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Quadrature surfaces as free boundaries

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Abstract. This paper deals with a free boundary problem connected with the concept "quadrature surface". Let $\Omega\subset\mathbf{R}^n$ be a bounded domain with a C^2 boundary and μ a measure compactly supported in Ω. Then we say $\partial\Omega$ is a quadrature surface with respect to μ if the following overdetermined Cauchy problem has a solution.

$$\Delta u=-\mu \text{ in } \Omega, \quad u=0 \text{ and } \frac{\partial u}{\partial \nu}=-1 \text{ on } \partial\Omega.$$

Applying simple techniques, we derive basic inequalities and show uniform boundedness for the set of solutions. Distance estimates as well as uniqueness results are obtained in special cases, e.g. we show that if $\partial\Omega$ and ∂D are two quadrature surfaces for a fixed measure μ and Ω is convex, then $D\subset\Omega$. The main observation, however, is that if $\partial\Omega$ is a quadrature surface for $\mu\geq 0$ and $x\in\partial\Omega$, then the inward normal ray to $\partial\Omega$ at x intersects the convex hull of $\operatorname{supp}\mu$. We also study relations between quadrature surfaces and quadrature domains. D is said to be a quadrature domain with respect to a measure μ if there is a solution to the following overdetermined Cauchy problem:

$$\Delta u=1-\mu \text{ in } D, \quad \text{and} \quad u=|\nabla u|=0 \text{ on } \partial D.$$

Finally, we apply our results to a problem of electrochemical machining.

0. Preliminaries

Consider a bounded domain Ω in $\mathbf{R}^n$ $(n\geq 2)$ with regular boundary and a signed measure μ, compactly supported in Ω. Then it is known that there is a measure μ' (balayage measure), carried by the surface $\partial\Omega$ and having the same potential as μ outside $\overline{\Omega}$. In this paper we are interested in domains Ω such that for a (fixed) measure μ in Ω the balayage measure μ' coincides with the surface measure $d\Sigma$ $(=d\Sigma_\Omega)$.

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Since μ and μ' generate the same potential off $\bar{\Omega}$ we obtain, by classical approximation technique,

$$\int_{\partial\Omega} h(x)\,d\Sigma = \langle h, \mu\rangle \quad \forall h \in H(\bar{\Omega}), \tag{0-1}$$

where $H(\bar{\Omega})$ denotes the set of functions harmonic on a neighbourhood of $\bar{\Omega}$. For convenience, from now on we say that $\partial\Omega$ is a *quadrature surface* with respect to μ and write $\partial\Omega\in QS(\mu)$ if (0-1) is satisfied. We will also assume that $\partial\Omega$ (considered as QS) is C^2. Our first task, then, will be to transfer (0-1) to an elliptic problem; and one easily obtains the following (Theorem 1.1):

$\partial\Omega\in QS(\mu)$ if and only if there is a solution to the following overdetermined Cauchy problem:

$$\Delta u = -\mu \text{ in } \Omega, \quad u=0 \text{ and } \frac{\partial u}{\partial \nu^-} = -1 \text{ on } \partial\Omega. \tag{0-2}$$

Here $-$ $(+)$ indicates the limit from the interior (exterior) and ν is the outward normal vector to $\partial\Omega$. It is much easier to handle problem (0-2) because of the machinery of elliptic partial differential equation.

The paper is divided into 5 sections. In Section 1 we transfer (0-1) to (0-2) and vice versa, and give some examples. Section 2 takes care of basic properties. Here we observe that bounded elements in $QS(\mu)$ are uniformly bounded, and if in addition μ is positive we obtain an upper bound for $\operatorname{diam}(\Omega)$ in terms of $\|\mu\|$ (see definition below) and $\operatorname{diam}(\operatorname{supp}\mu)$. For any measure $d\mu = f\,dx$, where dx is the Lebesgue measure and f is assumed to be bounded we show that $M\delta > 2(n-1)$, where $\delta = \operatorname{diam}(\text{smallest ball containing supp } \mu)$ and $M \ge f$. The latter is especially useful in proving non-existence of QS, e.g. if $\mu = f\,dx$, $f \le 1$ and $\operatorname{supp} f \subset B(0, n-1)$, then $QS(\mu)$ is empty. Section 2 is concluded with a result concerning the uniqueness problem: we show that if $\partial\Omega_i \in QS(\mu)$ for $i=1,2$ and Ω_1 is convex then $\Omega_2 \subset \Omega_1$; it also follows that if $\Omega_1 \cap \Omega_2$ is convex then $\Omega_1 = \Omega_2$.

Section 3 is mainly devoted to one theorem (Theorem 3.4) and its corollaries. The theorem says that if $\mu \ge 0$ and $x \in \partial\Omega \setminus W$ ($W =$ convex hull of $\operatorname{supp}\mu$), then the inward normal ray to $\partial\Omega$ at x meets W. A geometric consequence of this is that if $\operatorname{supp}\mu$ is contained in a hyperplane then Ω is symmetric with respect to that hyperplane; consequently if μ is a constant multiple of the Dirac measure, then Ω is a ball.

In Section 4 we show some connections between quadrature surfaces and quadrature domains. The main theorem here is that if Ω is convex and $\partial\Omega \in QS(\mu)$ and D is a quadrature domain for μ (see definition below), then $\sup \operatorname{dist}(x, \Omega) < 2$, where sup is taken over ∂D.

Section 5 is an application of Section 3 to problems of potential flow in electrochemical machining.

Let us now introduce some basic notations and definitions which are frequently used in this paper. $QS(\mu)$ and $H(\overline{\Omega})$ were defined earlier. Ω will always denote a bounded domain in $\mathbf{R}^n$ with a C^2 boundary. The solution of (0-2), when it exists, will be called the *associated potential* (AP) of $\partial\Omega$ (with respect to μ). We also extend the associated potential u of a quadrature surface $\partial\Omega$ to $\mathbf{R}^n$ by defining it to be zero outside Ω. For a measure μ with compact support we define $\hat{\mu}$ to be the Newtonian potential of μ with the normalization $\Delta\hat{\mu}=-\mu$ (in the sense of distributions) and $G\mu$ stands for the Green potential of μ with respect to Ω, i.e. $G\mu(x)=\int G(x,y)\,d\mu$, where G is the Green function of Ω; we also define $\|\mu\|=\int d|\mu|$, δ=diam(smallest ball containing supp μ) and δ_x denotes Dirac measure with support at x. $B(x,r)$ means the n-dimensional open ball with center x and radius r; $S(x,r)=\partial B(x,r)$. By $|\partial\Omega|$ we mean the $(n-1)$-dimensional Lebesgue measure of $\partial\Omega$, $A_n=|S(0,1)|$, $c_n=1/((n-2)A_n)$ $(n\geq 3)$ and $c_2=1/2\pi$. For a domain D in $\mathbf{R}^n$, Σ_D or $d\Sigma_D$ means the surface element of ∂D, $CH(D)$ denotes the convex hull of D and we say T (a hyperplane) is a supporting plane to D at $x\in\partial D$ if it intersects $\overline{D}$ and a closed half-space of T contains D; points in the set $\overline{D}\cap T$ are called contact points. We also define D to be a *quadrature domain* (QD) for μ (compactly supported measure in D) and write $D\in QD(\mu)$, if supp $\mu\subset D$ and there is a solution v to the following overdetermined Cauchy problem:

$$\Delta v = 1-\mu \text{ in } D, \quad v=|\nabla v|=0 \text{ on } \partial D.$$

The solution v is called the *modified Schwarz potential* (MSP) for Ω with respect to μ. We extend v to be zero in $\mathbf{R}^n\setminus\Omega$. We will also refer to the boundary point version of Hopf's maximum principle ([17; p. 65]) as the boundary point lemma. Let us state this.

Boundary point lemma. *Let Ω be a domain with C^2 boundary and $x\in\partial\Omega$. Suppose u is a non-constant subharmonic function in Ω, continuous at x and satisfies $u(x)=\sup_\Omega u$. Then the outer normal derivative of u at x, if it exists, satisfies*

$$\frac{\partial u}{\partial \nu}(x)>0.$$

Remark. Throughout this paper we assume that all domains have a C^2 boundary. When taking the (normal) derivative of a function (generally the AP) on the boundary we mean the limit from the interior of the domain. This assumption makes it clear that the AP will always be C^1 in the interior of the domain up to the boundary.

1. The AP and some examples

Let us first establish the existence of the AP for a QS and vice versa.

Theorem 1.1. *Let* $\operatorname{supp}\mu\subset\Omega$. *Then* $\partial\Omega\in QS(\mu)$ *if and only if there exists an associated potential* u *of* $\partial\Omega$ *with respect to* μ.

Proof. Let $\partial\Omega\in QS(\mu)$ and recall that

$$\widehat{\Sigma}(y)=\int_{\partial\Omega}K(x,y)\,d\Sigma(x),$$

where K is the Newtonian kernel with the normalization mentioned in the preliminaries. For $y\in\mathbf{R}^n\setminus\overline{\Omega}$, set $h_y(x)=K(x,y)$. Then h_y is harmonic in $\overline{\Omega}$ (as a function of x) and by the assumption

$$\widehat{\Sigma}(y)=\int_{\partial\Omega}h_y\,d\Sigma=\langle h_y,\mu\rangle=\langle K(\cdot,y),\mu\rangle=\hat{\mu}(y), \tag{1-1}$$

for $y\in\mathbf{R}^n\setminus\overline{\Omega}$. Thus, by the continuity of the single-layer potential, this is true also for $y\in\mathbf{R}^n\setminus\Omega$. Define now $u(x)=\hat{\mu}(x)-\widehat{\Sigma}_\Omega(x)$ for $x\in\Omega$. Then $\Delta u=-\mu$ (in the sense of distributions) and $u=0$ in $\mathbf{R}^n\setminus\Omega$ by (1-1). Since $\hat{\mu}$ is C^1 (even harmonic) in $\mathbf{R}^n\setminus\operatorname{supp}\mu$ and $u=0$ in $\mathbf{R}^n\setminus\Omega$, we obtain

$$\frac{\partial u}{\partial\nu^-}=\frac{\partial u}{\partial\nu^-}-\frac{\partial u}{\partial\nu^+}=\frac{\partial\widehat{\Sigma}_\Omega}{\partial\nu^+}-\frac{\partial\widehat{\Sigma}_\Omega}{\partial\nu^-}=-1; \tag{1-2}$$

the last equality follows by [12; p. 164]. This proves the only if part of the theorem.

Conversely, let u be the AP of $\partial\Omega$ with corresponding distribution μ and take $h\in H(\overline{\Omega})$. Then

$$\int_{\partial\Omega}h\,d\Sigma=\int_{\partial\Omega}\left(u\frac{\partial h}{\partial\nu^-}-\frac{\partial u}{\partial\nu^-}h\right)d\Sigma.$$

Since h and $(u-\hat{\mu})$ are harmonic in Ω we obtain, by applying the Green's identity,

$$\int_{\partial\Omega}h\,d\Sigma=\int_{\partial\Omega}\left(\hat{\mu}\frac{\partial h}{\partial\nu^-}-\frac{\partial\hat{\mu}}{\partial\nu^-}h\right)d\Sigma=\langle\mu,h\rangle.$$

The last equality follows from [13; see the assertion in Theorem 4.2]. $\square$

Remark. From now on we will leave the minus sign out when expressing the normal derivative (see (1-2)), but we still mean the limit from the interior.

Corollary 1.2. *Let $\partial\Omega \in QS(\mu)$. Then u (the AP) is the Green potential of μ with respect to Ω; if in addition μ is absolutely continuous with bounded density function then u is continuous.*

Proof. Let $G\mu$ be the Green potential of μ with respect to Ω then $u-G\mu$ is zero on $\partial\Omega$ and harmonic in Ω, therefore identically zero in Ω. For the second statement see [9; Theorem 6.22]. $\square$

It is not too easy to give explicit examples of QSs, where both the measure and the surface are explicitly given, but there are some rather trivial ones. However, any analytic closed surface is a quadrature surface for some unknown measure. The converse is also true if we have an a priori regularity of the surface (see [2]). But this is not true in general.

Let us assume $n \geq 3$. Then the simplest example of a QS is the sphere $S(x^0, R)$ whose associated potential u is

$$u = \frac{R}{n-2}\left(\frac{R^{n-2}}{|x-x^0|^{n-2}} - 1\right),$$

and the corresponding measure is $A_n R^{n-1} \delta_{x^0}$. To give another example consider the mass A_n uniformly distributed on the unit sphere and denote it by μ. Set now $\mu_\varrho = \varrho\mu$ where $1 < \varrho < 2$ is fixed and define $\Gamma_r = S(0,r) \cup S(0,r_1)$ where $0 < r < 1$ and $r_1^{n-1} + r^{n-1} = r_1 + r = \varrho$. Then $\Gamma_r \in QS(\mu_\varrho)$ and the corresponding AP is

$$u_r(x) = \begin{cases} \dfrac{1}{2-n}\left(\dfrac{r^{n-1}}{|x|^{n-2}} - \varrho + r_1\right), & r < |x| \leq 1, \\[2ex] \dfrac{1}{2-n}\left(\dfrac{r^{n-1}-\varrho}{|x|^{n-2}} + r_1\right), & 1 < |x| < r_1. \end{cases}$$

Remark. A rather interesting generalization of QSs is to let the normal derivative of the AP on the boundary be a function g continuous on $\mathbf{R}^n$ or—what amounts to the same thing—to consider weighted QS with weight g. An example in this case is the ellipsoidal conductor; we leave the details of this to the interested reader to work out, (see [12; pp. 188–191 and 195]). However, the quadrature formula in the case $n=3$ is as follows.

$$\int_{\partial E} gh \, d\Sigma = \frac{2a_1a_2a_3}{\sqrt{(a_1^2-a_3^2)(a_2^2-a_3^2)}} \int_{E_o} hf \, dx_1 \, dx_2, \quad \forall h \in H(\overline{E})$$

where

$$E = \left\{\sum x_j^2/a_j^2 < 1\right\}, \quad E_o = \{x_1^2/(a_1^2-a_3^2) + x_2^2/(a_2^2-a_3^2) < 1\},$$

$$f = (1 - x_1^2/(a_1^2-a_3^2) - x_2^2/(a_2^2-a_3^2))^{-1/2} \quad \text{and} \quad g = \left(\sum x_j^2/a_j^4\right)^{-1/2}.$$

2. Basic properties of QS and the AP

The first question to discuss in this section is the uniform boundedness, i.e. whether, given μ, bounded surfaces in $QS(\mu)$ are uniformly bounded or not? This question, in the context of QDs, was posed, for the first time, by M. Sakai [18] and he proved uniform boundedness for quadrature domains. There are other proofs for QDs, one due to B. Gustafsson [6] and another due to the author [21]; here we answer this question in the affirmative for QSs.

The following lemma will frequently be used in this section.

Lemma 2.1. *Let $\partial\Omega \in QS(\mu)$ and u denote the AP of $\partial\Omega$. Let D be a convex domain containing* $\operatorname{supp}\mu$ *and such that $\Omega\setminus D$ is not empty. Then*

$$\sup_{\partial D} u > \sup_{x\in\partial\Omega} d(x) =: d_0, \quad \textit{and} \quad \frac{\partial u}{\partial \nu}(x^0) < -1,$$

where $d(x)=\operatorname{dist}(x,D)$, ν is a normal vector to any supporting plane to D at x^0 which points away from D, and x^0 is a point on ∂D where u attains its maximum value on ∂D.

Proof. Let $z\in\partial\Omega$ and $y\in\partial D$ be two points such that $d_0=|z-y|$. We may, by rotation and translation, assume y is the origin and $z=(z_1,0')$ where $z_1=d_0$ and $0'$ is the origin in $\mathbf{R}^{n-1}$. Set $v=u+x_1$ in $\Omega'=\{x\in\Omega: x_1>0\}$. Then v is harmonic in Ω' and consequently it attains its maximum value in $\overline{\Omega}'$ at some $x'\in\partial\Omega'$. We claim now $x'\in\partial\Omega'\setminus\partial\Omega$. This together with the fact that u is harmonic in $\Omega\setminus\overline{D}$ implies

$$u(x^0) = \sup_{\partial D} u \geq u(x'),$$

where x^0 is a point corresponding to the maximum value of u on ∂D. Then, since $u=0$ on $\partial\Omega$, we obtain

$$\sup_{\partial D} u \geq u(x') = \sup_{\partial\Omega'}(u+x_1) > z_1 = d_0.$$

This proves the first statement (modulo the claim). The second statement will then follow by replacing D by any half-space whose boundary is a supporting plane to D at x^0 and then applying the boundary point lemma to the function $u+d$ in $\Omega\setminus\overline{D}$, where d, now, is the distance function to this supporting plane.

Now to complete the proof let us assume, in order to reach a contradiction, that the maximum value of v is attained on $\partial\Omega$. Then, since $u=0$ on $\partial\Omega$, this maximum value is attained at z; moreover the outward normal direction to Ω at z is the x_1 direction. Thus, by the boundary point lemma,

$$0 < \frac{\partial v}{\partial x_1}(z) = \frac{\partial u}{\partial x_1}(z) + 1 = -1+1 = 0.$$

Hence a contradiction. Thus the lemma is proved. $\square$

Theorem 2.2. *Bounded domains in $QS(\mu)$ are uniformly bounded.*

Proof. Let $\partial\Omega \in QS(\mu)$. Consider first the case $n=2$. By the definition, i.e. (0-1), we have $|\partial\Omega| = \langle 1, \mu \rangle$, which implies that the length of the curve $\partial\Omega$ is a fixed constant, independent of Ω. Thus Ω is contained in a fixed ball of diameter $\langle 1, \mu \rangle / 2$. Let now $n \geq 3$. Then fix a ball B containing $\operatorname{supp} \mu$ and let $\partial\Omega \in QS(\mu)$ with $\Omega \setminus B$ not empty. By Lemma 2.1

$$\sup_{\partial\Omega} \operatorname{dist}(x, B) \leq \sup_{\partial B} u.$$

Therefore it suffices to prove $\sup_{\partial B} u$ is uniformly bounded. By the proof of Theorem 1.1 $u = \hat{\mu} - \widehat{\Sigma}$ where $\widehat{\Sigma} > 0$ on $\mathbf{R}^n$ if $n \geq 3$. Hence

$$\sup_{\partial B} u < \sup_{\partial B} \hat{\mu} = \text{constant (independent of } \Omega),$$

which gives the desired result. $\square$

Let now $\partial\Omega \in QS(\mu)$, where $\mu \geq 0$ and recall that $\|\mu\| = \int d|\mu| = \int d\mu = |\partial\Omega|$, δ $=$diam(smallest ball containing $\operatorname{supp} \mu$) and $c_n = 1/(n-2)A_n$. Then, by direct calculation, we can obtain a distance estimate of $\partial\Omega$ to the convex hull of $\operatorname{supp} \mu$ in terms of $\|\mu\|$.

Theorem 2.3. *Let $n \geq 3$ and $\partial\Omega \in QS(\mu)$, where μ is a positive measure and define D_0 to be the convex hull of* $\operatorname{supp} \mu$. *Then*

$$d_0 \leq \max\Big(\delta, \inf_{0<\alpha<1} \beta(\alpha)\Big),$$

where $(1-\alpha)\beta^{n-1}(\alpha) = c_n \|\mu\| (\alpha^{2-n} - 2^{2-n})$ *and* $d_0 = \sup_{x \in \partial\Omega} \operatorname{dist}(x, D_0)$.

Proof. To prove the theorem suppose $d_0 \geq \delta$. Then we show $d_0 < \beta(\alpha)$ for $\alpha \in (0,1)$. Define $D(=D_\alpha)$ to be

$$D = \{x \in \mathbf{R}^n : \operatorname{dist}(x, D_0) < \alpha d_0\},$$

where $0 < \alpha < 1$. We first estimate $u(x)$ on ∂D by applying Theorem 1.1. Hence

$$\sup_{\Omega \cap \partial D} u \leq \sup_{\partial D} \hat{\mu} - \inf_{\Omega \cap \partial D} \widehat{\Sigma}_\Omega. \tag{2-1}$$

Since $\widehat{\Sigma}_\Omega$ is harmonic in Ω it attains its minimum value in Ω on the boundary. Thus

$$\inf_{\Omega \cap \partial D} \widehat{\Sigma}_\Omega \geq \inf_{\partial\Omega} \widehat{\Sigma}_\Omega = \inf_{\partial\Omega} \hat{\mu},$$

where in the equality we have used the assumption that $\hat{\mu}=\widehat{\Sigma}_\Omega$ in $\mathbf{R}^n\setminus\Omega$, i.e. $\partial\Omega\in QS(\mu)$. Thus a further reduction of (2-1), which is

$$\sup_{\partial D} u \le \sup_{\partial D} \hat{\mu} - \inf_{\partial\Omega} \hat{\mu}. \tag{2-2}$$

Now it is very easy to estimate the right side of the above inequality. The first term is

$$\hat{\mu}(x) \le c_n\|\mu\|(\alpha d_0)^{2-n} \quad \forall x\in\partial D.$$

The second term, since $d_0\ge\delta$, can be estimated from below as

$$\hat{\mu}(x) \ge c_n\|\mu\|\left(\frac{1}{d_0+\delta}\right)^{n-2} \ge c_n\|\mu\|(2d_0)^{2-n} \quad \forall x\in\partial\Omega.$$

Putting this into (2-2) we obtain

$$\sup_{x\in\partial D} u(x) \le c_n\|\mu\| d_0^{2-n}(\alpha^{2-n}-2^{2-n}). \tag{2-3}$$

Now by Lemma 2.1

$$(1-\alpha)d_0 \le \sup_{x\in\partial D} u(x)$$

which together with (2-3) results in

$$d_0^{n-1} \le \frac{c_n\|\mu\|}{1-\alpha}(\alpha^{2-n}-2^{2-n}),$$

for all $\alpha\in(0,1)$. This completes the proof. $\square$

Remark. For $n=2$ the result is slightly different. Since $|\partial\Omega|=\|\mu\|$, we have $d_0\le\frac{1}{2}\|\mu\|$. Similar calculations as in Theorem 2.3 show that

$$\beta^{n-1}(\alpha) = \frac{\|\mu\|}{2\pi(1-\alpha)}(\log 2-\log\alpha),$$

for any α, $0<\alpha<1$. We leave the details to the interested reader.

Our next result concerns diameter estimate of $\operatorname{supp}\mu$, when μ is an absolutely continuous measure with bounded density function for which $QS(\mu)$ is not empty. To be more precise:

Theorem 2.4. *Let $d\mu = f dx$ where f is a bounded function with compact support and dx denotes the Lebesgue measure. Assume $QS(\mu)$ is not empty, set $M = \sup f$ and recall that $\delta = diam$(smallest ball containing supp μ). Then*

$$M\delta > 2(n-1).$$

Proof. Suppose the conclusion in the theorem does not hold. So let $M\delta \leq 2(n-1)$ and define x^0 to be the center of the smallest ball containing $\operatorname{supp}\mu$. Set now $v(x) = u(x) + |x - x^0|$ where u is the AP of $\partial\Omega$. Then by the assumption (i.e. $M\delta \leq 2(n-1)$) v is subharmonic in Ω. Hence v attains its maximum value on the boundary, and consequently, since u is zero there, at point(s) with largest distance to x^0. Let x' be a point with this property, then the outward normal derivative of $|x - x^0|$ at x' is 1. Now by the boundary point lemma

$$0 < \frac{\partial v}{\partial \nu}(x') = -1 + 1 = 0,$$

a contradiction. Thus the theorem is proved. □

Theorem 2.4 shows that $QS(\mu)$ may be empty. As an example let μ be a positive measure bounded by one and $\operatorname{supp}\mu \subset B(0, n-1)$, then $QS(\mu)$ is empty.

Theorem 2.5. *With the same assumptions as in Theorem 2.4 the following statements are true*

(1) $Mr_0 > n$,

(2) $M\delta^3 > r_0^2(n-3)$, $(n \geq 4)$,

(3) $(M\delta)^3 > n^2(n-3)$, $(n \geq 4)$,

where δ and M are as in Theorem 2.4 and $r_0 = \sup_{x \in \partial\Omega} |x - x^0|$, where x^0 is the center of the smallest ball containing $\operatorname{supp}\mu$.

Proof. The proof is similar to that of Theorem 2.4, with the only difference that we set

$$v = u + \frac{|x - x^0|^2}{2r_0},$$

in the first case and

$$v = u - \frac{r_0^2}{|x - x^0|},$$

in the second case. (3) follows by combining (1) and (2). □

As we mentioned earlier, at the end of Section 1, QSs are not necessarily unique. It is, however, known that if the boundary of two convex domains (not necessary C^2) generate the same exterior potential then they are identical (see [11; p. 62]).

This will of course imply that if Ω_1 and Ω_2 are convex domains and their boundaries are in $QS(\mu)$ for a fixed μ, then they are identical. We will here improve this result in a certain direction; but unfortunately we have to assume that the boundaries are C^2.

Theorem 2.6. *Let $\partial\Omega_j \in QS(\mu)$ for $j=1,2$ and assume Ω_1 is convex. Then $\Omega_2 \subset \Omega_1$.*

Proof. Suppose $\Omega_2 \setminus \Omega_1$ is non-empty. Recalling that $u_2=0$ outside Ω_2 we obtain $\Delta u_2 = \Sigma_{\Omega_2} - \mu$ in Ω_1 and moreover

$$\sup_{\partial\Omega_1} u_2 > 0; \tag{2-4}$$

the latter follows by Lemma 2.1. Define now $u=u_2-u_1$ in Ω_1. Then u, being subharmonic in Ω_1, attains its maximum value on $\bar{\Omega}_1$ at $x^0 \in \partial\Omega_1$. Moreover $x^0 \in \Omega_2$; else

$$0 = u(x^0) = \sup_{x\in\partial\Omega_1} u(x) = \sup_{x\in\partial\Omega_1} u_2(x),$$

which contradicts (2-4). Now applying the boundary point lemma (this can be done since u is analytic in Ω_1 near x^0 and it is continuous at x^0) we obtain

$$\frac{\partial u}{\partial\nu}(x^0) > 0.$$

Hence

$$\frac{\partial u_2}{\partial\nu}(x^0) > -1,$$

which contradicts Lemma 2.1. Thus the theorem is proved. $\square$

Theorem 2.7. *Let $\partial\Omega_j \in QS(\mu)$ for $j=1,2$ and suppose $\Omega_1 \cap \Omega_2$ is convex. Then $\Omega_1 = \Omega_2$.*

The proof of this theorem is much the same as that of Theorem 2.6 and therefore omitted.

3. Symmetry principle and QS

Our aim here is to investigate QSs from a geometric point of view. For this we apply the symmetry principle, due originally to Alexandroff ([10; Ch. 7]), to our problem. We begin by introducing some notations.

Let a be a unit vector in $\mathbf{R}^n$ and define T_t $(=T_t^a)$ to be the hyperplane $a\cdot x=t$. Now consider a bounded domain Ω in $\mathbf{R}^n$, with C^2 boundary. Then for large t $T_t \cap \bar{\Omega}$

is empty and T_t moves continuously as we decrease t in direction $-a$ toward Ω until it intersects $\bar{\Omega}$ at some point(s) (we refer to this point(s) as contact point(s)). If we continue this moving process even after T_t hits $\bar{\Omega}$, we see that for every t there corresponds a cap $\Omega_t(=\Omega_t^a)=\{x\in\Omega : a\cdot x>t\}$, which has been cut off from Ω by T_t.

In doing so we have produced a cap which has interesting properties when reflected in T_t^a; we denote the reflected cap by $\mathrm{Ref}(\Omega_t)$ ($\mathrm{Ref}(\Omega_t^a)$). Let now $t_0=\sup\{t : T_t\cap\Omega\not\equiv\emptyset\}$, then for $t<t_0$ and near t_0 it is true that $\mathrm{Ref}(\Omega_t)\subset\Omega$ and as t decreases one of the following is obtained:

(1) T_t reaches a position where it is orthogonal to $\partial\Omega$ at some point on $\partial\Omega$,

(2) $\mathrm{Ref}(\Omega_t)$ becomes internally tangent to $\partial\Omega$ at some point not on T_t.

The first result to be obtained is that situation (2) above cannot arise for QSs of positive measure as long as T_t does not hit the convex hull of $\operatorname{supp}\mu$. Then, by [20], it follows that neither is situation (1) possible for QSs of positive measure if T_t does not hit the convex hull of $\operatorname{supp}\mu$. This implies huge restrictions on the shape of a QS when the corresponding measure is positive.

For convenience we will adopt the following notations and definitions. As before μ will stand for a measure and throughout this section we assume it to be positive ($\mu\geq 0$). We denote the convex hull of $\operatorname{supp}\mu$ by W ($=W_\mu$) and x^t will mean the reflection of x in T_t for a fixed direction a. A cap Ω_{t_1} obtained by this technique will be called an *optimal cap* if

$$t_1=\sup\{t : T_t \text{ is orthogonal to } \partial\Omega \text{ at some point}\}.$$

In this notation we have assumed that the direction a is known and fixed. We also assume that the restriction of the AP u to Ω has a C^2 extension to a neighborhood of $\partial\Omega$.

Remark. This moving plane technique has also been used in [4] and [16].

Lemma 3.1. *Let $\partial\Omega\in QS(\mu)$ and t_1 be as above for a fixed direction. Then*

$$\mathrm{Ref}(\Omega_t)\subset\Omega, \quad \forall t\in(t_1,t_0)$$

provided $\Omega_t\cap W=\emptyset$.

Proof. Suppose the conclusion in the lemma does not hold and set

$$t'=\sup\{t : \mathrm{Ref}(\Omega_t)\setminus\Omega\not\equiv\emptyset\}.$$

Then the reflected cap for $t=t'$ is in Ω and its boundary is internally tangent to $\partial\Omega$ at some point $x^0\notin T_{t'}$. Now define $\tilde{u}$ to be the reflection of u in $T_{t'}$ i.e. $\tilde{u}(x)=u(x^{t'})$. Then $u-\tilde{u}$ is superharmonic in $\mathrm{Ref}(\Omega_{t'})$ and nonnegative on $\partial(\mathrm{Ref}(\Omega_{t'}))$. Hence either $u \equiv \tilde{u}$ (which gives the result) or $(u-\tilde{u})>0$ on $\mathrm{Ref}(\Omega_{t'})$ and it attains its minimum value at points on the boundary where it is zero, and particularly at x^0. Now by the boundary point lemma

$$0 > \frac{\partial u}{\partial \nu}(x^0) - \frac{\partial \tilde{u}}{\partial \nu}(x^0) = -1+1 = 0,$$

where ν is the outward normal to Ω at x^0. Thus a contradiction is obtained and the lemma is proven. □

Lemma 3.2. *Let $\partial\Omega \in QS(\mu)$. Then situation* (1)*, above, cannot arise as long as $\overline{\Omega}_t \cap W = \emptyset$.*

Proof. See [20; pp. 307–309]. Observe that the proof presented in [20] is of local character and there will be no obstacle to apply it to our problem. □

Lemma 3.3. *Let $\partial\Omega \in QS(\mu)$ and Ω' be a cap for an arbitrarily fixed direction a. Then*

$$a \cdot \nabla u(x) < 0, \quad \forall x \in \overline{\Omega'} \setminus \partial\Omega,$$

provided $\overline{\Omega'} \cap W = \emptyset$.

Proof. The proof is similar to that of Lemma 3.1 and therefore omitted. □

Theorem 3.4. *Let $\partial\Omega \in QS(\mu)$ where $\mu \geq 0$. Then for any $x^0 \in \partial\Omega \setminus W$ the inward normal ray to $\partial\Omega$ at x^0 meets W.*

Proof. Let $x^0 \in \partial\Omega \setminus W$ and suppose the inward normal ray l at x^0 does not meet W. Now there exists a hyperplane T containing l such that $W \cap T = \emptyset$. The plane T is orthogonal to $\partial\Omega$ at x^0, and it cuts off Ω a cap which we denote by Ω_1. It is obvious that we can assume Ω_1 to be an optimal cap with respect to the direction a, where a is the normal vector to the plane T pointing away from W, otherwise we move the plane in direction a until such a position is obtained; but this contradicts Lemma 3.2. Thus the theorem follows. □

Remark. Similar results have been obtained for QDs (see [8]).

Corollary 3.5. *Let $\partial\Omega \in QS(\mu)$. Then any cap Ω' of Ω which does not contain W has the property that $\partial\Omega \cap \partial\Omega'$ is a graph, and consequently Ω' is simply connected.*

Proof. By Lemma 3.3 $\partial u / \partial a < 0$ in any such cap, which implies the desired result. □

Corollary 3.6. *Let $x\in\partial\Omega\setminus W$ and $K_x:=\{z\in\mathbf{R}^n:(z-x,y-x)\leq 0\ \ \forall y\in W\}$. Then $K_x\cap(\Omega\cup W)=\emptyset$.*

Proof. Let $z\in K_x$. Then by the definition $z\notin W$. So suppose $z\in\Omega$. Then the hyperplane T through $(x+z)/2$ and orthogonal to $z-x$ cuts off Ω a cap Ω' where $\overline{\Omega'}\cap W=\emptyset$. Now reflecting Ω' in T we obtain that the reflection of z is the point x which is not in Ω. This contradicts Lemma 3.1. Thus the corollary is proved. □

Corollary 3.7 (To Lemma 3.1 and 3.2). *Let $\partial\Omega\in QS(\mu)$ and suppose there is a hyperplane T containing* $\operatorname{supp}\mu$. *Then Ω is symmetric with respect to T.*

Corollary 3.8. *Let $\partial\Omega\in QS(\mu)$ and suppose $\mu=c\delta_x$ $(c>0)$. Then $\partial\Omega$ is a sphere centered at x and with radius r $(c=A_n r^{n-1})$.*

Remark. There is also another simple proof of Corollary 3.8 due to the author [22].

4. QS in comparison with QD

In this section we show that if $\partial\Omega\in QS(\mu)$ is convex and $D\in QD(\mu')$ where $d\mu'-d\mu=f\,dx$, $\operatorname{supp}\mu'\subset\Omega$ and $f\leq 1$, then $D\subset\Omega'=\{x:\operatorname{dist}(x,\Omega)<2\}$. For this we need the following lemma which is essentially due to L. A. Caffarelli [3].

Lemma 4.1. *Let $D\in QD(\mu')$ and Ω be a convex domain containing* $\operatorname{supp}\mu'$. *Then*

$$\sup_{\partial\Omega} v\geq \sup_{x\in\partial D}\tfrac{1}{2}(\operatorname{dist}(x,\Omega))^2=:\tfrac{1}{2}d_0^2,$$

where v is the MSP of D.

Proof. Let $y\in\partial\Omega$ and $z\in\partial D$ be two points such that $d_0=|z-y|$. We may, by rotation and translation, assume that y is the origin and $z=(z_1,0')$. Now set $D'=\{x\in D:x_1>0\}$ and let $\{z^j\}$ be a sequence in D converging to z and satisfying $v(z^j)>0$. (The existence of such points follows from the fact that v is subharmonic near the boundary and hence by the sub-mean value theorem $\int_B v\geq 0$ where B is any small ball with center at the boundary.) Define now

$$w(x)=v(x)-v(z^j)-\tfrac{1}{2}(x_1-z_1^j)^2\quad \text{ in } D',$$

and observe that $w(z^j)=0$. Then w, being harmonic in D', attains its positive maximum on the boundary of D'. Since $w<0$ on ∂D the maximum value is attained at $x_1=0$ and it is positive. Now letting $z^j\to z$ we obtain

$$\sup_{\partial\Omega} v\geq\sup_{x_1=0} v\geq\tfrac{1}{2}z_1^2=\tfrac{1}{2}d_0^2,$$

where the first inequality is a consequence of the maximum principle, applied to v in $D\setminus\bar{\Omega}$. Thus the lemma is proved. □

Theorem 4.2. *Let $\partial\Omega \in QS(\mu)$ with Ω convex. Let moreover $D\in QD(\mu')$, where* $\operatorname{supp}\mu'\subset\Omega$ *and $\mu'-\mu=f\,dx$ with $f\le 1$. Then*

$$d_0 := \sup_{x\in\partial D} \operatorname{dist}(x,\Omega)\le 2.$$

Proof. By assuming the contrary we reach a contradiction. So let $d_0>2$ and define u to be the AP of $\partial\Omega$ and v the MSP of D. Then, by Lemma 4.1, it follows that

$$\sup_{\partial\Omega} v \ge \tfrac{1}{2}d_0^2 > d_0. \tag{4-1}$$

The latter follows from the assumption that $d_0>2$. Since Ω is convex, the function $v+d$ is subharmonic in $D\setminus\Omega$, where $d(x)=\operatorname{dist}(x,\Omega)$. Therefore the maximum value of this function is attained at $x^0\in\partial(D\setminus\Omega)$, and by (4-1) $x^0\in\partial\Omega$. Moreover, by the boundary point lemma,

$$\frac{\partial v}{\partial\nu}(x^0)>1, \tag{4-2}$$

where ν is the inward normal to Ω. Our aim now is to prove the inequality opposite to (4-2). Recall $v=0$ outside D. Then $v-u$ is subharmonic in Ω and therefore the maximum is attained at x^0. Now by the boundary point lemma we obtain

$$\frac{\partial v}{\partial\nu}(x^0)<\frac{\partial u}{\partial\nu}(x^0)=1.$$

Since this contradicts (4-2) we conclude that the theorem is true. □

5. Applications to potential flow

In this section we apply results from Section 3 to the following free boundary problem:

Let D be a bounded domain in $\mathbf{R}^n$. Find another domain Ω containing D and a continuous function u satisfying

$$\begin{aligned}
\Delta u &= 0 && \text{in } \Omega\setminus D,\\
u &= 0 && \text{on } \partial\Omega \text{ (free boundary)},\\
-\frac{\partial u}{\partial\nu} &= 1 && \text{on } \partial\Omega,\\
u &= 1 && \text{on } D,
\end{aligned}$$

where ν is the outward normal to $\partial\Omega$ and it is assumed that $\partial\Omega$ is real analytic. This problem arises in the modeling of potential flow in the process of electrochemical machining for shaping hard metals. For its physical background we refer to [15]. This free boundary problem has been subject to intensive studies, and the interested reader can consult [1], [2] and the references therein.

In order to apply the results obtained earlier, especially in Section 3, we transfer this free boundary to a QS.

Lemma 5.1. *$h \geq 0$ be continuous in U (open set) and subharmonic in $U \cap \{h>0\}$. Then h is subharmonic in U.*

We omit the simple proof of this lemma.

Theorem 5.2. *Let u and Ω be a solution to the free boundary problem mentioned above and D the given domain. Then $\partial\Omega \in QS(\mu)$ where $\mu = -\Delta u \geq 0$.*

Proof. By the strong maximum principle $1-u>0$ in $\Omega \setminus \overline{D}$ and vanishes identically on D. Hence, by Lemma 5.1

$$0 \leq \Delta(1-u) = -\Delta u = \mu,$$

in the sense of distributions. Now Theorem 1.1 gives that $\partial\Omega \in QS(\mu)$. □

Considering the free boundary (above) as a QS we can apply results from Section 3 to obtain:

Theorem 5.3. *Let Ω, D and u be as in Theorem 5.2 and set $D' = CH(D)$. Then the following is true.*

(1) *For any $x \in \partial\Omega \setminus D'$ the inward normal ray to $\partial\Omega$ at x meets D'.*

(2) *Let T be any supporting plane to D' and Ω' be the cap cut off by T and such that $D' \cap \Omega' = \emptyset$. Then $\partial\Omega' \cap \partial\Omega$ is a graph.*

(3) *Let $x \in \partial\Omega$ and $K_x = \{z \in \mathbf{R}^n : (z-x, y-x) \leq 0,\ \forall y \in D'\}$. Then $K_x \cap (\Omega \cup D') = \emptyset$.*

(4) *$|\nabla u| \neq 0$ in $\Omega \setminus D'$.*

(5) *$|\nabla u| \geq 1$ on ∂D provided $D = D'$.*

Proof. (1) follows from Theorem 3.4, (2) follows from Corollary 3.5, (3) follows from Corollary 3.6, (4) follows from Lemma 3.3 and (5) follows from Lemma 2.1. □

Concluding remarks

To the author's best knowledge QSs (at least for $n \geq 3$), unlike their counterparts QDs, have not been the subject of investigations. It depends partly on the discontinuity of the normal derivative of the single layer potential at the boundary points and partly on the lack of an established theory, as in the case of volume potentials, for the single layer potentials. Another basic problem is the applicability of variational inequalities to this kind of problems, which, for the moment, seems to be unfeasible. This, on the contrary, has very successfully been applied to the domain problems QD and both existence and uniqueness results have been obtained for a wide class of measures (see [19], [6]). However an approach based on minimization is possible where the minimizing functional is

$$\int |\nabla u|^2 - 2\mu u + \chi_{\{u>0\}}.$$

Here we need to assume that μ is a bounded function with compact support. This is the subject of a forthcoming paper by the author and B. Gustafsson. A third obstacle is that it is not clear apriori that $\partial\Omega$, considered as a QS, is C^2; so definition (0-2) does not make sense in general. Observe, to overcome this difficulty, that if $\partial\Omega$ is C^1 then

$$\frac{\partial u}{\partial \nu} = -|\nabla u| \text{ on } \partial\Omega,$$

where u is the AP and ν the outward normal to $\partial\Omega$. It seems that if we replace the third condition in (0-2) by $|\nabla u|=1$, then there is no need to presume any regularity for the surface. The very first problem emerging now, is to prove regularity of the surface in order to use definition (0-2), which seems to be the simplest one to work with. Observe also that if we assume that $\partial\Omega$ is C^1 and that the solution u (to (0-2)) has a C^2 extension to a neighbourhood of $\partial\Omega$, then it follows by ([14; Theorem 2]) that $\partial\Omega$ is real analytic.

As to the closing, we want to mention that in $\mathbf{R}^2$ QSs have been studied by B. Gustafsson [7] and H. Shapiro and C. Ullemar [23]. The technique employed by these authors is purely complex analytic and by no means applicable to $\mathbf{R}^n$ for $n\geq 3$.

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