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Characterization of removable sets in strongly pseudoconvex boundaries

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I. Introduction

Let M be a complex-analytic manifold(1) of complex dimension $n\geq 2$.

Given an open domain $D\subset\subset M$, a proper closed subset K of the boundary bD of D is called *removable* in case $bD\setminus K$ is C^1-smooth and every continuous CR-function f on $bD\setminus K$ has a continuous extension F to $\overline{D}\setminus K$ which is holomorphic on D.

A general account of the subject of removable sets is given in [19], where one can also find most of the related references.

One main result in this area is the following characterization of removable sets in the two-dimensional case (see [19; II.10]):

Theorem 0. *Let M be a Stein manifold of dimension two and $D\subset\subset M$ a C^2-bounded strongly pseudoconvex domain such that $\overline{D}$ is $\mathcal{O}(M)$-convex. Then for a proper closed subset K of bD the following two conditions are equivalent:*

(a) K is removable;

(b) K is $\mathcal{O}(M)$-convex.

This connection, in dimension two, between removability and $\mathcal{O}(M)$-convexity is of considerable interest, and has been recently used by Forstnerič and Stout [6] to exhibit some new instances of polynomially convex sets in $\mathbf{C}^2$.

On the other hand, Theorem 0 does not hold for general $n\geq 2$ (it being only true, for $n\geq 3$, that $(b)\Rightarrow(a)$) and a result of this kind valid for $n\geq 2$ seems to be still unknown, due perhaps to the fact that the known proof of Theorem 0 depends on a version of a theorem of Słodkowski [18; Theorem 2.1] on two-dimensional

(1) Throughout the paper manifolds are assumed to be connected and with countable topology.

pseudoconvex domains (see also [16]), of which no suitable extensions to higher dimensions have presented themselves so far.(2)

This paper is devoted to fill this gap by providing a characterization of removable sets parallel to that of Theorem 0, valid for general $n \geq 2$. In fact we shall prove here the following theorem:

Theorem 1. *Let M be a Stein manifold of dimension $n \geq 2$ and $D \subset\subset M$ a C^2-bounded strongly pseudoconvex domain such that $\overline{D}$ is $\mathcal{O}(M)$-convex. Then for a proper closed subset K of bD the following two conditions are equivalent:*

(A) K is removable;

(B) The restriction map $Z_{\bar{\partial}}^{n,n-2}(M) \to Z_{\bar{\partial}}^{n,n-2}(K)$ has dense image, moreover $H_{\bar{\partial}}^{n,n-1}(K)=0$.

Remarks. I.1. In condition (B) it is assumed that $Z_{\bar{\partial}}^{n,n-2}(K)$ is equipped with the standard locally convex inductive limit topology derived from the inductive system of the Fréchet–Schwartz spaces $Z_{\bar{\partial}}^{n,n-2}(U)$, as U ranges through a fundamental system of open neighbourhoods of K in M. Since the canonical projections $Z_{\bar{\partial}}^{n,n-2}(M) \to H_{\bar{\partial}}^{n,n-2}(M)$ and $Z_{\bar{\partial}}^{n,n-2}(K) \to H_{\bar{\partial}}^{n,n-2}(K)$ are surjective topological homomorphisms, hence continuous and open maps, it follows that the restriction map $Z_{\bar{\partial}}^{n,n-2}(M) \to Z_{\bar{\partial}}^{n,n-2}(K)$ has dense image if and only if the same is true of the induced map $H_{\bar{\partial}}^{n,n-2}(M) \to H_{\bar{\partial}}^{n,n-2}(K)$. Therefore for $n \geq 3$, since M being Stein implies that $H_{\bar{\partial}}^{n,n-2}(M)=0$, condition (B) amounts to having ${}^{\sigma}H_{\bar{\partial}}^{n,n-2}(K)=0$ and $H_{\bar{\partial}}^{n,n-1}(K)=0$, where the suffix σ means the associated separated space, i.e. the quotient space by the closure of zero. Furthermore, as the Dolbeault isomorphisms are topological isomorphisms (see [2, or 3]), condition (B) can also be translated in terms of Čech cohomology, by saying that the restriction map $H^{n-2}(M; \Omega^n) \to H^{n-2}(K; \Omega^n)$ has dense image and $H^{n-1}(K; \Omega^n)=0$, where Ω^n denotes the sheaf of germs of holomorphic n-forms on M.

I.2. Since M is Stein, it is possible to find finitely many functions $f_1, \ldots, f_r \in \mathcal{O}(M)$, $(r \leq 2n+1)$ whose differentials generate the holomorphic cotangent space of M at every point. It follows that an exact sequence $0 \to \mathcal{R} \to \mathcal{O}^r \to \Omega^n \to 0$ holds on M, with $\mathcal{R}$ being a locally free sheaf of $\mathcal{O}$-modules on M of rank $r-1$, and hence isomorphisms of sheaves $\mathcal{O}^r \cong \mathcal{R} \oplus \Omega^n$ and $(\Omega^n)^r \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{R}, \Omega^n) \oplus \mathcal{O}$ hold on M. Moreover, since K is compact, if $\mathcal{S}$ is a coherent analytic sheaf on M, it is known that a Runge open set $O \subset\subset M$ containing K and a positive integer s can be found,

(2) A generalization of Słodkowski's theorem to the context of a Stein manifold of any dimension $n \geq 2$ can be found in [14], but it is one which does not serve the purpose of finding a n-dimensional generalization of Theorem 0.

such that an exact sequence $\mathcal{O}^s \to \mathcal{S} \to 0$ holds on O. Using these facts one can check that condition (B) is also equivalent to either of the following two conditions:

(B_1) The restriction map $H^{n-2}(M;\mathcal{O}) \to H^{n-2}(K;\mathcal{O})$ has dense image and $H^{n-1}(K;\mathcal{O})=0$;

(B_2) For every coherent analytic sheaf, $\mathcal{S}$, on M, the restriction map $H^{n-2}(M;\mathcal{S}) \to H^{n-2}(K;\mathcal{S})$ has dense image and $H^{n-1}(K;\mathcal{S})=0$.

In particular, going back to differential forms, one infers that, since M is Stein, condition (B) is equivalent to the parallel condition in which $(n, n-2)$-forms and $(n, n-1)$-forms are replaced by $(0, n-2)$-forms and $(0, n-1)$-forms, respectively. The reason why we have stated the condition in terms of $(n, n-2)$-forms and $(n, n-1)$-forms, rather than in terms of $(0, n-2)$-forms and $(0, n-1)$-forms, is that doing so makes the condition still sufficient for removability in a more general setting where the ambient manifold is no longer Stein (see Section V below).

I.3. A quite general situation in which condition (B) holds is when K is $(n-2)$-*convex in* M in the following sense (see [11, 13]): given arbitrarily an open neighbourhood U of K in M, one can find a $\mathcal{C}^\infty$ strongly $(n-2)$-plurisubharmonic proper function $u: M \to \mathbf{R}$ such that $K \subset \{z \in M : u(z) < 0\} \subset\subset U$. For $n \geq 3$ we do not know if the converse is true too, i.e. if condition (B) implies that K is $(n-2)$-convex in M. On the contrary, for $n=2$ this is the case, as follows from the fact that a compact subset K of a two-dimensional Stein manifold, such that $H^1(K;\mathcal{O})=0$, is holomorphically convex, in the sense that the evaluation map $K \to \mathrm{sp}(\mathcal{O}(K))$ is bijective (see [5] and [10], taking into account also the fact that the vanishing of $H^2(K;\mathcal{O})$ is automatic(3)).

In view of the preceding remarks it is plain that for $n=2$ condition (B) means exactly that K should be $\mathcal{O}(M)$-convex, and hence for $n=2$ Theorem 1 reduces to Theorem 0.

On the other hand for $n \geq 3$ the following improvement of Theorem 1 is valid:

Corollary 1. *For $n \geq 3$ Theorem 1 is true for every $\mathcal{C}^2$-bounded strongly pseudoconvex domain $D \subset\subset M$, i.e. without the assumption that $\overline{D}$ should be $\mathcal{O}(M)$-convex. Moreover conditions (A) and (B) are also equivalent to the following further condition:*

$$H^{0,1}_{\bar\partial}(M \setminus K) = 0.$$

(3) Indeed, as every non-compact n-dimensional complex-analytic manifold X is $(n-1)$-complete (see [7]) and so $H^n(X;\mathcal{S})=0$, for every coherent analytic sheaf, $\mathcal{S}$, on X, if $K \subset X$ is any compact set, an inductive limit consideration gives at once that $H^n(K;\mathcal{S})=0$.

We shall also establish in this paper another result, in the spirit of Theorem 1, to the effect of characterizing the removable singularities of a suitable subclass of CR-functions.

Given D and K as at the beginning, call K *weakly removable* in case the following holds: if f is a continuous CR-function on $bD\setminus K$ that satisfies the moment condition

$$\int_{bD\setminus K} f\omega = 0, \tag{I.4}$$

for every $\mathcal{C}^\infty$ $\bar\partial$-closed $(n,n-1)$-form, ω, defined on a neighbourhood of $\bar D$ and such that $(\operatorname{supp}(\omega))\cap K$ is empty, then f has a continuous extension F to $\bar D\setminus K$ which is holomorphic on D.

Clearly, if K is removable, it is weakly removable too; however the converse statement is false. Indeed, by [15; Theorem 2], if $D\subset\subset \mathbf{C}^n$ is a $\mathcal{C}^2$-bounded domain, a sufficient condition in order that a proper closed subset K of bD may be weakly removable is that the $(2n-2)$-dimensional Hausdorff measure of K should be zero; however it is easy to exhibit examples showing that the same condition is not sufficient for K to be removable.

The characterization of weakly removable sets is as follows:

Theorem 2. *Let M be a Stein manifold of dimension $n\geq 2$ and $D\subset\subset M$ a $\mathcal{C}^2$-bounded strongly pseudoconvex domain. Then for a proper closed subset K of bD the following two conditions are equivalent:*

(α) *K is weakly removable;*

(β) $H^{n,n-1}_{\bar\partial}(K)=0$.

Note that, in view of the remarks after the statement of Theorem 1, condition (β) is equivalent to having $H^{n-1}(K;\mathcal{O})=0$ and also to having $H^{n-1}(K;\mathcal{S})=0$ for every coherent analytic sheaf, $\mathcal{S}$, on M; moreover Theorem 2 implies the following new result in dimension two:

Corollary 2. *Let M be a Stein manifold of dimension two and $D\subset\subset M$ a $\mathcal{C}^2$-bounded strongly pseudoconvex domain. Then for a proper closed subset K of bD the following two conditions are equivalent:*

(α) *K is weakly removable;*

(β) *K is holomorphically convex.*(4)

(4) I am indebted to E. M. Chirka who drew my attention to a slip in my original formulation of this corollary.

The known proof of Theorem 0 is function-theoretic in nature; on the contrary the proofs of Theorem 1 and of Theorem 2 presented here are mainly based on cohomological methods.

We shall expose first the proof of Theorem 2 and then those of Theorem 1 and of Corollary 1.

Later we shall state and prove two further theorems, to the effect of extending the validity of the implication $(B)\Rightarrow(A)$ of Theorem 1 and of the implication $(\beta) \Rightarrow (\alpha)$ of Theorem 2 to more general settings.

II. Proof of Theorem 2

Consider the cohomology space $H^{0,1}_{\bar\partial}(M\setminus K)$ endowed with the standard topology. We shall prove that conditions (α) and (β) are both equivalent to the following topological condition:

(γ) *The space $H^{0,1}_{\bar\partial}(M\setminus K)$ is separated, i.e. the zero element is closed in it.*

We first prove that $(\beta)\Leftrightarrow(\gamma)$. By the refined version [3; VII.4.1] of the Serre duality theorem, (γ) is equivalent to the condition that the cohomology space with compact supports $H^n_c(M\setminus K;\Omega^n)$ should be separated. Now, since M is Stein, it is known that $H^{n-1}_c(M;\Omega^n)=0$ and $H^n_c(M;\Omega^n)$ is separated (see [3; VII.4.4]); hence the exact sequence with compact supports

$$\dots \to 0 \to H^{n-1}(K;\Omega^n) \xrightarrow{\delta} H^n_c(M\setminus K;\Omega^n) \xrightarrow{i_*} H^n_c(M;\Omega^n) \to 0$$

implies at once, by the continuity of the map i_* induced by inclusion, that if (β) holds, $H^n_c(M\setminus K;\Omega^n)$ is separated. Conversely, if $H^n_c(M\setminus K;\Omega^n)$ is separated, then, by [3; VII.4.1] again, i_* turns out to coincide, up to topological isomorphisms, with the transpose of the restriction map $\varrho\colon \mathcal{O}(M)\to\mathcal{O}(M\setminus K)$. The latter map is bijective, by the connectedness of $M\setminus K$ and the Hartogs theorem, and consequently it is a topological isomorphism, since the source space and the target space are Fréchet (see [8; p.162]). It follows that i_* is a topological isomorphism too, which implies that $H^{n-1}(K;\Omega^n)=0$. Thus the equivalence $(\beta)\Leftrightarrow(\gamma)$ is proved.

The proof that $(\alpha)\Leftrightarrow(\gamma)$ requires more effort.

If X is any complex-analytic manifold of dimension n, we denote, as usual, by $\mathcal{D}'(X)$ the double complex of currents on X, endowed with the standard Fréchet–Schwartz topology. Then the cohomology bigraded space $H_{\bar\partial}(X)$ can be regarded, as a topological vector space, as the $\bar\partial$-cohomology space derived from the complex $\mathcal{D}'(X)$.

In particular throughout the continuation we shall regard $H^{0,1}_{\bar\partial}(M\setminus K)$ as the space $H^{0,1}_{\bar\partial}(\mathcal{D}'(M\setminus K))$.

We shall need the following fact:

II.1. *Let* $T\in\mathcal{D}'^{p,q}(X)$ $(0\le p,q\le n)$; *then the following two properties are equivalent:*

(i) $T(\omega)=0$ *for every* $\mathcal{C}^\infty$ $\bar\partial$*-closed* $(n-p,n-q)$*-form,* ω*, on* X *with compact support*;

(ii) $\bar\partial T=0$ *and the class of* T *in* $H^{p,q}_{\bar\partial}(X)$ *is in the closure of zero.*

Indeed (II.1) amounts to saying that the left kernel of the bilinear map

$$H^{p,q}_{\bar\partial}(X)\times H^{n-p,n-q}_{\bar\partial}(A_c(X))\to\mathbf{C}$$

induced by the natural pairing $<T,\omega>=T(\omega)$ of a current and a compactly supported form coincides with the closure of zero in $H^{p,q}_{\bar\partial}(X)$. This follows from the version [3; VII.4.2] of the Serre duality theorem.

Now let us consider the family of all the closed subsets of $M\setminus K$ which are relatively compact in M. This is a paracompactifying family in $M\setminus K$ (see [4]) and we denote it by Φ. Let $\mathcal{D}'_\Phi(M\setminus K)$ denote the subcomplex of $\mathcal{D}'(M\setminus K)$ of the currents on $M\setminus K$ whose supports belong to Φ, and consider the $\bar\partial$-cohomology space $H^{0,1}_{\bar\partial}(\mathcal{D}'_\Phi(M\setminus K))$. We shall use the following fact:

II.2. *The linear map* $i_*\colon H^{0,1}_{\bar\partial}(\mathcal{D}'_\Phi(M\setminus K))\to H^{0,1}_{\bar\partial}(M\setminus K)$ *induced by inclusion is injective.*

As a matter of fact, consider the short exact sequence

$$0\to\mathcal{D}'_\Phi(M\setminus K)\xrightarrow{i}\mathcal{D}'(M\setminus K)\xrightarrow{\pi}\frac{\mathcal{D}'(M\setminus K)}{\mathcal{D}'_\Phi(M\setminus K)}\to 0$$

and the induced long sequence of $\bar\partial$-cohomology

$$0\to Z^{0,0}_{\bar\partial}(\mathcal{D}'_\Phi(M\setminus K))\xrightarrow{i}Z^{0,0}_{\bar\partial}(\mathcal{D}'(M\setminus K))\xrightarrow{\pi}Z^{0,0}_{\bar\partial}\left(\frac{\mathcal{D}'(M\setminus K)}{\mathcal{D}'_\Phi(M\setminus K)}\right)\xrightarrow{\delta}$$
$$H^{0,1}_{\bar\partial}(\mathcal{D}'_\Phi(M\setminus K))\xrightarrow{i_*}H^{0,1}_{\bar\partial}(M\setminus K)\to\dots\,.$$

The regularity theorem for $\bar\partial$, the connectedness of $M\setminus K$ and the Hartogs theorem imply that $Z^{0,0}_{\bar\partial}(\mathcal{D}'_\Phi(M\setminus K))=0$ and $Z^{0,0}_{\bar\partial}(\mathcal{D}'(M\setminus K))=\mathcal{O}(M\setminus K)=\mathcal{O}(M)$. Moreover by the same reasons one has

$$Z^{0,0}_{\bar\partial}\left(\frac{\mathcal{D}'(M\setminus K)}{\mathcal{D}'_\Phi(M\setminus K)}\right)=\frac{\mathcal{O}(M)+\mathcal{D}'^{\,0,0}_\Phi(M\setminus K)}{\mathcal{D}'^{\,0,0}_\Phi(M\setminus K)}\cong\mathcal{O}(M).$$

Indeed, if $T\in\mathcal{D}'^{0,0}(M\setminus K)$ and $\bar\partial T\in\mathcal{D}'^{0,1}_{\Phi}(M\setminus K)$, it follows that there exists $f\in\mathcal{O}(M)$ such that $T-f\in\mathcal{D}'^{0,0}_{\Phi}(M\setminus K)$. Therefore in the preceding $\bar\partial$-cohomology sequence the projection π is bijective, which implies the validity of (II.2).

Next we prove the implication $(\gamma)\Rightarrow(\alpha)$. Thus consider a CR-function f on $bD\setminus K$ that satisfies the moment condition (I.4). Hence we have in particular $\int_{bD} f\omega=0$ for every $\mathcal{C}^\infty$ $\bar\partial$-closed $(n,n-1)$-form, ω, on $M\setminus K$ with compact support.

Since bD is strictly Levi-convex, the local extension property of CR-functions is valid at each point of bD; therefore it is possible to define a $\mathcal{C}^\infty$ function $\tilde f\colon D\to\mathbf{C}$ with the following property: for each point $z\in bD\setminus K$ there exists an open neighbourhood J of z in $M\setminus K$ such that $\tilde f|_{D\cap J}$ is holomorphic and has continuous boundary values f on $bD\cap J$. Then consider the $\mathcal{C}^\infty$ $\bar\partial$-closed $(0,1)$-form η on $M\setminus K$, supported in Φ, given by

$$\eta=\bar\partial\tilde f \text{ on } D,\quad \eta=0 \text{ on } (M\setminus K)\setminus D,$$

and let $T_\eta\in\mathcal{D}'^{0,1}_{\Phi}(M\setminus K)$ denote the current defined by η. For every ω as above one has

$$T_\eta(\omega)=\int_{M\setminus K}\eta\wedge\omega=\int_D\bar\partial\tilde f\wedge\omega=\int_{bD}f\omega=0;$$

hence, by (II.1) and the assumption that (γ) holds, the class of T_η in $H^{0,1}_{\bar\partial}(M\setminus K)$ is the zero class. Then, by (II.2), also the class of T_η in $H^{0,1}_{\bar\partial}(\mathcal{D}'^{0,1}_{\Phi}(M\setminus K))$ is the zero class and therefore there exists a distribution $U\in\mathcal{D}'^{0,0}_{\Phi}(M\setminus K)$ such that $\bar\partial U=T_\eta$.

Now, since $T_\eta=0$ on a neighbourhood of $(M\setminus K)\setminus D$, the regularity theorem for $\bar\partial$ implies that U coincides, on a neighbourhood of $(M\setminus K)\setminus D$, with a holomorphic function; moreover, since $\operatorname{supp}(U)\in\Phi$ and $M\setminus\bar D$ is connected, it follows that this holomorphic function is null. Therefore

$$F=\tilde f-U|_D$$

is the required holomorphic extension of f to D, which concludes the proof that $(\gamma)\Rightarrow(\alpha)$.

There remains to prove that $(\alpha)\Rightarrow(\gamma)$. Indeed we shall prove the equivalent fact that, if (γ) is not valid, then (α) is not valid as well.

Thus assume that (γ) is not valid. It follows, in view of (II.1), that there exists a $\bar\partial$-closed current $T\in\mathcal{D}'^{0,1}(M\setminus K)$ which is not $\bar\partial$-exact, but satisfies $T(\omega)=0$ for every $\mathcal{C}^\infty$ $\bar\partial$-closed $(n,n-1)$-form ω on $M\setminus K$ with compact support. Let $\widehat K_{\bar D}$ denote the $\mathcal{O}(\bar D)$-hull of K. Since D is strongly pseudoconvex, we have $\widehat K_{\bar D}\setminus K\subset D$; moreover we can, by pushing bD away from D with a small perturbation that leaves K fixed pointwise, obtain a pseudoconvex domain $\Delta\subset\subset M$ such that $D\subset\Delta$ and

$bD \cap b\Delta = K$. Since $\widehat{K}_{\overline{D}}$ is a Stein compactum, arguing as above in the proof that $(\beta) \Rightarrow (\gamma)$, with $\widehat{K}_{\overline{D}}$ in place of K, gives that the space $H^{0,1}_{\bar\partial}(M \setminus \widehat{K}_{\overline{D}})$ is separated. It follows in the first place, as $H^{0,1}_{\bar\partial}(M \setminus K)$ is assumed not to be separated, that $\widehat{K}_{\overline{D}} \setminus K$ is non-empty, and then, by (II.1) applied to $M \setminus \widehat{K}_{\overline{D}}$, that the restriction of the current T to $M \setminus \widehat{K}_{\overline{D}}$ projects into the zero class of $H^{0,1}_{\bar\partial}(M \setminus \widehat{K}_{\overline{D}})$, i.e. there exists a distribution $V \in \mathcal{D}'^{0,0}(M \setminus \widehat{K}_{\overline{D}})$ such that

$$T = \bar\partial V \quad \text{on } M \setminus \widehat{K}_{\overline{D}}.$$

On the other hand, since Δ is pseudoconvex and so $H^{0,1}_{\bar\partial}(\Delta) = 0$, there exists also a distribution $W \in \mathcal{D}'^{0,0}(\Delta)$ such that

$$T = \bar\partial W \quad \text{on } \Delta.$$

It follows that there exists a holomorphic function $f \in \mathcal{O}(\Delta \setminus \widehat{K}_{\overline{D}})$ such that

$$V - W = f \quad \text{on } \Delta \setminus \widehat{K}_{\overline{D}}.$$

Since T is not $\bar\partial$-exact in $M \setminus K$, no holomorphic extension of f to Δ may exist. Therefore, if we prove that f satisfies the moment condition (I.4), we shall get the desired conclusion that (α) is not valid.

Thus consider a $\mathcal{C}^\infty$ $\bar\partial$-closed $(n, n-1)$-form ω defined on a neighbourhood of $\overline{D}$, such that $\operatorname{supp}(\omega) \cap K = \emptyset$. Since $\overline{D}$ is a Stein compactum, we can find a Stein open neighbourhood $N \subset\subset M$ of $\overline{D}$ and a $\mathcal{C}^\infty (n, n-2)$-form on N, ψ, with $\bar\partial\psi = \omega$ on N. Then, if $h: M \to \mathbf{R}$ is a $\mathcal{C}^\infty$ cutoff function, $h = 1$ outside of a neighbourhood $N' \subset\subset N$ of K that does not meet $\operatorname{supp}(\omega)$ and $h = 0$ on a smaller neighbourhood of K, on N we have $\omega = h\omega = h\bar\partial\psi = \psi \wedge \bar\partial h + \bar\partial(h\psi)$, and hence $\int_{bD} f\omega = \int_{bD} f\psi \wedge \bar\partial h$. Therefore it suffices to prove that $\int_{bD} f\omega = 0$ in the case that ω be a $\mathcal{C}^\infty$ $\bar\partial$-closed $(n, n-1)$-form on the whole M, with compact support disjoint with K.

Let $\chi: M \to \mathbf{R}$ be a $\mathcal{C}^\infty$ function such that $\chi = 1$ on a neighbourhood of $M \setminus D$ and $\operatorname{supp}(\chi) \cap \operatorname{supp}(\omega) \cap \widehat{K}_{\overline{D}} = \emptyset$. Then we have

$$\begin{aligned}\int_{bD} f\omega = \int_{bD} f\chi\omega = \int_{D} f\bar\partial\chi \wedge \omega = \int_{\Delta} f\bar\partial\chi \wedge \omega = \int_{\Delta \setminus \widehat{K}_{\overline{D}}} f\bar\partial\chi \wedge \omega \\ = [V - W](\bar\partial\chi \wedge \omega) = -T(\chi\omega) - W(\bar\partial\chi \wedge \omega).\end{aligned}$$

On the other hand, since Δ is pseudoconvex, we can find a $\mathcal{C}^\infty$ $(n, n-2)$-form v on Δ such that $\omega = \bar\partial v$ on Δ, and hence $W(\bar\partial\chi \wedge \omega) = W(-\bar\partial(\bar\partial\chi \wedge v)) = T(\bar\partial\chi \wedge v)$. It follows that

$$\int_{bD} f\omega = -T(\chi\omega + \bar\partial\chi \wedge v),$$

and since $\chi\omega+\bar{\partial}\chi\wedge v$ is a $\mathcal{C}^\infty$ $\bar{\partial}$-closed $(n,n-1)$-form on M with compact support disjoint with K, by the assumption on T we conclude that $\int_{bD} f\omega=0$.

The proof of Theorem 2 is then completed.

II.3. *Remark.* The proofs given above that $(\beta)\Leftrightarrow(\gamma)$ and that (II.2) holds have depended on the fact that the compact set K was a proper closed subset of bD only in that this implied the connectedness of $M\setminus K$. Hence it is true, more generally, that every co-connected compact subset of a Stein manifold of dimension $n\geq 2$ has the properties that $(\beta)\Leftrightarrow(\gamma)$ and that (II.2) holds.

III. Proof of Theorem 1

We need to consider again the paracompactifying family Φ in $M\setminus K$ of all the closed subsets of $M\setminus K$ which are relatively compact in M. In fact we shall prove that conditions (A) and (B) are both equivalent to the following algebraic condition:

$$(C) \qquad H^1_\Phi(M\setminus K;\mathcal{O})=0.$$

We refer to [4] for basic information on sheaf cohomology with general families of supports. We recall that, by the generalized version [17; Theorem 1] of the Dolbeault theorem, there is an algebraic isomorphism

$$H^1_\Phi(M\setminus K;\mathcal{O})\cong H^{0,1}_{\bar{\partial}}(\mathcal{D}'_\Phi(M\setminus K)).$$

We shall use another formulation of (C), which requires us to consider the sheaf cohomology space $H^2_K(M;\mathcal{O})$ with supports in K, i.e. whose family of supports is that of all the closed subsets of K. We recall that in general, given a topological space X, a sheaf $\mathcal{A}$ of Abelian groups on X and a closed set $Y\subset X$, the cohomology $H^*_Y(X;\mathcal{A})$ with supports in Y is algebraically isomorphic to the relative cohomology $H^*(X,X\setminus Y;\mathcal{A})$ (see [4; II.12]).

Now, there is an exact cohomology sequence

$$\dots\to H^1_c(M;\mathcal{O})\to H^1_\Phi(M\setminus K;\mathcal{O})\xrightarrow{\delta} H^2_K(M;\mathcal{O})\xrightarrow{j} H^2_c(M;\mathcal{O})\to\dots,$$

and since M is Stein and $n\geq 2$, $H^1_c(M;\mathcal{O})=0$; therefore we see that (C) is equivalent to

$(C)'$ *The linear map* $j\colon H^2_K(M;\mathcal{O})\to H^2_c(M;\mathcal{O})$ *induced by inclusion of supports is injective.*

We shall first prove that $(B)\Leftrightarrow(C)'$. To this end we shall apply the relative duality theorem of Serre type [3; VII.4.15]. We state, for the convenience of the reader, the version of that theorem sufficient for our purposes.

III.1. *Let X be a complex-analytic manifold of dimension $n\geq 1$, $Y\subset X$ a compact set and p,q integers such that $0\leq p,q\leq n$. Then the spaces $H^p_Y(X;\Omega^q)$ and $H^{n-p}(Y;\Omega^{n-q})$ have natural topologies such that the associated separated spaces are in topological duality, that is*

$$^\sigma H^p_Y(X;\Omega^q)\cong \operatorname{Hom Cont}(^\sigma H^{n-p}(Y;\Omega^{n-q}),\ \mathbf{C}).$$

Moreover $H^p_Y(X;\Omega^q)$ is separated if and only if $H^{n-p+1}(Y;\Omega^{n-q})$ is separated.

Note that the topology of $H^{n-p}(Y;\Omega^{n-q})$ is the locally convex inductive limit topology derived from the inductive system formed by the spaces $H^{n-p}(U;\Omega^{n-q})$, as U ranges through a fundamental system of open neighbourhoods of Y; whereas the topology of $H^p_Y(X;\Omega^q)$ coincides up to isomorphism with that of the p-dimensional cohomology space derived from the Fréchet–Schwartz complex $\{C^{q,r},d^{(r)}\}_{r\in\mathbf{Z}}$ defined as $C^{q,r}=A^{q,r}(X)\oplus A^{q,r-1}(X\setminus Y)$ and $d^{(r)}(\omega,\theta)=(\bar\partial\omega,\omega|_{X\setminus Y}-\bar\partial\theta)$, for every $(\omega,\theta)\in C^{q,r}$. In particular one has that the linear map $H^p_Y(X;\Omega^q)\to H^p_c(X;\Omega^q)$ induced by inclusion of supports is a continuous map. Indeed this map can be viewed as that induced at the cohomology level by the linear map $C^{q,p}\to A^{q,p}_c(X)$ such that $(\omega,\theta)\mapsto\chi\omega+\bar\partial\chi\wedge\theta$, where $\chi\colon X\to\mathbf{R}$ be any compactly supported $\mathcal{C}^\infty$ function with $\chi=1$ on a neighbourhood of Y.

We shall also need the following fact:

III.2. *The cohomology space $H^1_K(M;\mathcal{O})=0$.*

This follows from the connectedness of $M\setminus K$ and the Hartogs theorem, using the exact sequence of relative cohomology

$$0\to\mathcal{O}(M)\to\mathcal{O}(M\setminus K)\to H^1_K(M;\mathcal{O})\to H^1(M;\mathcal{O})=0\to\dots\ .$$

Now, to prove that $(B)\Leftrightarrow(C)'$, let us first consider the case $n\geq 3$. In this case (B) is equivalent to having $0={}^\sigma H^{n-2}(K;\Omega^n)=H^{n-1}\ (K;\Omega^n)$, and, since $H^2_c(M;\mathcal{O})=0$, $(C)'$ is equivalent to having $H^2_K(M;\mathcal{O})=0$. Then, if (B) is valid, it follows at once from (III.1) that ${}^\sigma H^2_K(M;\mathcal{O})=0$ and that $H^2_K(M;\mathcal{O})$ is separated, which means that $(C)'$ is valid. Conversely, if $(C)'$ is valid, first it follows from (III.1) that ${}^\sigma H^{n-2}(K;\Omega^n)=0$ and $H^{n-1}(K;\Omega^n)$ is separated, and then it follows from (III.2) and (III.1) that $H^{n-1}(K;\Omega^n)=0$; hence (B) is valid. Thus the equivalence $(B)\Leftrightarrow(C)'$ is proved for $n\geq 3$.

Next let us consider the case $n=2$. By (III.1) and the absolute Serre duality theorem, there is a commutative diagram

$$\begin{array}{ccc} {}^\sigma H^2_K(M;\mathcal{O}) & \xrightarrow{\ \ {}^\sigma j\ \ } & {}^\sigma H^2_c(M;\mathcal{O}) \\ \wr\downarrow & & \downarrow\wr \\ \operatorname{Hom Cont}(\Omega^2(K),\mathbf{C}) & \xrightarrow{\ \varrho^*\ } & \operatorname{Hom Cont}(\Omega^2(M),\mathbf{C}) \end{array},$$

in which ${}^{\sigma}j$ is the linear map induced by $j\colon H^2_K(M;\mathcal{O})\to H^2_c(M;\mathcal{O})$ and the projections onto the associated separated spaces, and ϱ^* is the transpose of the restriction map $\varrho\colon\Omega^2(M)\to\Omega^2(K)$. Then, if (B) is valid, i.e. K is $\mathcal{O}(M)$-convex, we have that ϱ^* is injective and $H^1(K;\Omega^2)=0$. The latter fact implies, in view of (III.1), that $H^2_K(M;\mathcal{O})$ is separated and, since M is Stein, $H^2_c(M;\mathcal{O})$ is separated too; therefore ${}^{\sigma}j=j$. Moreover, since ϱ^* is injective, so too is j, and hence we see that $(B)\Rightarrow(C)'$.

Conversely, assume that $(C)'$ is valid. In the first place it follows, since j is continuous and $H^2_c(M;\mathcal{O})$ is separated, that $H^2_K(M;\mathcal{O})$ is separated too, and then, by the above commutative diagram, that ϱ^* is injective. The latter fact implies that $\varrho(\Omega^2(M))$ is a dense subspace of $\Omega^2(K)$. Indeed, since the space $\Omega^2(K)$ is locally convex, if $\varrho(\Omega^2(M))$ were not dense in $\Omega^2(K)$, there would exist a convex open and non-empty subset U of $\Omega^2(K)$ not meeting $\varrho(\Omega^2(M))$ and then, by a version of the Hahn–Banach theorem (see [8; p.54]), one could find an $x'\in\operatorname{Hom\,Cont}(\Omega^2(K),\mathbf{C})$ such that $\operatorname{Ker}(x')\supset\varrho(\Omega^2(M))$ and $x'\neq0$ on U, hence $\varrho^*(x')=0$, with x' being not identically null on $\Omega^2(K)$. Moreover the separation of $H^2_K(M;\mathcal{O})$ also implies, by (III.1), that of $H^1(K;\Omega^2)$ and then, by (III.1) and (III.2), it follows that $H^1(K;\Omega^2)=0$. Hence (B) is valid.

The proof that $(B)\Leftrightarrow(C)$ is then completed.

Now we prove that $(C)\Rightarrow(A)$. As it is already known that $(C)\Rightarrow(B)$, and trivially (B) implies condition (β) of Theorem 2, if we just prove that (C) also implies that every continuous CR-function f on $bD\setminus K$ satisfies the moment condition (I.4), the conclusion will be a straightforward consequence of the implication $(\beta)\Rightarrow(\alpha)$ of Theorem 2. Thus assume that (C) is valid. Arguing as in the proof of Theorem 2, it suffices to show that $\int_{bD}f\omega=0$ for each $\mathcal{C}^\infty$ $\bar\partial$-closed $(n,n-1)$-form on M, ω, with compact support disjoint with K. As a matter of fact, the current on $M\setminus K$ given by $\alpha\mapsto\int_{bD}f\alpha$, for each compactly supported $\mathcal{C}^\infty$ $(n,n-1)$-form, α, on $M\setminus K$, is $\bar\partial$-closed, since f is a CR-function, and is obviously supported in Φ; hence, by the assumption that (C) holds, it is the $\bar\partial$-differential of a distribution $U\in\mathcal{D}'^{0,0}_{\Phi}(M\setminus K)$. Therefore we have

$$\int_{bD}f\omega=\bar\partial U(\omega)=-U(\bar\partial\omega)=-U(0)=0,$$

and we conclude that $(C)\Rightarrow(A)$.

Finally we prove that $(A)\Rightarrow(C)$. Thus assume that (A) is valid. Let $\widehat K$ denote the $\mathcal{O}(M)$-hull of K. The hypotheses on D imply that $\widehat K=\widehat K_{\overline D}=\mathcal{O}(\overline D)$-hull of K and $\widehat K\cap bD=K$; moreover, since (A) holds, $bD\setminus K$ is connected, which implies that $D\setminus\widehat K$ is connected too (see [1, or 13]). Therefore we can, by pushing bD away from D with a small perturbation leaving K fixed pointwise, obtain a pseudoconvex domain $\Delta\subset\subset M$ with $D\subset\Delta$, $bD\cap b\Delta=K$ and with $\Delta\setminus\widehat K$ being connected. Now, as (A) is valid, the restriction map $\mathcal{O}(\Delta)\to\mathcal{O}(\Delta\setminus\widehat{K})$ is surjective, and hence, since Δ is pseudoconvex, the exact sequence of relative cohomology

$$0\to\mathcal{O}(\Delta)\to\mathcal{O}(\Delta\setminus\widehat{K})\to H^1_{\widehat{K}\cap\Delta}(\Delta;\mathcal{O})\to H^1(\Delta;\mathcal{O})=0\to\dots$$

implies that $H^1_{\widehat{K}\cap\Delta}(\Delta;\mathcal{O})=0$. As a consequence, by applying also the exact sequence of relative cohomology with supports in $\widehat{K}$

$$\dots\to H^1_{\widehat{K}}(M;\mathcal{O})\to H^1_{\widehat{K}\cap\Delta}(\Delta;\mathcal{O})\to H^2_{\widehat{K}}(M,\Delta;\mathcal{O})\to H^2_{\widehat{K}}(M;\mathcal{O})\to\dots,$$

we infer that the canonical linear map $H^2_{\widehat{K}}(M,\Delta;\mathcal{O})\to H^2_{\widehat{K}}(M;\mathcal{O})$ is injective. On the other hand there is an algebraic isomorphism $H^2_K(M;\mathcal{O})\xrightarrow{\cong}H^2_{\widehat{K}}(M,\Delta;\mathcal{O})$ induced by inclusion of supports (see [4; II.12.1]), and, since $\widehat{K}$ is an $\mathcal{O}(M)$-convex Stein compactum, arguing as above, in the proof that $(B)\Rightarrow(C)'$ for $n=2$, gives that the canonical linear map $H^2_{\widehat{K}}(M;\mathcal{O})\to H^2_c(M;\mathcal{O})$ is injective. Therefore we conclude that $(C)'$ is valid.

The proof of Theorem 1 is then completed.

Remarks. III.3. The proof given above that $(B)\Leftrightarrow(C)$ has depended on the fact that the compact set K was a proper closed subset of bD only in that this implied the connectedness of $M\setminus K$. Hence it is true, more generally, that every co-connected compact subset of a Stein manifold of dimension $n\geq2$ has the property that $(B)\Leftrightarrow(C)$. In particular a co-connected compact subset K of a two-dimensional Stein manifold M is an $\mathcal{O}(M)$-convex Stein compactum if and only if $H^1_\Phi(M\setminus K;\mathcal{O})=0$.

III.4. An alternative route to the conclusion that $(A)\Rightarrow(C)$, whose basic idea has been kindly communicated to the author by E. L. Stout, is as follows.

One has to prove that, on the assumption that (A) holds, if α is any $\mathcal{C}^\infty$ $\bar\partial$-closed $(0,1)$-form on $M\setminus K$, supported in Φ, there exists a function $g\in\mathcal{C}^\infty_\Phi(M\setminus K)$ with $\bar\partial g=\alpha$. Let Δ be as above and denote by $\Phi(\overline{D})$ the family of supports in $M\setminus\overline{D}$ of all the closed subsets of $M\setminus\overline{D}$ whose closure in M is compact, so that $\Phi(\overline{D})=\Phi\cap(M\setminus\overline{D})$. Now, since Δ is pseudoconvex, there exists $f_1\in\mathcal{C}^\infty(\Delta)$ with $\bar\partial f_1=\alpha$ on Δ. On the other hand, since $\overline{D}$ is an $\mathcal{O}(M)$-convex Stein compactum, by the preceding remark one has that $H^1_{\Phi(\overline{D})}(M\setminus\overline{D};\mathcal{O})=0$; hence there exists also $f_2\in\mathcal{C}^\infty_{\Phi(\overline{D})}(M\setminus\overline{D})$ with $\bar\partial f_2=\alpha$ on $M\setminus\overline{D}$. Then f_2-f_1 is a holomorphic function on $\Delta\setminus\overline{D}$ and, since $bD\setminus K$ is strictly Levi-convex, it extends to an $f\in\mathcal{O}(\Delta\setminus D)$. The latter function in turn extends, by hypothesis, to an $F\in\mathcal{O}(\Delta)$, and hence a function $g\in\mathcal{C}^\infty_\Phi(M\setminus K)$, such that $\bar\partial g=\alpha$, as is required, can be defined by

$$g=f_1+F \text{ on } \Delta,\quad g=f_2 \text{ on } M\setminus\overline{D}.$$

IV. Proof of Corollary 1

We have seen that conditions (A) and (B) of Theorem 1 are both equivalent to condition $(C)'$ and that for $n\geq 3$ the latter reduces to having $H^2_K(M;\mathcal{O})=0$. But since M is Stein, we get from the exact sequence of relative cohomology

$$\ldots \to H^1(M;\mathcal{O})=0\to H^1(M\setminus K;\mathcal{O})\to H^2_K(M;\mathcal{O})\to H^2(M;\mathcal{O})=0\to\ldots$$

that there is an algebraic isomorphism $H^1(M\setminus K;\mathcal{O})\cong H^2_K(M;\mathcal{O})$. Therefore, by the Dolbeault isomorphism too, it follows that for $n\geq 3$ and $\bar{D}$ being $\mathcal{O}(M)$-convex conditions (A) and (B) are equivalent to having $H^{0,1}_{\bar\partial}(M\setminus K)=0$.

Now suppose that $D\subset\subset M$ be any $\mathcal{C}^2$-bounded strongly pseudoconvex domain, with $\bar{D}$ nonnecessarily being $\mathcal{O}(M)$-convex. Anyhow, due to the strict Levi-convexity of bD, there exists a Stein open neighbourhood M' of $\bar{D}$, such that $\bar{D}$ is $\mathcal{O}(M')$-convex, and hence the above implies that for $n\geq 3$, if K is a proper closed subset of bD, then condition (A) is valid if and only if $H^{0,1}_{\bar\partial}(M'\setminus K)=0$.

On the other hand we have $M'\setminus K=(M\setminus K)\cap M'$ and $(M\setminus K)\cup M'=M$, and since M and M' are Stein, we get from the Mayer–Vietoris sequence

$$\ldots\to H^{0,1}_{\bar\partial}(M)=0\to H^{0,1}_{\bar\partial}(M\setminus K)\oplus H^{0,1}_{\bar\partial}(M')\to H^{0,1}_{\bar\partial}(M'\setminus K)\to H^{0,2}_{\bar\partial}(M)=0\to\ldots$$

that there is an algebraic isomorphism $H^{0,1}_{\bar\partial}(M\setminus K)\cong H^{0,1}_{\bar\partial}(M'\setminus K)$. Therefore for $n\geq 3$ condition (A) is equivalent to having $H^{0,1}_{\bar\partial}(M\setminus K)\cong H^1(M\setminus K;\mathcal{O})=0$, and consequently is also equivalent to condition (C), no matter whether $\bar{D}$ be or not be $\mathcal{O}(M)$-convex.

Finally, since in the proof of Theorem 1 the equivalence $(B)\Leftrightarrow(C)$ has been established without using the assumption that $\bar{D}$ was $\mathcal{O}(M)$-convex, we conclude that for $n\geq 3$ the equivalences $(A)\Leftrightarrow(B)\Leftrightarrow H^{0,1}_{\bar\partial}(M\setminus K)=0$ are valid apart from that assumption.

IV.1. *Remark.* We have already pointed out in Remark III.3 that the equivalence $(B)\Leftrightarrow(C)$ is valid for every co-connected compact subset K of a Stein manifold M of dimension $n\geq 2$. The same is true of the equivalence $(B)\Leftrightarrow(C)\Leftrightarrow H^{0,1}_{\bar\partial}(M\setminus K)=0$, in the case that $n\geq 3$.

V. Sufficient conditions for removability and weak removability

In Theorems 1 and 2 the hypotheses that M be Stein and D be strongly pseudoconvex are needed to prove the implications $(A)\Rightarrow(B)$ and $(\alpha)\Rightarrow(\beta)$. However such hypotheses are unnecessarily restrictive for the purpose of proving only sufficient conditions of removability and weak removability. Indeed in this Section we want to show that, for the validity of the converse implications $(B)\Rightarrow(A)$ and $(\beta)\Rightarrow(\alpha)$, the contexts of Theorem 1 and of Theorem 2 can be replaced by considerably wider contexts.

In the first place we can establish the following generalization of the implication $(B)\Rightarrow(A)$ of Theorem 1:

Theorem 3. *Let M be a non-compact complex-analytic manifold of dimension $n\geq 2$, such that $H^1_c(M;\mathcal{O})=0$. Let $D\subset\subset M$ be an open domain and $K\subset bD$ a compact set, such that $bD\setminus K$ is C^1-smooth. Assume that the following condition holds:*

(∗) *There is a compact set $E\subset M$ such that $\overline{D}\cap E=K$, $M\setminus(\overline{D}\cup E)$ is connected, the restriction map $Z^{n,n-2}_{\bar\partial}(M)\to Z^{n,n-2}_{\bar\partial}(E)$ has dense image, and, moreover, that $H^{n,n-1}_{\bar\partial}(E)=0$.*

Then K is removable.

Proof. This theorem can be derived from our previous result [11; Theorem 1.1], which provides a fruitful generalization of the so called Hartogs–Bochner theorem. The version of the result sufficient for the purpose can be stated as follows:

V.1. *Let X be a non-compact complex-analytic manifold of dimension $n\geq 2$ and let Φ be a paracompactifying family of closed subsets of X, not the one of all closed subsets of X. Assume that $H^1_\Phi(X;\mathcal{O})=0$. Then, if $D\subset X$ is an open domain such that $\overline{D}\in\Phi$, and bD is connected and C^1-smooth, it follows that every continuous CR-function on bD extends to a continuous function on $\overline{D}$ which is holomorphic on D.*

We point out that what is really needed for the proof of (V.1) is the connectedness of $X\setminus\overline{D}$, rather than of bD, and hence (V.1) remains true under such weaker assumption.

We shall prove Theorem 3 by applying (V.1) to the case where $X=M\setminus E$ and Φ is the family of all the closed subsets of $M\setminus E$ which are relatively compact in M. Thus what we have to show is that, if (∗) holds, the following two conditions are satisfied:

(1) $M\setminus E$ is connected;

(2) $H^1_\Phi(M\setminus E;\mathcal{O})=0$.

Now, since $H^{n-1}(E;\Omega^n)\cong H^{n,n-1}_{\bar\partial}(E)=0$, (III.1) implies that ${}^\sigma H^1_E(M;\mathcal{O})=0$ too; moreover, since $H^n(E;\Omega^n)=0$ (see I, footnote 3), (III.1) also implies that $H^1_E(M;\mathcal{O})$ is separated. Therefore $H^1_E(M;\mathcal{O})=0$, which, in view of the exact sequence of relative cohomology

$$0\to\mathcal{O}(M)\to\mathcal{O}(M\setminus E)\to H^1_E(M;\mathcal{O})\to\dots,$$

implies at once the validity of (1). Next, to prove (2), we argue essentially as in Section III for the proof that $(B)\Rightarrow(C)$ when $n=2$. By (III.1) and the absolute Serre duality theorem there is a commutative diagram

$$\begin{array}{ccc} {}^\sigma H^2_E(M;\mathcal{O}) & \xrightarrow{\quad {}^\sigma j \quad} & {}^\sigma H^2_c(M;\mathcal{O}) \\ \wr\downarrow & & \downarrow\wr \\ \operatorname{Hom}\operatorname{Cont}({}^\sigma H^{n-2}(E;\Omega^n),\mathbf{C}) & \xrightarrow{({}^\sigma\varrho)^*} & \operatorname{Hom}\operatorname{Cont}({}^\sigma H^{n-2}(M;\Omega^n),\mathbf{C}) \end{array},$$

in which ${}^\sigma j$ is the linear map induced by inclusion of supports and projections onto the associated separated spaces and $({}^\sigma\varrho)^*$ is the transpose of the linear map ${}^\sigma\varrho\colon {}^\sigma H^{n-2}(M;\Omega^n)\to{}^\sigma H^{n-2}(E;\Omega^n)$ induced by restriction and projections onto the associated separated spaces. Since the restriction map $Z^{n,n-2}_{\bar\partial}(M)\to Z^{n,n-2}_{\bar\partial}(E)$ is assumed to have dense image, it is plain the same is true of the induced map ${}^\sigma\varrho\colon {}^\sigma H^{n-2}(M;\Omega^n)\to{}^\sigma H^{n-2}(E;\Omega^n)$, and hence, by the Hahn–Banach theorem (see [8, p.54]), $({}^\sigma\varrho)^*$ is injective. It follows that ${}^\sigma j$ is injective too. On the other hand, since $H^{n-1}(E;\Omega^n)=0$, (III.1) implies that $H^2_E(M;\mathcal{O})$ is separated, and therefore ${}^\sigma j=\pi\circ j$, with $\pi\colon H^2_c(M;\mathcal{O})\to{}^\sigma H^2_c(M;\mathcal{O})$ being the canonical projection; hence j is injective. Finally we apply the exact sequence

$$\ldots\to H^1_c(M;\mathcal{O})\to H^1_\Phi(M\setminus E;\mathcal{O})\to H^2_E(M;\mathcal{O})\xrightarrow{j} H^2_c(M;\mathcal{O})\to\ldots;$$

since, by assumption, $H^1_c(M;\mathcal{O})=0$, and it has been shown that j is injective, the validity of (2) follows.

The theorem is proved.

V.2. *Remark.* We could have already resorted to (V.1) previously for the proof of the implication $(B)\Rightarrow(A)$ of Theorem 1; however we have wanted to show an alternative route to the same conclusion based on Theorem 2.

Next we prove the following generalization of the implication $(\beta)\Rightarrow(\alpha)$ of Theorem 2:

Theorem 4. *Let M be an arbitrary non-compact complex-analytic manifold of dimension $n\geq 2$. Let $D\subset\subset M$ be an open domain and $K\subset bD$ a compact set, such that $bD\setminus K$ is C^1-smooth. Assume that the following condition holds:*
(**) *There is a compact set $E\subset M$ such that $\bar D\cap E=K$ and $H^{n,n-1}_{\bar\partial}(E)=0$.*
Then K is weakly removable.

We point out that neither bD nor $bD\setminus K$ are assumed to be connected. Thus the theorem is a wide generalization of the result of Weinstock [20] that characterizes the boundary values of functions holomorphic on bounded domains of $\mathbf{C}^n$ with nonnecessarily connected boundaries. It also includes [15; Theorem 2] as a particular case, since it has been proved that, if $K\subset\mathbf{C}^n$ is a compact set whose $(2n-2)$-dimensional Hausdorff measure vanishes, then $H_{\bar\partial}^{n,n-1}(K)=0$ (see [12]).

Proof. It is no loss of generality to assume that there be no connected components of $M\setminus(\bar D\cup E)$ with compact closure in M. Indeed, if such components existed, C_i, $i\in\mathcal{I}$, say, we could pick points $P_i\in C_i$, $i\in\mathcal{I}$, and replace M by $M\setminus\{P_i\}_{i\in\mathcal{I}}$ as the ambient manifold.

Let us consider again the paracompactifying family Φ in $M\setminus E$ of all the closed subsets of $M\setminus E$ which are relatively compact in M. We need to consider also the family of all the subsets of $M\setminus E$ which are closed in M, i.e. of all the closed subsets of $M\setminus E$ each of which lies outside of a neighbourhood, in M, of E. We denote the latter family by Ψ. It is also a paracompactifying family in $M\setminus E$ and it is the *dual family of* Φ, in the sense that a closed subset C of $M\setminus E$ belongs to Ψ if and only if $C\cap S$ is compact for each $S\in\Phi$ (see [2]). Then consider the double complexes $A_\Psi(M\setminus E)$ and $\mathcal{D}'_\Phi(M\setminus E)$ of $\mathcal{C}^\infty$ differential forms on $M\setminus E$ supported in Ψ and of currents on $M\setminus E$ supported in Φ, respectively. $A_\Psi(M\setminus E)$ can be made into a topological vector space as the locally convex inductive limit of the Fréchet–Schwartz subspaces $A_G(M)\subset A(M)$ of the $\mathcal{C}^\infty$ forms on M whose supports are contained in G, as G ranges through a countable family of members of Ψ whose complements in M form a fundamental sequence of open neighbourhoods of E. It turns out that $A_\Psi(M\setminus E)$, with this topology, is a bigraded space of type $(\mathcal{LF})$ and the strong dual of a Fréchet–Schwartz space (see [8]). On the other hand it can be checked, by a reasoning analogous to that in the proof of [17; Proposition 4], that there is a canonical bijective linear map

$$L:\mathcal{D}'_\Phi(M\setminus E)\to\operatorname{Hom\,Cont}(A_\Psi(M\setminus E),\mathbf{C}),$$

defined by $T\mapsto L_T$, $L_T(\omega)=T(\omega)$; hence one may consider $\mathcal{D}'_\Phi(M\setminus E)$ as a bigraded space of Fréchet–Schwartz type, with the topology induced by L. Then, arguing as in [3; VII], one can establish a duality theorem of Serre type to the effect that, for all integers p,q with $0\le p,q\le n$, L induces a topological isomorphism

$${}^\sigma H_{\bar\partial}^{p,q}(\mathcal{D}'_\Phi(M\setminus E))\xrightarrow{\cong}\operatorname{Hom\,Cont}({}^\sigma H_{\bar\partial}^{n-p,n-q}(A_\Psi(M\setminus E),\mathbf{C})),$$

and that $H_{\bar\partial}^{p,q}(\mathcal{D}'_\Phi(M\setminus E))$ is separated if and only if so is $H_{\bar\partial}^{n-p,n-q+1}(A_\Psi(M\setminus E))$. It follows that the left kernel of the canonical bilinear map

$$H_{\bar\partial}^{p,q}(\mathcal{D}'_\Phi(M\setminus E))\times H_{\bar\partial}^{n-p,n-q}(A_\Psi(M\setminus E))\to\mathbf{C}$$

coincides with the closure of zero in $H_{\bar\partial}^{p,q}(\mathcal{D}'_\Phi(M\setminus E))$; therefore the following relative version of (II.1) is valid:

V.3. *Let* $T \in \mathcal{D}'^{p,q}_{\Phi}(M \setminus E)$ $(0 \le p, q \le n)$*; then the following two properties are equivalent*:

(i) $T(\omega)=0$ *for every* $\mathcal{C}^\infty$ $\bar\partial$*-closed* $(n-p, n-q)$*-form,* ω*, on* $M \setminus E$ *with support in* Ψ;

(ii) $\bar\partial T=0$ *and the class of* T *in* $H^{p,q}_{\bar\partial}(\mathcal{D}'_{\Phi}(M \setminus E))$ *is in the closure of zero.*

This is true for every compact subset E of M, i.e. apart from the assumption that $H^{n,n-1}_{\bar\partial}(E)=0$; whereas such assumption implies in addition the following fact:

V.4. *The space* $H^{0,1}_{\bar\partial}(\mathcal{D}'_{\Phi}(M \setminus E))$ *is separated.*

As a matter of fact, from the exact sequence of relative $\bar\partial$-cohomology

$$\ldots \to H^{n,n-1}_{\bar\partial}(E) \to H^{n,n}_{\bar\partial}(A_{\Psi}(M \setminus E)) \to H^{n,n}_{\bar\partial}(M) \to \ldots$$

we infer, also in view of Section I, footnote 3, that $H^{n,n}_{\bar\partial}(A_{\Psi}(M \setminus E))=0$, which, by the above, implies (V.4).

That being stated, let there be given a continuous CR-function f on $bD \setminus K$ that satisfies the moment condition (I.4). Consider the current on $M \setminus K$ given by $\alpha \mapsto \int_{bD} f\alpha$, for each $\mathcal{C}^\infty$ $(n, n-1)$-form, α, on $M \setminus K$ with $(\mathrm{supp}(\alpha)) \cap K=\emptyset$, and let T denote its restriction to $M \setminus E$. Then $\mathrm{supp}(T) \subset bD \setminus K$, hence $T \in \mathcal{D}'^{0,1}_{\Phi}(M \setminus E)$, moreover $T(\omega)=0$ for every for every $\mathcal{C}^\infty$ $\bar\partial$-closed $(n, n-1)$-form, ω, on $M \setminus E$ with support in Ψ; therefore, on account of (V.3) and (V.4), T it is the $\bar\partial$-differential of a distribution $U \in \mathcal{D}'^{0,0}_{\Phi}(M \setminus E)$. Now, since $\mathrm{supp}(T) \subset bD \setminus K$, the regularity theorem for $\bar\partial$ implies that there exist holomorphic functions $F \in \mathcal{O}(D)$, $F' \in \mathcal{O}(M \setminus (\bar D \cup E))$ such that $U|_D=F$ and $U|_{M \setminus (\bar D \cup E)}=F'$; moreover, since $\mathrm{supp}(U) \in \Phi$ and no connected components of $M \setminus (\bar D \cup E)$, being not relatively compact in M, can be the interior of a me mber of Φ, it follows that $F'=0$. Finally there remains to prove that F extends continuously to $\bar D \setminus K$ with boundary values f on $bD \setminus K$. This is an entirely local matter, which can be settled as in [9; Lemma 5.4].

The theorem is proved.

Remarks. V.5. We have outlined a direct proof of (V.3), along the lines of [17] and [3; VII], for the convenience of the reader. Alternatively one can derive (V.3) from the duality results of [2] relative to dual families of supports. The required condition which allows one to do so (see [2; pp. 188-189]) is fulfilled, since, if $M^*=M \cup \{\infty\}$ is the one-point compactification of M and $u: M^* \to [0,1]$ is a Uryshon function with $E=\{P \in M^* : u(P)=0\}$, $\{\infty\}=\{P \in M^* : u(P)=1\}$, then it follows that $\Phi=\{C \subset M \setminus E : C$ is closed and $\sup_C u<1\}$ and $\Psi=\{C \subset M \setminus E : C$ is closed and $\inf_C u>0\}$.

V.6. Indeed the above procedure to prove Theorem 4, based on the dual families of supports Φ and Ψ in $M \setminus E$, could also have been used before to prove the implication $(\beta)\Rightarrow(\alpha)$ of Theorem 2. However the hypothesis of Theorem 2 that the ambient manifold is Stein has made it possible to avoid such procedure and invoke instead the more familiar duality theorems with usual supports.

V.7. Finally we wish to point out that Theorem 3 and Theorem 4 can be refined in the following way: if bD is smooth of class $\mathcal{C}^v$, $(1\leq v\leq\infty)$ and the CR-function f is of class $\mathcal{C}^u$ $(0\leq u\leq v)$, then the extension F is of class $\mathcal{C}^u$ on $\bar{D}\backslash K$. Such a refinement is a local matter, which can be treated as in [9; Theorem 5.2]. Moreover, if bD is smooth of class $\mathcal{C}^\infty$, it is also possible to establish versions of Theorem 3 and Theorem 4 for a CR-distribution t on $bD\backslash K$ (in the case of Theorem 4 t has to verify the moment condition $t(\omega)=0$ for every ω as in Section I).

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