of surjections $q_i\colon \mathcal{Q}_i \to \mathcal{Q}_{i-1}$ for $i=1,2,\ldots$ of locally free sheaves $\mathcal{Q}_i$ of rank i, together with a sequence of maps $v_i\colon \mathcal{W} \to \mathcal{Q}_i$ such that, for $i=1,2,\ldots$, the following diagram is commutative:

$$\begin{tikzcd} \mathcal{W} \arrow[d, "v_i"'] \arrow[dr, "v_{i-1}"] & \\ \mathcal{Q}_i \arrow[r, "q_i"] & \mathcal{Q}_{i-1}\,. \end{tikzcd}$$

We denote by $\mathcal{K}_i$ the kernel of the map q_i and by $\mathcal{E}_i$ the cokernel of the map v_i. Then $\mathcal{K}_i$ is locally free of rank 1.

By Lemma (1.5), the following inequalities hold:

$$\operatorname{rk} v_i \geq \operatorname{rk} v_{i-1} \geq \operatorname{rk} v_i - 1. \tag{2.1.1}$$

Similarly, if x is a point of X, then the following inequalities hold:

$$\mathrm{rk}\, v_i(x) \geq \mathrm{rk}\, v_{i-1}(x) \geq \mathrm{rk}\, v_i(x) - 1, \tag{2.1.2}$$

and, moreover, the following three conditions are equivalent:

(i) The first inequality of (2.1.2) is strict.

(ii) The map $\mathcal{E}_i(x) \to \mathcal{E}_{i-1}(x)$ induced by q_i, v_i and v_{i-1} is an isomorphism.

(iii) Equality holds in the second inequality of (2.1.2).

(2.2) *Definition.* An integer $i \geq 1$ will be called a *gap* for the Wronski system at the point x if the three equivalent conditions of (2.1) hold. The s'th gap at x will be denoted $g_s(x)$. An integer $i \geq 1$ will be called a *generic gap* for the system if $\mathrm{rk}\, v_i > \mathrm{rk}\, v_{i-1}$. Denote by g_s the s'th generic gap.

Note that if the v_i's are of generically constant rank and ξ is an associated point of X, then i is a gap at ξ if and only if i is a generic gap.

(2.3) Lemma. *Let x be a point of X. Consider a sequence $1, 2, \ldots, h$ where h is a positive integer. Then the rank of v_h is equal to the number of generic gaps in the sequence. Similarly, the rank of $v_h(x)$ is equal to the number of gaps at x in the sequence. Moreover, for every integer s the following conditions are equivalent*:

(i) $\mathrm{rk}\, v_h(x) \geq s$, (ii) $\dim \mathcal{E}_h(x) \leq h - s$, (iii) $h \geq g_s(x)$.

Proof. By (2.1.1), the following two inequalities hold:

$$0 \leq \mathrm{rk}\, v_i - \mathrm{rk}\, v_{i-1} \leq 1, \tag{2.3.1}$$

and equality holds in the second inequality if and only if i is a generic gap. Clearly, the first assertion of the lemma follows upon addition of the inequalities (2.3.1) for $i = 1, \ldots, h$.

The second assertion of the lemma follows by applying similarly the inequalities (2.1.2).

The equivalence of the conditions (i) and (iii) follows from the second assertion of the lemma. That the conditions (i) and (ii) are equivalent follows from the equation $\mathrm{rk}\, v_h(x) + \dim \mathcal{E}_h(x) = h$. The latter equations hold because $\mathcal{Q}_h$ is locally free of rank h. Thus the third assertion of the lemma has been proved.

(2.4) Proposition. *The number of generic gaps is finite and equal to the rank of v_i for all sufficiently large i. Similarly, for any point x, the number of gaps at x is finite and equal to $\mathrm{rk}\, v_i(x)$ for all sufficiently large i. Moreover, the number of gaps at x is at most equal to the number of generic gaps. Finally, the s'th gap at x is greater than or equal to the s'th generic gap, that is,*

$$g_s(x) \geq g_s.$$

Proof. Clearly, the rank $\operatorname{rk} v_i$ is bounded above by the largest integer t, such that $\bigwedge^t \mathcal{W} \neq 0$. Hence it follows from the first assertion of Lemma (2.3) that the number of generic gaps is finite and equal to $\operatorname{rk} v_i$ for all sufficiently large i. Hence the first assertion holds. The proof of the second assertion is similar.

Moreover, the third assertion follows from the first two assertions and the inequality (1.4.1). Finally, to prove the last assertion, consider the sequence $1, 2, \ldots, h$ where $h := g_s(x)$. By the choice of h, the sequence contains s gaps at x. Therefore, by Lemma (2.3) and the inequality (1.4.1), the sequence contains at least s generic gaps. In particular, the asserted inequality $g_s \leq h$ holds.

(2.5) Proposition. (1) *The following two conditions on points x of X are equivalent:*

(i) *The number of gaps at x is equal to the number of generic gaps.*

(ii) *For all sufficiently big i there exists a neighborhood of x over which the cokernel $\mathcal{E}_i$ is free and the image of v_i is free of rank equal to* $\operatorname{rk} v_i$.

Denote by U the subset of points x where the two conditions are satisfied. Then U is an open subset of X. Moreover, if i is sufficiently big, then over U the cokernel $\mathcal{E}_i$ is locally free and the image of v_i is locally free of rank equal to $\operatorname{rk} v_i$.

(2) *Assume that $\mathcal{W}$ is locally free of rank r. Then the following two conditions on points x of X are equivalent:*

(iii) *The number of gaps at x is equal to r.*

(iv) *The map $v_i(x)$ is injective for some value of i.*

Moreover, if there exists a point in X satisfying the latter conditions, then for every point x, the four conditions are equivalent and the number of generic gaps is equal to r.

(3) *Assume that $\mathcal{W}$ is locally free. If the map v_i is injective for some value of i, then the four conditions are equivalent, and they hold for every associated point of X. Conversely, if condition* (iv) *holds for every associated point of X, then the map v_i is injective when i is sufficiently big.*

Proof. (1) Denote by U_i the set of points x for which

$$\operatorname{rk} v_i = \operatorname{rk} v_i(x). \tag{2.5.1}$$

By Lemma (1.4), U_i is open and over U_i the cokernel $\mathcal{E}_i$ is locally free and the image of v_i is locally free of rank equal to $\operatorname{rk} v_i$. Now, when i is sufficiently big, say for $i \geq i_0$, the left hand side of (2.5.1) is constant and equal to the number of generic gaps by Proposition (2.4). Hence it follows from (2.1.2) and (1.4.1) that $U_i \subseteq U_{i+1}$ for $i \geq i_0$. Let x be a point of X. By Proposition (2.4), (i) is satisfied if and only if x belongs to the union $\bigcup_{i \geq i_0} U_i$. By Lemma (1.4), (ii) is satisfied if and only if x belongs to U_i for all sufficiently big i. Therefore (i) and (ii) are equivalent, and U

is equal to the union $\bigcup_{i\geq i_0} U_i$. In particular, U is open. Moreover, U is equal to U_i when i is sufficiently big, because X is noetherian. Therefore the last assertion of (1) holds.

(2) The map $v_i(x)$ is a map of vector spaces, and its source has rank r. Clearly, if $v_i(x)$ is injective, then $v_j(x)$ is injective for all $j\geq i$. Thus (iv) holds if and only if the rank of $v_i(x)$ is equal to r when i sufficiently big. Hence it follows from Proposition (2.4) that (iii) and (iv) are equivalent.

Assume that $v_i(x)$ is injective for some point x. Then, clearly, the rank of v_i is r. Consequently, by Proposition (2.4), the number of generic gaps is equal to r. Therefore, the conditions (iii) and (i) are equivalent for all points x. Hence, by (1), all *four conditions* are equivalent.

(3) The map v_i is a map of locally free sheaves. Therefore, by (1.2)(2), the map v_i is injective if and only if the map $v_i(\xi)$ is injective for every associated point ξ. The assertions of (3) follow easily.

Thus the Proposition has been proved.

(2.6) *Remark.* The main difference between the theory of gaps for smooth curves in any characteristic and for Riemann surfaces is that the latter have generic gap sequence $1, 2, ..., r$ for any linear system of dimension r. It is therefore interesting to have criteria for when a gap sequence of a Wronski system is the sequence $1, 2, ..., r$.

Assume that $\mathcal{W}$ is locally free of rank r. We shall say that a gap sequence is *classical* if it is the sequence $1, 2, ..., r$. Clearly, if the sequence of generic gaps is classical, then $\operatorname{rk} v_i = i$ for $i=0, ..., r$ and $\operatorname{rk} v_i = r$ for $i\geq r$. Conversely, if $\operatorname{rk} v_r = r$, then it follows from the inequalities of (2.1.1) that the sequence of generic gaps is classical. The condition $\operatorname{rk} v_r = r$ is satisfied if v_r is injective. Indeed, if v_r is injective, then the exterior power $\wedge^r v_r$ is injective, cf. Remark (1.2).

Similarly, it follows from the inequalities (2.1.2) that the sequence of gaps at a point x is classical, if and only if the map $v_r(x)$ is injective. The latter condition holds for all associated points of X, if and only if v_r is injective, cf. Remark (1.2)(2). As a consequence, the map v_r is injective, if and only if the sequence of gaps at every associated point of X is classical.

(2.7) *Definition.* A point x in X is called a *Weierstrass point* for the Wronski system if there are non-negative integers g such that $\operatorname{rk} v_g > \operatorname{rk} v_g(x)$. Let x be a Weierstrass point, and let g be the smallest integer such that $\operatorname{rk} v_g > \operatorname{rk} v_g(x)$. Then $\operatorname{rk} v_g$ is called the *rank* of the Weierstrass point.

(2.8) Lemma. *Let x be a point in X and g a positive integer. Assume that the following equation holds*:

$$\operatorname{rk} v_{g-1} = \operatorname{rk} v_{g-1}(x).$$

Then, $\operatorname{rk} v_g > \operatorname{rk} v_g(x)$ *if and only if* g *is a generic gap and not a gap at* x.

Proof. The assertion follows easily from the inequalities (2.1.1), (2.1.2) and (1.4.1).

(2.9) Proposition. *Let* x *be a point of* X. *Then* x *is a Weierstrass point if and only if the sequence of gaps at* x *is different from the sequence of generic gaps. Moreover, if* x *is a Weierstrass point, then the rank of* x *is equal to the smallest integer* s *such that the* s*'th generic gap is not a gap at* x. *Finally, the set of Weierstrass points is a closed subset of* X.

Proof. Assume first that x is a Weierstrass point. Let g be the smallest integer such that $\operatorname{rk} v_g > \operatorname{rk} v_g(x)$. Then, clearly, $\operatorname{rk} v_{g-1} = \operatorname{rk} v_{g-1}(x)$. Therefore, by Lemma (2.8), the integer g is a generic gap and not a gap at x.

Assume conversely that the sequence of generic gaps is different from the sequence of gaps. Then, since the number of gaps at x is at most equal to the number of generic gaps by Proposition (2.4), there are generic gaps that are not gaps at x. Let $g = g_s$ be the first generic gap which is not a gap at x. Then it follows from Lemma (2.8), by induction on i, that $\operatorname{rk} v_i = \operatorname{rk} v_i(x)$ for $i = 1, \ldots, g-1$ and that $\operatorname{rk} v_g > \operatorname{rk} v_g(x)$. In particular, x is a Weierstrass point. Moreover, we note that by definition of the rank of a Weierstrass point, the rank of x is equal to $\operatorname{rk} v_g$. By Lemma (2.3), $\operatorname{rk} v_g$ is equal to the number of generic gaps that are at most equal to g, whence, the rank is equal to s.

Thus the first two assertions of the Proposition has been proved. To prove the last assertion, consider the complement of the set of Weierstrass points. By the definition, the complement consists of the points x satisfying for all i the following condition,

$$\operatorname{rk} v_i = \operatorname{rk} v_i(x). \tag{2.9.1}$$

Clearly, when i is sufficiently big, say for $i \geq i_0$, the left hand side of (2.9.1) is constant. It follows from the inequality (1.4.1) and the first inequality in (2.1.2) that the condition (2.9.1) holds for all $i \geq i_0$ if and only if it holds for i_0. Hence the complement of the set of Weierstrass points is equal to the set of points x such that the condition (2.9.1) holds for $i = 0, \ldots, i_0$. It follows from Lemma (1.4) that the condition (2.9.1) is an open condition. Therefore, the complement is open. Thus the last assertion of the Proposition holds.

(2.10) *Note.* Proposition (2.9) shows that the Weierstrass points are those with exceptional gap sequences. In the classical situation, a gap at a point x is defined as an integer n such that there are no meromorphic functions with a pole of exactly order n at x and no other poles. As is well known, and shown using the wronskian determinant, the gaps at almost all points are $1, 2, \ldots, g$ (where g is the genus of the curve), the remaining points being the Weierstrass points. It was therefore natural, as Schmidt did [35], to define the Weierstrass points of any non-singular curve as being the points with exceptional gap sequences, even when the general gap sequence is different from $1, 2, \ldots, g$. Proposition (2.9) shows that we could have used a similar definition here. It is however preferable to use our definition, because it relates the Weierstrass points to the ranks of maps of sheaves. One example that illustrates this advantage is the inequalities of Proposition (2.4). The first one of these, for $s=1$, was observed by Schmidt [35] and he suggested that the remaining would hold. These were however first proved by Matzat [30]. From our point of view these inequalities simply reflect the upper semi-continuity of ranks of sheaves. Another example of the advantage of our definition is given by the assertions of Proposition (2.5). These assertions state that the two key properties of the introduction can be formulated in terms of rank conditions.

In Section 5 we shall make frequent use of this interpretation of the key properties to prove that the most common Wronski systems have the properties.

3. Wronskians

The main reason for introducing Wronski systems is that they, in a natural way, give rise to certain maps called wronskians that generalize the classical Wronski determinants. We shall in this section recall (from [25]) the main properties of the wronskians associated to a Wronski system and show that the wronskians vanish exactly at the Weierstrass points defined in the previous section.

(3.1) Theorem. *Fix a Wronski system as in* (2.1). *Moreover, fix a sequence* $n_1, n_2, \ldots$ *such that each* n_i *is equal to* 0 *or* 1. *Set*

$$r_i := n_1 + \ldots + n_i \quad \textit{for } i = 1, 2, \ldots\ .$$

Assume for $i=1, 2, \ldots$ *that*

$$\operatorname{rk} v_i \leq r_i. \tag{3.1.1}$$

Denote by $i_1, i_2, \ldots$ *the increasing sequence of indices* i *for which* $n_i=1$. *Then*:

For every positive integer h *there is a canonical map,*

$$w_h \colon \bigwedge^{r_h} \mathcal{W} \to \mathcal{K}_{i_1} \otimes \ldots \otimes \mathcal{K}_{i_{r_h}}. \tag{3.1.2}$$

If one of the inequalities (3.1.1) *for* $i=1, \ldots, h$ *is strict, then the map* w_h *is equal to zero. Assume moreover that every map* $v_1, \ldots, v_h$ *of the Wronski system is of* *generically constant rank. Then conversely, if* w_h *is equal to zero, then one of the inequalities* (3.1.1) *for* $i=1,\dots,h$ *is strict.*

The formation of the maps w_h *of* (3.1.2) *commutes with pull-back as follows:*

For any morphism of schemes $f\colon X'\to X$*, the pull-back of the Wronski system* (2.1) *is a Wronski system on* X' *and* f^*w_h *is equal to the map* w_h *obtained from the pull-back of the system.*

Finally, assume for $i=1,\dots,h$ *that the inclusion map* $\iota_i\colon \mathcal{K}_i\to\mathcal{Q}_i$ *has a contraction* $\varrho_i\colon \mathcal{Q}_i\to\mathcal{K}_i$*. Then*

$$w_h=\bigwedge^r(\varrho_{i_1}v_{i_1},\dots,\varrho_{i_r}v_{i_r}),$$

where $r:=r_h$ *and* $(\varrho_{i_1}v_{i_1},\dots,\varrho_{i_r}v_{i_r})$ *is the map* $\mathcal{W}\to\mathcal{K}_{i_1}\oplus\dots\oplus\mathcal{K}_{i_r}$ *induced by the maps* $\varrho_{i_j}v_{i_j}$.

Proof. All the assertions of the Theorem are proved in Section 1 of [25].

(3.2) *Definition.* Fix a Wronski system as in (2.1). Define the *wronskians* of the system as the maps w_h of Theorem (3.1) corresponding to the sequence of n_i's defined by the equations,

$$\operatorname{rk} v_i=n_1+\dots+n_i \quad \text{for } i=1,2,\dots .$$

Clearly, $n_i=1$ if and only if i is a generic gap of the Wronski system. Hence the sequence $i_1,i_2,\dots$ of Theorem (3.1) is the sequence $g_1,g_2,\dots$ of generic gaps of the Wronski system. Consequently, the h'th wronskian is a map,

$$w_h\colon \bigwedge^r\mathcal{W}\to\mathcal{K}_{g_1}\otimes\dots\otimes\mathcal{K}_{g_r}, \tag{3.2.1}$$

where $r:=\operatorname{rk} v_h$. Note that the g_j's in (3.2.1) are precisely the generic gaps less than or equal to h, cf. Lemma (2.3).

It follows from the theorem that if $f\colon X'\to X$ is a morphism such that every map f^*v_i is of generically constant rank on X'—in particular, if X' is integral—then $f^*w_h=0$ if and only if at least one of the following inequalities is strict:

$$\operatorname{rk} f^*v_i\le\operatorname{rk} v_i \quad \text{for } i=1,\dots,h.$$

Thus we have partly obtained a functorial description of the zero-scheme $Z(w_h)$ of the wronskian. In particular we have obtained the following description of the points of $Z(w_h)$:

(3.3) Corollary. *A point x of X belongs to the zero scheme $Z(w_h)$ of the wronskian w_h if and only if at least one of the inequalities $\operatorname{rk} v_i(x) \leq \operatorname{rk} v_i$ for $i=1,\dots,h$ is strict, that is, if and only if x is a Weierstrass point of rank at most equal to $\operatorname{rk} v_h$.*

(3.4) *Note.* It follows from (3.3) that the Weierstrass points of the Wronski system are exactly the union of the underlying sets of the schemes $Z(w_i)$ for $i=1,2,\dots$.

Assume that every map v_i of the Wronski system is of generically constant rank. Then the gap sequence at any associated point ξ of X is equal to the generic gap sequence. Therefore, by Proposition (2.9), no associated point is a Weierstrass point. In particular, the zero schemes $Z(w_h)$ of the wronskians are proper subschemes of X.

4. Gaps of Wronski systems obtained from derivations

We shall extend the theory of higher derivations developed in [25]. In particular we shall study the properties of the Wronski systems associated to a higher derivation. The results are central to our proof of the two key properties of gaps for the Wronski systems associated to principal parts on a smooth family of curves with geometrically reduced fibers. The local, algebraic, approach presented here represents a generalization of the methods for function fields used by F. K. Schmidt [34] and [35], and continued by B. Matzat [30].

(4.1) Setup. Fix a (noetherian) ring k and a (noetherian) k-algebra A. Let D be a higher k-derivation in A. Recall, cf. [25, Section 3], or [29, Chapter 9, Theorem 27.2, p. 206], that D can be defined as a family of k-linear maps $D_i: A \to A$ satisfying the conditions,

$$D_0(f) = f, \quad D_i(fg) = \sum_{j=0}^{i} D_j(f)\, D_{i-j}(g) \quad \text{for } i = 0, 1, \dots ,$$

or, equivalently, as a map of k-algebras

$$D: A \to A[[u]],$$

whose degree zero part D_0 is the identity on A. The elements f in A such that $D_i(f)=0$ for $i=1,2,\dots$ will be called *derivation constants.* It is easily seen that the derivation constants form a k-subalgebra of A.

As in [loc. cit], the higher derivation $D=\{D_i\}$ will be called *iterative*, if

$$D_i D_j = \binom{i+j}{j} D_{i+j} \quad \text{for all } i \text{ and } j.$$

Clearly, if D is iterative, then $i!D_i=(D_1)^i$. In particular, over the field of rational numbers an iterative derivation $D=\{D_i\}$ is completely determined by the (usual) derivation D_1, and the ring of derivation constants is the kernel of D_1.

(4.2) *Definition.* Fix a finitely generated free k-module V, and a k-linear map $\gamma\colon V\to A$. Denote by $D_V\colon A\otimes V\to A[[u]]$ the A-linear map defined by $D_V(a\otimes f):=aD(\gamma f)$. Moreover, denote by

$$D_{V,i}\colon A\otimes V\to A[[u]]/(u^i)$$

the composition of the map D_V and the projection $A[[u]]\to A[[u]]/(u^i)$. Clearly, the quotient $A[[u]]/(u^i)$ is a free A-module of rank i and the following diagrams are commutative:

(4.2.1)
$$\begin{tikzcd} A\otimes V \arrow[d, "D_{V,i}"'] \arrow[dr, "D_{V,i-1}"] & \\ A[[u]]/(u^i) \arrow[r, "q_i"] & A[[u]]/(u^{i-1}), \end{tikzcd}$$

where q_i is the natural projection. Consequently, the corresponding diagrams of modules on $\operatorname{Spec} A$ form a Wronski system. The kernel of q_i is the free A-module of rank 1 generated by the image of u^{i-1} in $A[[u]]/(u^i)$. Hence, the wronskians of the system are maps,

(4.2.2)
$$w_h\colon A\otimes\bigwedge^r V\to K_{g_1}\otimes\ldots\otimes K_{g_r}=Au^{g_1+\cdots+g_r-r},$$

where r is the rank of $D_{V,h}$ and $g_1,g_2,\ldots$ is the increasing sequence of generic gaps of the system, see (3.2).

(4.3) Lemma. *The derivation D extends uniquely to any localization of A. Moreover, if $\mathfrak{q}$ is an associated prime ideal of A, and $\kappa(\mathfrak{q}):=A_\mathfrak{q}/\mathfrak{q}A_\mathfrak{q}$ is the corresponding residue class field, then D extends uniquely to $\kappa(\mathfrak{q})$.*

Proof. The first assertion follows easily by the description of D as a map of k-algebras $A\to A[[u]]$.

Therefore, to prove the second assertion, we may assume that A is a local ring with $\mathfrak{q}$ as its maximal ideal. Then, clearly, it suffices to prove that the composite map $\bar{D}\colon A\to A[[u]]\to A/\mathfrak{q}[[u]]$ vanishes on $\mathfrak{q}$.

Since $\mathfrak{q}$ is an associated prime we have that $\mathfrak{q}$ is the annihilator, $\mathfrak{q}=\operatorname{Ann} g$, of an element g in A. Let l be the integer such that all elements $D_i(g)$ are in $\mathfrak{q}^l$, but not all are in $\mathfrak{q}^{l+1}$. Such an integer exists by Krull's intersection theorem, since $D_0(g)=g$

is non-zero. Moreover, let i be the smallest index such that $D_0(g),\ldots,D_{i-1}(g)$ are in $\mathfrak{q}^{l+1}$, but $D_i(g)\notin\mathfrak{q}^{l+1}$. Assume now, by way of contradiction, that there is an element $f\in\mathfrak{q}$ such that $\overline{D}(f)\neq 0$. Then one of the elements $D_j f$ is outside $\mathfrak{q}$, and hence invertible. Let j be the smallest index such that all elements $D_0(f),\ldots,D_{j-1}(f)$ belong to $\mathfrak{q}$, but $D_j(f)$ is invertible. Consider the coefficient

$$D_{i+j}(f)D_0(g)+\ldots+D_j(f)D_i(g)+\ldots+D_0(f)D_{i+j}(g)$$

of u^{i+j} in the product $D(f)D(g)$. The coefficient is equal to zero, because $D(f)D(g)=D(fg)=D(0)=0$. On the other hand, all the terms in the above sum are in $\mathfrak{q}^{l+1}$ except the term $D_j(f)D_i(g)$ which is in $\mathfrak{q}^l\setminus\mathfrak{q}^{l+1}$. This is impossible, so we must have $\overline{D}(f)=0$ for all $f\in\mathfrak{q}$.

Thus the Lemma has been proved.

(4.4) *Definition.* Let t be an independent variable over $A[[u]]$, and denote by $D_t\colon A\to A[[t]]$ the k-derivation defined by

$$D_t(f)=\sum_{i=0}^{\infty} D_i(f)\,t^i.$$

Clearly, D induces a higher k-derivation $D^{\mathrm{coef}}\colon A[[t]]\to A[[t,u]]$ defined by

$$D^{\mathrm{coef}}\left(\sum_{i=0}^{\infty} f_i t^i\right)=\sum_{i=0}^{\infty} D(f_i)t^i=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} D_j(f_i)\,t^i u^j.$$

On the other hand, a higher A-derivation $\mathrm{Can}\colon A[[t]]\to A[[t,u]]$ is defined by substitution,

$$\mathrm{Can}\left(\sum_{i=0}^{\infty} f_i t^i\right):=\sum_{i=0}^{\infty} f_i(t+u)^i.$$

(4.5) Lemma. *The derivation D is iterative if and only if the two composite maps,*

$$A\xrightarrow{D_t}A[[t]]\xrightarrow{D^{\mathrm{coef}}}A[[t,u]]$$

and

$$A\xrightarrow{D_t}A[[t]]\xrightarrow{\mathrm{Can}}A[[t,u]],$$

are equal.

Proof. The proof is a simple computation.

(4.6) *Remark.* Assume that D is iterative. It is an easy consequence of Lemma (4.5) that the unique extension of D to the residue class field $\kappa(\mathfrak{q})$ at an associated prime $\mathfrak{q}$, cf. (4.3), is again iterative.

(4.7) Lemma. *Assume that D is iterative. Let f, $f_1, \dots, f_n$, $g_1, \dots, g_n$ be elements of A. Assume that the following equation of power series in $A[[t]]$ holds:*

$$D_t(f) = g_1 D_t(f_1) + \dots + g_n D_t(f_n). \tag{4.7.1}$$

Then the following equations of power series hold:

$$0 = D_i(g_1) D_t(f_1) + \dots + D_i(g_n) D_t(f_n) \quad \text{for all } i > 0. \tag{4.7.2}$$

Proof. It follows from Equation (4.7.1) that the following equation holds in the ring of power series $A[[t,u]]$:

$$\sum_{j=1}^{n} g_j D(f_j)(u+t) = \sum_{j=1}^{n} D(g_j) D(f_j)(u+t). \tag{4.7.3}$$

Indeed, evaluate Can and D^{coef} on the right hand side of Equation (4.7.1). By Lemma (4.5), $D^{\text{coef}} D_t = \operatorname{Can} D_t$. Hence it follows from Equation (4.7.1) that the results of the evaluations are equal. Clearly, evaluation of Can yields the left hand side of Equation (4.7.3). Evaluation of D^{coef} yields the right hand side of Equation (4.7.3), again because $D^{\text{coef}} D_t = \operatorname{Can} D_t$. Thus Equation (4.7.3) holds.

Substitute $t \mapsto t-u$ in Equation (4.7.3) to obtain the following equation in $A[[t,u]]$:

$$\sum_{j=1}^{n} g_j D_t(f_j) = \sum_{j=1}^{n} D(g_j) D_t(f_j). \tag{4.7.4}$$

The asserted Equations (4.7.2) follow by equating the coefficients of u^i for $i>0$ in Equation (4.7.4).

(4.8) Proposition. *Assume that A is an integral domain and k is a field. Let $V \subseteq A$ be a k-linear subspace of finite dimension. Then the following two conditions are equivalent:*

(i) *The map $D_V : A \otimes_k V \to A[[u]]$ is injective.*

(ii) *The map $D_{V,i} : A \otimes_k V \to A[[u]]/(u^i)$ is injective when i is sufficiently big.*

Moreover, if D is iterative and k is of characteristic zero, then the above conditions are equivalent to the following:

(iii) *The map $D_{V,i} : A \otimes_k V \to A[[u]]/(u^i)$ is injective when $i \geq \operatorname{rk} V$.*

Proof. Denote by K the field of fractions of A. Clearly, the map in (i) or (ii) is injective, if and only if the map defined similarly by replacing A by K and D by its extension to K is injective. Hence we may assume that A is a field.

Clearly, (i) and (ii) are equivalent, and (iii) implies (ii). The proof of the remaining implication can be found in the proof of [25, Lemma (3.5), pp. 142–143].

(4.9) Proposition. *Assume that A is an integral domain and k is a field. Denote by K the field of fractions of A. Then the following conditions are equivalent:*

(i) *For every finite dimensional k-linear subspace V of A, the induced map $D_V: A \otimes_k V \to A[[u]]$ is injective.*

(ii) *For every finite dimensional k-linear subspace V of A, the induced map $D_{V,i}: A \otimes_k V \to A[[u]]/(u^i)$ is injective when i is sufficiently big.*

(iii) *The map $D_A: A \otimes_k A \to A[[u]]$ is injective.*

(iv) *The map $D_K: K \otimes_k K \to K[[u]]$, defined similarly using the extension of D to K, is injective.*

Moreover, the conditions imply that the extension of D to K has k as field of derivation constants. Assume that D is iterative. Then conversely, if the extension of D to K has k as field of derivation constants, then the above conditions hold.

Proof. It follows from Proposition (4.8) that (i) and (ii) are equivalent. The equivalence of (i) and (iii) is immediate. Clearly, (iv) implies (iii). To prove the converse, assume that (iii) holds. Let z be an element in the kernel of the map in (iv). Then, since K is the field of fractions of A, there exists a non-zero element b of A such that the product of $1 \otimes b$ and z is contained in $K \otimes_k A$. Clearly, multiplication by $1 \otimes b$ on the source of the map in (iv) commutes with multiplication by Db on the target. Therefore, since multiplication by $1 \otimes b$ is an automorphism of $K \otimes_k K$, we may assume that z belongs to $K \otimes_k A$. Then z belongs to the kernel of the restricted map $K \otimes_k A \to K[[u]]$. The restricted map is injective, because the map of (iii) is injective. Hence z is equal to zero. Thus (iv) holds. Hence the conditions (i)–(iv) have been shown to be equivalent.

Assume now that the conditions hold. Denote by C the ring of derivation constants for the extension of D to K. If a belongs to C, then the image of $1 \otimes a$ under the map in (iv) is the constant power series a. Therefore, the image of $K \otimes_k C$ under the map in (iv) is of dimension 1 over K. Since the map in (iv) is injective, it follows that C is of dimension 1 over k. Hence $C = k$.

Assume finally that D is iterative and that the extension of D to K has k as field of derivation constants. By the equivalence of the conditions of the lemma, it suffices to prove for any finite dimensional k-linear subspace V of K that the map $K \otimes_k V \to K[[u]]$ is injective. The latter assertion is equivalent to the following: If $f_1, \ldots, f_n$ are linearly independent over k, then the power series $D(f_1), \ldots, D(f_n)$ are linearly independent over K.

To prove the latter assertion we proceed by induction on n. Let $f_1, \ldots, f_{n+1}$ be linearly independent over k. Assume, by way of contradiction, that the power series $D(f_1), \ldots, D(f_{n+1})$ are linearly dependent over K. By induction, the n power series $D(f_1), \ldots, D(f_n)$ are linearly independent over K. Therefore, there exists a linear relation of the form

$$D(f_{n+1}) = g_1 D(f_1) + \dots + g_n D(f_n), \tag{4.9.1}$$

where $g_1, \dots, g_n$ are elements of K. Equating the constant terms in the latter relation yields the equation

$$f_{n+1} = g_1 f_1 + \dots + g_n f_n. \tag{4.9.2}$$

Moreover, by Lemma (4.7), Equation (4.9.1) implies the following equations,

$$D_i(g_1)D(f_1) + \dots + D_i(g_n)D(f_n) = 0 \quad \text{for all } i > 0.$$

By the induction hypothesis, the latter equations imply that $D_i(g_j)=0$ for $i>0$, that is, the coefficients g_j are derivation constants. Therefore, by hypothesis, the coefficients g_j are elements of k. Thus Equation (4.9.2) implies that the elements $f_1, \dots, f_{n+1}$ are linearly k-independent, in contrast to the assumption.

Hence the Proposition has been proved.

(4.10) *Note.* It is apparent from the preceding result why the iterative derivations are so important. Only the restrictive hypothesis that the derivations are on a field limits the usefulness of the results. As the next two result show we can, in many useful situations, limit our attention to the case of fields.

(4.11) Theorem. *Let A be a k-algebra, and D an iterative k-derivation on A. Moreover, let V be a free k-submodule of A. Assume for every associated prime $\mathfrak{q}$ of A that the following two conditions hold:*

(1) *The intersection $V \cap \mathfrak{q}$ is equal to $\mathfrak{p}V$,*

(2) *The extension of D to $\kappa(\mathfrak{q})$, given in* (4.3), *has $\kappa(\mathfrak{p})$ as field of derivation constants,*

where $\mathfrak{p} := k \cap \mathfrak{q}$ is the contraction of $\mathfrak{q}$ to k. Then, the map of (4.2),

$$D_{V,i} \colon A \otimes_k V \to A[[u]]/(u^i),$$

is injective when i is sufficiently big. Moreover, if the field $\kappa(\mathfrak{q})$ is of characteristic zero for every associated prime $\mathfrak{q}$ of A, then the map $D_{V,i}$ is injective when $i \geq \operatorname{rk} V$.

Proof. The map $D_{V,i}$ is a map of free A-modules. Hence, by (1.2)(2), $D_{V,i}$ is injective if the map of fibers $D_{V,i}(\mathfrak{q})$ is injective for every associated prime $\mathfrak{q}$ of A. Let $\mathfrak{q}$ be an associated prime of A, and denote by $\mathfrak{p} := \mathfrak{q} \cap k$ the contraction. It follows from Lemma (4.3) that D extends uniquely to a derivation $D(\mathfrak{q})$ on $\kappa(\mathfrak{q})$. Clearly, the extension $D(\mathfrak{q})$ is iterative and $\kappa(\mathfrak{p})$-linear.

Consider the fibers $V(\mathfrak{p}){:=}V\otimes_k \boldsymbol{\kappa}(\mathfrak{p})$ and $A(\mathfrak{p}){:=}A\otimes_k \boldsymbol{\kappa}(\mathfrak{p})$. Let

$$\gamma(\mathfrak{q})\colon V(\mathfrak{p}) \to \boldsymbol{\kappa}(\mathfrak{q})$$

be the $\boldsymbol{\kappa}(\mathfrak{p})$-linear map defined as the composition of the map $V(\mathfrak{p}){\to}A(\mathfrak{p})$ and the canonical map $A(\mathfrak{p}){\to}\boldsymbol{\kappa}(\mathfrak{q})$. Clearly, the map $D_{V,i}(\mathfrak{q})$ is equal to the map of (4.2) obtained from the extended derivation $D(\mathfrak{q})$ and the map $\gamma(\mathfrak{q})$ above. It follows from Condition (1) of the theorem that the map $\gamma(\mathfrak{q})$ is injective. Moreover, by Condition (2) of the theorem, the derivation $D(\mathfrak{q})$ has $\boldsymbol{\kappa}(\mathfrak{p})$ as field of derivation constants. Therefore, the injectivity of $D_{V,i}(\mathfrak{q})$ when i is sufficiently big is a consequence of the last assertion of Lemma (4.9), applied to the derivation $D(\mathfrak{q})$ and the subspace $V(\mathfrak{p})$ of $\boldsymbol{\kappa}(\mathfrak{q})$. Moreover, the last assertion of the theorem follows from the part of Lemma (4.8) that asserts that (ii) implies (iii).

(4.12) *Note.* Clearly, Condition (1) of Theorem (4.11) holds for an associated prime $\mathfrak{q}$ if and only if the composite map $\gamma(\mathfrak{q})\colon V(\mathfrak{p}){\to}A(\mathfrak{p}){\to}\boldsymbol{\kappa}(\mathfrak{q})$ (defined in the proof of the theorem) is injective. In particular, Condition (1) holds if A is flat over k and the fiber $A(\mathfrak{p})$ is an integral domain. Indeed, the contraction $\mathfrak{p}$ is an associated prime of k and $\mathfrak{q}$ corresponds to an associated prime of the fiber $A(\mathfrak{p})$, because A is flat over k (see [28, p. 58]). Hence the first map $V(\mathfrak{p}){\to}A(\mathfrak{p})$ is injective by (1.2)(1) and the second map is the inclusion of an integral domain in its field of fractions. Thus the composite map $\gamma(\mathfrak{q})$ is injective.

The above theorem has a wide variety of applications in geometry. In many situations, Condition (2) is guaranteed by the geometry of the spaces involved. However, for use in the next section we will only need the following more special result.

(4.13) Proposition. *Assume that A is an integral domain of dimension* 1 *and k is a field. Let $\mathfrak{m}$ be a prime ideal of A. Assume that $\mathfrak{m}$ is k-rational, that is, a prime ideal such that the composition $k{\to}A{\to}A/\mathfrak{m}$ is an isomorphism. Denote by $a{\mapsto}\bar{a}$ the resulting map $A{\to}k$. Consider the composite map induced by the k-derivation $D\colon A{\to}A[[u]]$,*

$$\overline{D}\colon A \to A[[u]] \to k[[u]],$$

where the second map is defined by reducing the coefficients modulo $\mathfrak{m}$. Assume that there exists an element z in A such that $\overline{D}_i(z){\neq}0$ for some $i{>}0$. Then:

(1) *The map $\overline{D}$ is injective.*

(2) *For every finite dimensional k-linear subspace V of A, the map of* (4.2)*,*

$$D_{V,i}\colon A\otimes_k V \to A[[u]]/(u^i),$$

is split injective in a neighborhood of $\mathfrak{m}$ *when* i *is sufficiently big. In particular, the map* $D_{V,i}$ *is injective when* i *is sufficiently big.*

(3) *The field* k *is the field of derivation constants for the extension of* D *to the field of fractions of* A.

(4) *Assume moreover that* D *is iterative and* k *is of characteristic zero. Then* $D_{V,i}$ *is injective when* $i \geq \operatorname{rk} V$.

Proof. The assertions (1), (2) and (4) are the contents of [25, Lemma (3.5), p. 142]. By Proposition (4.9), the assertion (3) is a consequence of (2). Hence the assertions of the lemma hold.

5. Gaps of Wronski systems of principal parts

For a linear system on a smooth family of curves there is a natural Wronski system coming from the sheaves of principal parts. This particular Wronski system was used to study Weierstrass points on a smooth curve in [23] and [24] and to study Plücker type formulas for families of curves in [25]. In this section we recall the properties of the Wronski system associated to principal parts given in [25], and show that the Weierstrass points that come from this Wronski system generalize the classical concept of Weierstrass points. More importantly we shall show how the results of the previous section can be used to analyze the key properties, of the introduction, for the Wronski systems coming from principal parts of a smooth family of curves with geometrically irreducible fibers.

(5.1) Setup. Assume that X is a smooth family of curves over a base scheme S, that is, assume that there is given a smooth map $f\colon X \to S$ whose geometric fibers are curves. Fix an invertible $\mathcal{O}_X$-module $\mathcal{L}$, a locally free $\mathcal{O}_S$ module $\mathcal{V}$, and an $\mathcal{O}_S$-linear map,

$$\gamma\colon \mathcal{V} \to f_*\mathcal{L}. \tag{5.1.1}$$

Consider the associated Wronski system, see [25, (2.2.1), p. 138],

$$\begin{tikzcd} \mathcal{V}_X \arrow[d, "v_i"'] \arrow[dr, "v_{i-1}"] & \\ \mathcal{P}^{i-1}_{X/S}(\mathcal{L}) \arrow[r, "q_i"] & \mathcal{P}^{i-2}_{X/S}(\mathcal{L}). \end{tikzcd} \tag{5.1.2}$$

Recall (cf. [25] (2.1)) that $\mathcal{P}^i_{X/S}(\mathcal{L})$ is the sheaf of i'th order principal parts, defined as $p_*(\mathcal{O}_{\Delta_i} \otimes q^*\mathcal{L})$ where p and q are the two projections from $X \times X$ to X and Δ_i is the subscheme of $X \times X$ defined by the $(i+1)$'st power of the ideal $\mathcal{I}$ of the diagonal.

The restrictions of p and q to the closed subscheme Δ_i are topologically the same map. Therefore, we can identify $\mathcal{P}^i_{X/S}(\mathcal{L})$ and $q_*(\mathcal{O}_{\Delta_i}\otimes q^*\mathcal{L})$ as abelian sheaves. Hence, by adjunction, we obtain an $\mathcal{O}_S$-linear map, $d^i_{\mathcal{L}}\colon \mathcal{L}\to\mathcal{P}^i_{X/S}(\mathcal{L})$. For $\mathcal{L}:=\mathcal{O}_X$ we obtain a sheaf of algebras, $\mathcal{P}^i_{X/S}:=p_*(\mathcal{O}_{\Delta_i})$, and the adjunction map $\iota^i\colon \mathcal{O}_X\to p_*(\mathcal{O}_{\Delta_i})$ is an inclusion of algebras. Clearly, the map $d^i_X:=d^i_{\mathcal{O}_X}$ is a map of $\mathcal{O}_S$-algebras. From the above maps we define the map $\delta^i_X\colon \mathcal{O}_X\to\mathcal{P}^i_{X/S}$ as the difference $\delta^i_X:=d^i_X-\iota^i_X$.

Consider on $X\times X$ the exact sequence,

$$0\to\mathcal{I}^i\to\mathcal{O}_{X\times X}\to\mathcal{O}_{\Delta_{i-1}}\to 0. \tag{5.1.3}$$

The sequence remains exact when tensored by $q^*\mathcal{L}$, and from the long exact sequence of higher direct images of p_* we obtain an exact sequence,

$$p_*q^*\mathcal{L}\to\mathcal{P}^{i-1}_{X/S}(\mathcal{L})\to R^1p_*(\mathcal{I}^i\otimes q^*\mathcal{L})\to R^1p_*q^*\mathcal{L}\to 0, \tag{5.1.4}$$

which is exact to the right, because the restriction of p to the closed subscheme Δ_{i-1} of $X\times X$ is a (topological) homeomorphism.

The base change map $f^*f_*\mathcal{L}\to p_*q^*\mathcal{L}$ is an isomorphism, because f is flat. The vertical map v_i in (5.1.2) is the composition of the first map of (5.1.4) and the map $f^*\gamma\colon f^*\mathcal{V}\to f^*f_*\mathcal{L}=p_*q^*\mathcal{L}$. The horizontal map in (5.1.2) is the canonical surjection $\mathcal{P}^{i-1}_{X/S}(\mathcal{L})\to\mathcal{P}^{i-2}_{X/S}(\mathcal{L})$. It is well known, cf. [loc. cit.], that the canonical surjection has as kernel the invertible sheaf $(\Omega^1_{X/S})^{\otimes(i-1)}\otimes\mathcal{L}$. In particular, the sheaf $\mathcal{P}^{i-1}_{X/S}(\mathcal{L})$ is locally free of rank i. Hence, the maps in (5.1.2) define a Wronski system as in (2.1). The wronskians of the system are maps of the form,

$$w_h\colon \bigwedge^r\mathcal{V}_X\to\left(\Omega^1_{X/S}\right)^{\otimes(g_1+\dots+g_r-r)}\otimes\mathcal{L}^{\otimes r}, \tag{5.1.5}$$

where $r=\operatorname{rk}v_h$ and $g_1,\dots,g_r$ is the increasing set of generic gaps that are less than or equal to h, see (3.2). Denote by $\mathcal{E}_i(\mathcal{L})$ the cokernel of the first map in (5.1.4). Then, from the exact sequence (5.1.4), we obtain a commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_i(\mathcal{L}) & \longrightarrow & R^1p_*(\mathcal{I}^i\otimes q^*\mathcal{L}) & \longrightarrow & R^1p_*q^*\mathcal{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{E}_{i-1}(\mathcal{L}) & \longrightarrow & R^1p_*(\mathcal{I}^{i-1}\otimes q^*\mathcal{L}) & \longrightarrow & R^1p_*q^*\mathcal{L} & \longrightarrow & 0. \end{array} \tag{5.1.6}$$

Note that the $\mathcal{E}_i(\mathcal{L})$ are quotients of the cokernels $\mathcal{E}_i$ of the maps v_i of the Wronski system (5.1.2). Clearly, if the map γ of (5.1.1) is surjective, then $\mathcal{E}_i=\mathcal{E}_i(\mathcal{L})$.

(5.2) *Note.* From diagram (5.1.6) we can obtain an interpretation of gaps at a point x in terms of the poles of meromorphic functions at x similar to that given on Riemann surfaces. Indeed, assume in addition to the hypotheses of (5.1) that f is proper and $R^1 f_*(\mathcal{L})$ is locally free. Let x be a point of X. Denote by Z_x the fiber product

$$Z_x := X \times_S \boldsymbol{\kappa}(x).$$

Then Z_x is a proper, smooth curve over $\boldsymbol{\kappa}(x)$, with x as a canonical $\boldsymbol{\kappa}(x)$-rational point.

Since $\mathcal{P}^{i-1}_{X/S}$ is a locally free $\mathcal{O}_X$-module, it follows that the module $\mathcal{O}_{X\times X}/\mathcal{I}^i$ is flat over X. Hence, it follows from the exact sequence (5.1.3) that $\mathcal{I}^i$ is flat over X. Thus the principle of base change [20, Theorem 12.11, p. 290] applies to the map $p\colon X\times X \to X$ and the module $\mathcal{I}^i \otimes \mathcal{L}$. Since the relative dimension is equal to 1, we obtain isomorphisms,

$$R^1 p_*(\mathcal{I}^i \otimes q^*\mathcal{L})(x) \xrightarrow{\sim} H^1(Z_x, (\mathcal{I}^i \otimes q^*\mathcal{L})|Z_x).$$

As noted above, the module $\mathcal{O}_{X\times X}/\mathcal{I}^i$ is flat over X. Therefore, from the exact sequence (5.1.3) we obtain by restricting to Z_x the exact sequence

$$0 \to \mathcal{I}^i_{Z_x} \to \mathcal{O}_{Z_x} \to \mathcal{O}_{Z_x}/\mathcal{I}^i_{Z_x} \to 0.$$

The quotient $\mathcal{O}_{Z_x}/\mathcal{I}^i_{Z_x}$ is supported at the point x, and at x the ideal $\mathcal{I}^i_{Z_x}$ is equal to $\mathfrak{m}_x^i$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{Z_x,x}$, see [19, Corollaire (16.4.12), p. 22].

Since Z_x is smooth, it follows that the ideal $\mathcal{I}^i_{Z_x}$ defines the Cartier divisor ix, that is, $\mathcal{I}^i_{Z_x} = \mathcal{O}_{Z_x}(-ix)$. Hence we have that

$$(\mathcal{I}^i \otimes q^*\mathcal{L})|Z_x = \mathcal{L}_{Z_x}(-ix).$$

Moreover, $R^1 f_*\mathcal{L}$ is locally free by assumption, and $R^1 p_* q^*\mathcal{L} = f^* R^1 f_*\mathcal{L}$ by flat base change. Therefore, we obtain from Diagram (5.1.6) a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_i(\mathcal{L})(x) \arrow[r] \arrow[d] & H^1(Z_x, \mathcal{L}_{Z_x}(-ix)) \arrow[r] \arrow[d] & H^1(Z_x, \mathcal{L}_{Z_x}) \arrow[r] \arrow[d, equal] & 0. \\
0 \arrow[r] & \mathcal{E}_{i-1}(\mathcal{L})(x) \arrow[r] & H^1(Z_x, \mathcal{L}_{Z_x}(-(i-1)x)) \arrow[r] & H^1(Z_x, \mathcal{L}_{Z_x}) \arrow[r] & 0.
\end{tikzcd}$$

It follows from the latter diagram that the surjection,

(5.2.1) $$H^1(Z_x, \mathcal{L}_{Z_x}(-ix)) \to H^1(Z_x, \mathcal{L}_{Z_x}(-(i-1)x)),$$

is an isomorphism if and only if the surjection,

$$\mathcal{E}_i(\mathcal{L})(x) \to \mathcal{E}_{i-1}(\mathcal{L})(x),$$

is an isomorphism. As mentioned in (5.1), if the map γ of (5.1.1) is surjective, then $\mathcal{E}_i(\mathcal{L})$ is equal to the cokernel $\mathcal{E}_i$ of the map v_i of the Wronski system (5.1.2). Hence we have obtained the following result:

Proposition. *Assume that the linear map* $\gamma\colon \mathcal{V} \to f_*\mathcal{L}$ *of* (5.1.1) *is surjective. Then, for the associated Wronski system* (5.1.2) *of principal parts, the integer* i *is a gap at* x*, if and only if the map* (5.2.1) *is an isomorphism.*

As a consequence, our definition of gaps generalizes the classical definition.

(5.3) *Definition.* Consider the Wronski system (5.1.2). Denote by U the set of points x of X for which the map $v_i(x)$ is injective when i is sufficiently big.

Moreover, denote for every s in S by $\gamma(s)'$ the following $\boldsymbol{\kappa}(s)$-linear map,

$$\gamma(s)'\colon \mathcal{V}(s) \to (f_*\mathcal{L})(s) \to f_{s*}(\mathcal{L}|X_s) = H^0(X_s, \mathcal{L}|X_s), \tag{5.3.1}$$

induced by γ and base-change. It follows from [25, Lemma (4.4), p. 145] that if the linear map γ is injective, then the map $\gamma(s)'$ of (5.3.1) is injective for every associated point s of S.

The map γ will be said to define a *linear system on the family* X/S, if the map $\gamma(s)'$ is injective for all points s of S.

(5.4) *Note.* To explain the relation between the Wronski system (5.1.2) associated to principal parts and the Wronski systems of the form (4.2.1) we choose an open affine subset $\operatorname{Spec} k$ of S and an open affine subset $\operatorname{Spec} A$ of $f^{-1}\operatorname{Spec} k$ such that $\mathcal{L}$ is trivial on $\operatorname{Spec} A$ and such that there exists an element z in A whose differential dz generate $\Omega^1_{A/k}$ (see [19, IV, 16.10.6 and 6.11] or [1, Lemma 5.6, p. 150]). We let $X = \operatorname{Spec} A$ and $\zeta = \delta^i_X z = d^i_X z - \iota^i_X z$, where δ^i_X, d^i_X and ι^i_X were defined in (5.1). Recall from [25, (2.4), (2.5) and (3.4)], that $\mathcal{P}^i_{X/S}$ is a free A-module with basis $1, \zeta, \ldots, \zeta^i$. Hence we can define maps $D_j\colon A \to A$ for $j = 0, 1, 2, \ldots$ by the equation,

$$d^i_X f = D_0 f + D_1 f\zeta + \ldots + D_i f\zeta^i.$$

Moreover, recall that the map $D\colon A \to A[[u]]$ defined by $Df = \sum_{i=0}^{\infty} D_i f u^i$ is an iterative higher derivation. The Wronski system (5.1.2) becomes the Wronski system (4.2.1) associated to D when we identify the A-module $A[[u]]/(u^i)$ with $\mathcal{P}^{i-1}_{X/S}$ via the map that sends u to ζ. As a consequence of this identification the following fundamental result follows from Proposition (4.13) (see [25, Lemma (4.5), pp. 145–146]):

Proposition. *In addition to the hypotheses of* (5.1), *assume that X/S has geometrically irreducible fibers. Let s be a point of S. Assume that the map $\gamma(s)'$ of* (5.3.1) *is injective. Then the fiber $f^{-1}s$ is contained in U. Moreover, if $\kappa(s)$ is of characteristic* 0 *and ξ is the generic point of the fiber $f^{-1}s$, then the map $v_i(\xi)$ is injective when $i \geq \operatorname{rk} \mathcal{V}$.*

From the latter result we obtain the following result that shows that the two key properties of gaps referred to in the introduction hold for linear systems on a smooth family with geometrically irreducible fibers:

(5.5) Theorem. *In addition to the hypotheses of* (5.1), *assume that X/S has geometrically irreducible fibers. Assume moreover that the linear map γ is injective. Then:*

(1) *The map v_i is injective for all sufficiently big i. In particular, the number of generic gaps is equal to the rank of $\mathcal{V}$, and the set U is equal to the open set defined by any of the equivalent conditions of Proposition* (2.5). *Moreover, the set U contains the fiber over every associated point of S.*

(2) *Assume that the characteristic is equal to zero at every associated point of S. Then v_i is injective for $i \geq \operatorname{rk} \mathcal{V}$. In particular, the sequence of generic gaps is classical, that is, it is the sequence $1, 2, \ldots, \operatorname{rk} \mathcal{V}$.*

(3) *Assume that the map γ defines a linear system on X/S. Then $U = X$. In particular, the number of gaps at any point of X is equal to the rank of $\mathcal{V}$.*

Proof. Let ξ be an associated point of X. To prove the injectivity of v_i, asserted in (1) for i sufficiently big and in (2) for $i \geq \operatorname{rk} \mathcal{V}$, it suffices by Remark (1.2) to prove that the map $v_i(\xi)$ is injective. Set $s := f\xi$. Then s is an associated point of S and ξ is the generic point of its fiber because f is flat (see [28, Corollary, p. 58]). It follows from what we said in (5.3) that the map $\gamma(s)'$ is injective. Therefore, the asserted injectivity follows from the Proposition of Note (5.4).

By Proposition (2.5), the second assertion of (1) is a consequence of the first. The last assertion of (1) follows from the Proposition of Note (5.4).

We proved above the injectivity asserted in (2). Clearly, the second assertion of (2) is a consequence.

The first assertion of (3) follows immediately from the Proposition of Note (5.4). The second assertion follows from (1) and Proposition (2.5).

Thus the theorem has been proved.

6. Wronski systems on Gorenstein curves

For families of curves with singular fibers the relative principal parts of Section 5 are not locally free and therefore do not give Wronski systems as in (2.1). In this section we shall show that for a single Gorenstein curve there is a natural replacement of the principal parts, and we obtain natural Wronski systems. We take here as a definition of a Gorenstein curve that the module of regular (meromorphic) differentials is invertible. Our theory can be applied to families with Gorenstein fibers that have a simultaneous resolution of singularities by a smooth family of curves. However, for simplicity, we shall treat the case of a curve defined over a perfect field only. The wronskians obtained from our Wronski systems are globally defined, and they generalize to arbitrary characteristics the theory of R. Lax and C. Widland [39] for Gorenstein curves in characteristic zero. A similar globalization of the wronskians was obtained by L. Gatto [17] in characteristic zero.

(6.1) The results of Section 5 apply in particular to the case of a smooth curve Y over a field k. So, if $\mathcal{L}$ is an invertible module on Y, and V is a finite dimensional k-subspace of $H^0(Y,\mathcal{L})$ there is a Wronski system,

(6.1.1)
$$\begin{tikzcd} V_Y \arrow[d, "v_i"'] \arrow[dr, "v_{i-1}"] & \\ \mathcal{P}^{i-1}_{Y/k}(\mathcal{L}) \arrow[r, "q_i"] & \mathcal{P}^{i-2}_{Y/k}(\mathcal{L}). \end{tikzcd}$$

The wronskians of the system are maps of the form,

$$w_h\colon \bigwedge^r V_Y \to (\Omega^1_{Y/k})^{\otimes(g_1+\dots+g_r-r)} \otimes \mathcal{L}^{\otimes r}, \tag{6.1.5}$$

where $r=\operatorname{rk} v_h$ and $g_1,\dots,g_r$ is the increasing set of generic gaps that are less than or equal to h. Note that the Proposition of (5.2) applies when Y is proper and the linear system defined by V is complete, that is, when $V=H^0(Y,\mathcal{L})$. Moreover, when Y is geometrically irreducible, theorem (5.5) applies; in particular, the number of gaps at any point of Y is equal to the number of generic gaps.

Let K be the field of rational functions on Y. Let q be a closed point of Y, and let z be a rational function on Y such that the differential dz generates $\Omega^1_{Y/k}$ at q; for instance z could be a function of order 1 at q. Then dz trivializes $\Omega^1_{Y/k}$ in a neighborhood of q. In such a neighborhood, the sheaf of algebras $\mathcal{P}^n_{Y/k}$ is free, with a basis given by the powers $1,\delta z,(\delta z)^2,\dots,(\delta z)^n$, where $\delta z=\delta^n_Y z$ is defined in (5.4).

(6.2) Consider now an integral algebraic curve X defined over a perfect field k. Let $\pi\colon Y\to X$ be the normalization. Then Y is smooth over k, and π is an isomorphism outside the finitely many singular points of X. Let K be the common field of rational functions of X and Y. Consider the vector spaces over K of meromorphic differentials $\Omega^1_{K/k}$ and meromorphic jets (or principal parts) $P^n_{K/k}$. The latter is of dimension $n+1$ over K, generated as in (6.1) by the powers $1, \delta z, \dots, (\delta z)^n$.

As defined by Serre [37, p. 76], a meromorphic differential $\omega \in \Omega^1_{K/k}$ is said to be *regular* at a closed point p in X, if

$$\sum_{q \in \pi^{-1}p} \operatorname{Res}_q(f\omega) = 0 \quad \text{for all } f \in \mathcal{O}_{X,p}.$$

The residue is that of Tate, see [1, p. 171]. The residue $\operatorname{Res}_q(\omega)$ vanishes at all closed points q of Y at which ω is regular. In particular, if ω is regular at all points in the fiber $\pi^{-1}p$ of closed point p of X, then it is regular at p. Moreover, if ω is regular at p, then it is regular at all closed points in a neighborhood of p. We denote by $\widetilde{\Omega}_X$ the $\mathcal{O}_X$-module of regular meromorphic differentials. Thus, over an open subset U of X, the sections of $\widetilde{\Omega}_X$ are the meromorphic differentials that are regular at all closed points of U. Note that the module $\widetilde{\Omega}_X$ contains the direct image $\pi_*\Omega^1_{Y/k}$ as a submodule.

(6.3) *Definition.* A non-zero function t of K will be called a *parameter* on an open subset U of X if t is regular at all points of $\pi^{-1}U$ and such that the differential dt is a basis for $\Omega^1_{\pi^{-1}U/k}$ as an $\mathcal{O}_{\pi^{-1}U}$-module. If p is a closed point of X, then there exists a function t which is a parameter on some open neighborhood of p in X. Indeed, by the approximation lemma for valuations, there exists a function t which is of order 1 at all points q in the fiber $\pi^{-1}p$. Then the differential dt generates the stalk of $\Omega^1_{Y/k}$ at all points q of $\pi^{-1}p$. Therefore, the differential dt generates $\Omega^1_{Y/k}$ in some open subset V of Y containing the fiber $\pi^{-1}p$. Since the map $\pi: Y \to X$ is closed, there is an open neighborhood U of p in X such that $\pi^{-1}U \subseteq V$. Then, clearly, t is a parameter on U.

Now, let p be a closed point of X. Consider non-zero functions t and g of K such that t is a parameter in a neighborhood of p and such that the meromorphic differential dt/g is regular at p. Fix n and form the meromorphic jet in $P^n_{K/k}$:

$$\delta t/g, \tag{6.3.1}$$

where $\delta t = \delta^n_K t$ as defined in (5.1). A meromorphic jet in $P^n_{K/k}$ will be called *regular* at p, if it belongs to the $\mathcal{O}_{X,p}$-subalgebra of $P^n_{K/k}$ generated by all jets of the form (6.3.1). Clearly, if a meromorphic jet is regular at a point p, then it is regular at all points in a neighborhood of p. We denote by $\widetilde{\mathcal{P}}^n_X$ the $\mathcal{O}_X$-module of regular (meromorphic) jets. Thus, over an open subset U of X, the sections of $\widetilde{\mathcal{P}}^n_X$ are the meromorphic jets that are regular at all closed points of U. Obviously, the surjection $P^n_{K/k} \to P^{n-1}_{K/k}$ induces a surjection $\widetilde{\mathcal{P}}^n_X \to \widetilde{\mathcal{P}}^{n-1}_X$. By definition we have that $\widetilde{\mathcal{P}}^0_X = \mathcal{O}_X$.

Assume from now that the $\mathcal{O}_X$-module $\widetilde{\Omega}_X$ of regular (meromorphic) differentials is invertible.

(6.4) Proposition. *Let z be a parameter on an open subset U of X, and let h be any non-zero function. Assume that the meromorphic differential dz/h is a generator for the restriction to U of the $\mathcal{O}_X$-module $\widetilde{\Omega}^1_X$ of regular differentials. Then the jets in $P^n_{K/k}$,*

$$1, (\delta z/h), ..., (\delta z/h)^n, \tag{6.4.1}$$

form a free basis for the restriction of $\widetilde{P}^n_X$ to U. Moreover, if f is a function in $\Gamma(U, \mathcal{O}_X)$, then the total differential $d^n f$ is regular on U.

Proof. Since z is a parameter on U, the jets $1, \delta z, ..., (\delta z)^n$ form a basis for $\mathcal{P}^n_{\pi^{-1}U/k}$, see (5.4). Consequently, the latter jets, and hence also the jets (6.4.1), form a K-basis for $P^n_{K/k}$. Thus we have to prove that, if a given meromorphic jet is regular on U, resp. is of the form $d^n f$, then, when expanded in terms of the basis (6.4.1), the coefficients belong to $\Gamma(U, \mathcal{O}_X)$.

The question is local on U. Therefore, we may assume that $U = \operatorname{Spec} A$ is affine. Moreover, to prove the assertion for regular jets, it suffices to consider a jet of the form (6.3.1), where t is a parameter on U and dt/g is regular on U.

The preimage $\pi^{-1}U$ is equal to $\operatorname{Spec} B$, where B is the integral closure of A in K. We note first that h belongs to the conductor of B/A. Indeed, for any $b \in B$, the differential bdz is regular on U. Hence there is an equation, $bdz = adz/h$ for some $a \in A$. Thus $hb = a \in A$, and consequently h belongs to the conductor.

Consider first the jet $\delta t/g$ of (6.3.1). The differential dt/g is assumed to be regular on U. Hence there is an equation,

$$dt/g = adz/h \quad \text{where } a \in A. \tag{6.4.2}$$

As t is a parameter on U, the differential dt is a generator for the B-module $\Omega^1_{B/k}$. It follows from Equation (6.4.2) that $dz = h/(ga)\, dt$. Hence the function $h/(ga)$ belongs to B. In particular, the function h/g belongs to B.

Now, in $P^n_{B/k}$ we have an expansion,

$$\delta t = b_1 \delta z + b_2 (\delta z)^2 + ... + b_n (\delta z)^n, \tag{6.4.3}$$

where the coefficients b_i belong to B (in the notion of (5.4) we have that $b_i = D_i(t)$, where the $\{D_i\}$ is the higher derivation associated to z). The coefficient b_1 is determined by the equation $dt = b_1 dz$, and hence it follows from (6.4.2) that $b_1 = ga/h$. From the expansion (6.4.3) and the determination of b_1, we obtain the expansion,

$$\delta t/g = a(\delta z/h) + (b_2 h^2/g)(\delta z/h)^2 + ... + (b_n h^n/g)(\delta z/h)^n.$$

The coefficient a was an element of A. Moreover, the coefficient $b_i h^i/g$ for $i \geq 2$ is equal to $h^{i-2}hb_i(h/g)$. As observed above, the function h/g belongs to B. Therefore, the coefficient belongs to A, because h was in the conductor of B/A. Hence, in the expansion of $\delta t/g$ in terms of the basis (6.4.1), all the coefficients belong to A.

Consider similarly the expansion of $d^n f$. In $P^n_{B/k}$ we have the expansion,

$$d^n f = f + b_1 \delta z + \ldots + b_n (\delta z)^n,$$

where the coefficients b_i belong to B. From the latter expansion we obtain the expansion in the basis (6.4.1),

$$d^n f = f + (hb_1)(\delta z/h) + \ldots + (h^n b_n)(\delta z/h)^n.$$

Since f is in A and h is in the conductor of B/A, the coefficients belong to A. Therefore, the total differential $d^n f$ belongs to the A-algebra generated by $\delta z/h$. In particular, $d^n f$ is regular on U.

Hence the proposition has been proved.

(6.5) The definitions and results above extend to the case of principal parts twisted by a given invertible $\mathcal{O}_X$-module $\mathcal{L}$.

Let L denote the space of meromorphic sections of $\mathcal{L}$. Then L is of dimension 1 over K, and $P^n_{K/k}(L)$ is the space of meromorphic sections of $\mathcal{P}^n_{X/k}(\mathcal{L})$.

Recall that if a section s of $\mathcal{L}$ over U trivializes $\mathcal{L}$ (as an $\mathcal{O}_U$-module), then the total differential $d^n s = d^n_{\mathcal{L}} s$ of (5.1) trivializes $\mathcal{P}^n_{U/k}(\mathcal{L})$ as a $\mathcal{P}^n_{U/k}$-module. Consequently, if a local section s of $\mathcal{L}$ generates $\mathcal{L}$ in a neighborhood of p, then every meromorphic jet ω in $P^n_{K/k}(L)$ is of the form $\varphi d^n s$, with a unique jet φ in $P^n_{K/k}$. Call a meromorphic jet ω of $P^n_{K/k}(L)$ *regular* at the closed point p, if the corresponding jet φ of $P^n_{K/k}$ is regular at p. As $d^n(fs) = d^n f d^n s$, it follows from the last part of Proposition (6.4) that the notion of regularity is independent of the choice of s. Denote by $\widetilde{\mathcal{P}}^n_X(\mathcal{L})$ the $\mathcal{O}_X$-module of regular jets of $P^n_{K/k}(L)$. Thus, over an open subset U of X, the sections of $\widetilde{\mathcal{P}}^n_X(\mathcal{L})$ are the meromorphic jets that are regular at all closed points of U.

Then, from Proposition (6.4) we obtain the following:

(6.6) Corollary. *The $\mathcal{O}_X$-module $\widetilde{\mathcal{P}}^n_X(\mathcal{L})$ is locally free of rank $n+1$, and there are exact sequences,*

$$0 \to \widetilde{\Omega}^{\otimes n}_X \otimes \mathcal{L} \to \widetilde{\mathcal{P}}^n_X(\mathcal{L}) \to \widetilde{\mathcal{P}}^{n-1}_X(\mathcal{L}) \to 0.$$

Moreover, the total differential $d^n_{\mathcal{L}}$ induces a k-linear map of sheaves $d^n\colon \mathcal{L}\to\widetilde{\mathcal{P}}^n_X(\mathcal{L})$.

(6.7) Assume now that a finite dimensional k-subspace V of $H^0(X,\mathcal{L})$ is given. Then we obtain $\mathcal{O}_X$-linear maps v_i fitting into commutative diagrams,

(6.7.1)
$$\begin{tikzcd} V_X \arrow[d, "v_i"'] \arrow[dr, "v_{i-1}"] & \\ \widetilde{\mathcal{P}}^{i-1}_X(\mathcal{L}) \arrow[r] & \widetilde{\mathcal{P}}^{i-2}_X(\mathcal{L}). \end{tikzcd}$$

It follows from Corollary (6.6) that the system is a Wronski system. The wronskians of the system are maps of the form,

$$w_h\colon \bigwedge^r V_X \to \mathcal{L}^{\otimes r}\otimes\widetilde{\Omega}_X^{\otimes(i_1+\ldots+i_r-r)},$$

where $r=\operatorname{rk} v_h$ and $i_1,\ldots,i_r$ is the increasing set of generic gaps that are less than or equal to h.

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