

On a theorem of Baernstein

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Abstract In the paper [B2] Baernstein constructs a simply connected domain Ω in the plane for which the conformal mapping f of Ω into the unit disc Δ satisfies

$$\int_{\mathbf{R} \cap \Omega} |f'(z)|^p |dz| = \infty$$

for some $p \in (1, 2)$ where \mathbf{R} is the real line

This gives a counterexample to a conjecture stating that for any simply connected domain Ω in the plane all the above integrals are finite for any $1 < p < 2$

In this paper we give a conceptual proof of the basic estimate of Baernstein

1 Introduction

Let us consider the following problem. Let Ω be a simply connected domain and f be the conformal mapping from Ω into the unit disc Δ . Assume that L is a straight line which intersects the domain Ω , Hayman and Wu [HW] showed that for any configuration as above,

$$\int_{L \cap \Omega} |f'(z)| |dz| \leq C,$$

where C is a universal constant. Later Garnett, Gehring and Jones [GGJ] simplified Hayman and Wu's proof and gave an improved value for the constant C . Fernández, Heinonen and Martio in [FHM] gave another proof of the same result with a better constant $C = 4\pi^2$, and a conjecture is offered for the best constant. In the same paper they showed that there exists a positive number p between 1 and 2, such that

$$\int_{L \cap \Omega} |f'(z)|^p |dz| \leq C,$$

⁽¹⁾ I would like to thank Professor Albert Baernstein II for his helpful comments and suggestions concerning this work

where C and p are constants independent of the configuration. It is not difficult to see that the line L may be taken to be the real axis \mathbf{R} . The question is then for which exponents p is it true that $f'(z) \in L^p(\mathbf{R} \cap \Omega)$, for any f and Ω ? Taking Ω to be $\Delta \setminus (-1, 0]$ one sees that $f'(z) \in L^2(\mathbf{R} \cap \Omega)$ can fail. Baernstein [B1] conjectured that $f'(z) \in L^p(\mathbf{R} \cap \Omega)$ would be true for any $1 < p < 2$.

Baernstein in [B2] showed that his own conjecture is not true. He constructed a simply connected domain Ω such that if we consider the conformal mapping f from Ω into the unit disc, there exists a positive value p between 1 and 2, such that,

$$\int_{\mathbf{R} \cap \Omega} |f'(z)|^p |dz| = \infty$$

We pass to describe briefly the work done by Baernstein in [B2]. His domain Ω is the complement of an infinite tree T clustering to the real line. The fixed aperture at every branching of the tree T is $\frac{1}{3}\pi$.

Let us consider the domain $\Theta = \mathbf{C} \setminus ((-\infty, 1] \cup (0, e^{i\pi/3}])$, where $(0, e^{i\pi/3}]$ is the segment joining these two points. We are going to call $a = e^{i\pi/3}$, and consider the conformal mappings $F_i(z)$, $i=1, 2$; mapping Θ onto the domain $H = \mathbf{C} \setminus (-\infty, 0]$, such that $F_1(1) = 0$, $F_2(a) = 0$ and $\lim_{z \rightarrow \infty} |F_i(z)/z| = 1$, $i=1, 2$.

If we consider,

$$\gamma = \lim_{z \rightarrow 1} \left| \frac{F_1(z)}{z-1} \right|, \quad \beta = \lim_{z \rightarrow a} \left| \frac{F_2(z)}{z-a} \right|,$$

then Baernstein's theorem states that,

Theorem

$$\gamma^{1/2} + \beta^{1/2} > \sqrt{2}$$

In his paper Baernstein proves this result after numerical evidence given to him by Donald Marshall, who computed the values of γ and β using Trefethen's program [T], see also [H, p 422], for finding parameters for Schwarz-Christoffel transformations. He starts with the 4-place decimal approximation to the parameters given by the computer and confirm by Calculus the validity of the theorem, then mentions that it would be desirable to have a conceptual proof of the theorem.

In this paper, we present such a conceptual proof, in it our main tool is the method of the extremal metric. The idea of how to obtain lower bounds for γ and β using extremal metric was inspired by the paper of Jenkins and Oikawa [JO], in which they obtain a sharp version of Ahlfors' distortion theorem, and then use it to give simpler proofs of some well known results of Hayman.

2 Proof of the theorem

2 1 Estimating $\gamma = |F'_1(1)|$

Let ϱ be a small positive number and consider the discs $D_\varrho^{(1)} = \{z: |z-1| < \varrho\}$, and $D_{1/\varrho}^{(1)} = \{z: |z-1| < 1/\varrho\}$. Let $\Theta_\varrho^{(1)}$ be the doubly connected domain

$$\Theta_\varrho^{(1)} = (\Theta \cap D_{1/\varrho}^{(1)}) \setminus \bar{D}_\varrho^{(1)}$$

Let $H_\varrho^{(1)}$ be the image under $F_1(z)$ of $\Theta_\varrho^{(1)}$, by the normalization properties of the function $F_1(z)$, it is not difficult to show that for any positive ε , there exists a small positive $\varrho(\varepsilon)$ such that,

$$\{z: |z| < (1-\varepsilon)/\varrho(\varepsilon)\} \cap H \subset F_1(D_{1/\varrho(\varepsilon)}^{(1)}) \subset \{z: |z| < (1+\varepsilon)/\varrho(\varepsilon)\} \cap H$$

and

$$\{z: |z| < |F'_1(1)|(\varrho(\varepsilon) - \varepsilon)\} \cap H \subset F_1(D_{\varrho(\varepsilon)}^{(1)}) \subset \{z: |z| < |F'_1(1)|(\varrho(\varepsilon) + \varepsilon)\} \cap H$$

Consider now the module problem for the family of curves Γ joining $\partial D_{\varrho(\varepsilon)}^{(1)}$ with $\partial D_{1/\varrho(\varepsilon)}^{(1)}$ in $\Theta_{\varrho(\varepsilon)}^{(1)}$. Using the conformal invariance of the module and the comparison property for the modules, we have that

$$M(\Gamma, \Theta_{\varrho(\varepsilon)}^{(1)}) \leq \frac{2\pi}{\ln((1-\varepsilon)/\varrho(\varepsilon)(\varrho(\varepsilon) + \varepsilon)|F'_1(1)|)}$$

This provides us with an upper bound for the module, our goal is to obtain a lower bound for the same module. For this we consider the conformal mapping $\Phi(z) = \ln(z-1)$,

$$\Phi(z): \Theta_{\varrho(\varepsilon)}^{(1)} \rightarrow S_{\varrho(\varepsilon)}^{(1)},$$

where $S_{\varrho(\varepsilon)}^{(1)}$ is the quadrangle in the Figure 2 1a

Let $\tilde{\Gamma}$ be the family of curves in $S_{\varrho(\varepsilon)}^{(1)}$ joining the pair of sides opposite to the vertical sides. By the conformal invariance of the module we have the following equality

$$M(\Gamma, \Theta_{\varrho(\varepsilon)}^{(1)}) = M(\tilde{\Gamma}, S_{\varrho(\varepsilon)}^{(1)}),$$

where $\bar{\Gamma}$ is the family of curves in $S_{\varrho(\varepsilon)}^{(1)}$ joining the pair of vertical sides. Since the families of curves $\bar{\Gamma}$ and $\tilde{\Gamma}$ are conjugate in the quadrangle $S_{\varrho(\varepsilon)}^{(1)}$, we have that

$$M(\bar{\Gamma}, S_{\varrho(\varepsilon)}^{(1)}) = 1/M(\tilde{\Gamma}, S_{\varrho(\varepsilon)}^{(1)}),$$

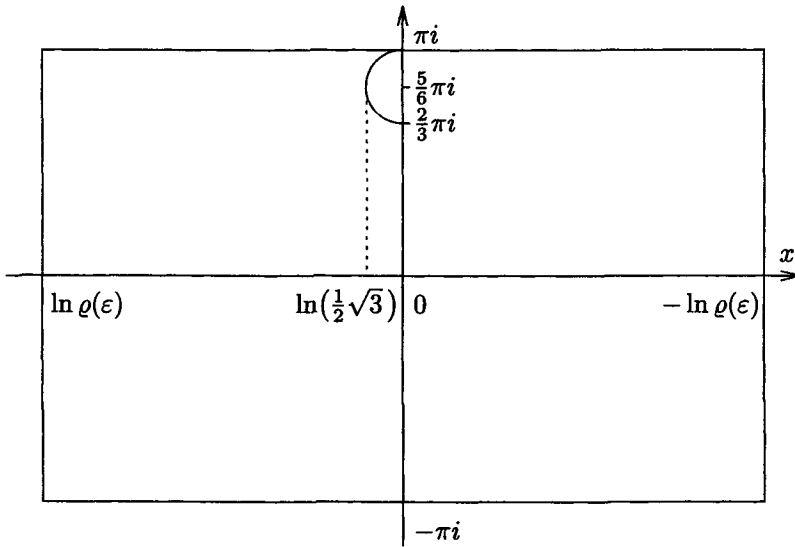


Figure 2 1a

therefore, to obtain a lower bound for $M(\Gamma, \Theta_{\rho(\varepsilon)}^{(1)})$, we need an upper bound of $M(\tilde{\Gamma}, S_{\rho(\varepsilon)}^{(1)})$

The idea of how to obtain the right upper bound for $M(\tilde{\Gamma}, S_{\rho(\varepsilon)}^{(1)})$ was suggested by [JO] For any value of x in the interval $\ln \rho(\varepsilon) < x < -\ln \rho(\varepsilon)$, let $\sigma(x)$ denote the maximal open subinterval of $\text{Re}\{z\}=x$ in $S_{\rho(\varepsilon)}^{(1)}$ such that the two components of $(S_{\rho(\varepsilon)}^{(1)} \setminus \sigma(x))$ have the two vertical sides as boundary components Let $\theta(x)$ denote the length of $\sigma(x)$, $\theta_1(x)$ the length of the part of the segment $\sigma(x)$ below the x -axis, and $\theta_2(x)$ the length of the part above the x axis As it can be easily seen, $\theta_1(x)=\pi$ for any x in the interval $\ln \rho(\varepsilon) < x < -\ln \rho(\varepsilon)$ For $\theta_2(x)$ we have

$$\theta_2(x) = \begin{cases} \pi, & \text{if } \ln \rho(\varepsilon) < x < \ln(\frac{1}{2}\sqrt{3}), \\ \theta_2(x), & \text{if } \ln(\frac{1}{2}\sqrt{3}) \leq x < 0, \\ \pi, & \text{if } 0 \leq x < -\ln \rho(\varepsilon) \end{cases}$$

Let the interval $[\ln(\frac{1}{2}\sqrt{3}), 0)$ be divided into n consecutive half closed subintervals $\Delta_j = [\ln(\frac{1}{2}\sqrt{3})(1-j/n), \ln(\frac{1}{2}\sqrt{3})(1-(j+1)/n))$, $j=0, \dots, n-1$ of equal length, and for each $j=0, \dots, n-1$ let

$$\theta_2^{(s)} = \min_{t \in \Delta_j} \theta_2(t),$$

and define for any $x \in [\ln(\frac{1}{2}\sqrt{3}), 0)$, $\theta_2^{(s)}(x) = \theta_2^{(s)}$ if $x \in \Delta_j$, $j=0, \dots, n-1$ It is clear

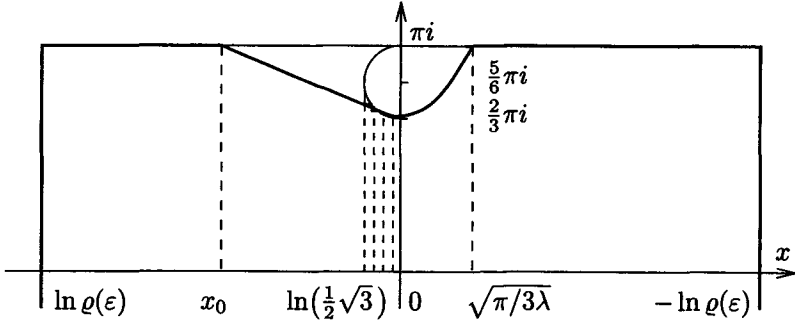


Figure 2 1b $x_0 = \ln(\frac{1}{2}\sqrt{3}) + \Delta_1 - 2\pi + \theta_2^{(s)}$

that such minimum exists and $\theta_2^{(s)}$ is a step function on the interval $[\ln(\frac{1}{2}\sqrt{3}), 0)$. At the right end point \bar{x} of any interval Δ_j the step function $\theta_2^{(s)}(x)$ has a negative jump, then we draw the ray given by $\bar{x} - \lambda, \theta_2^{(s)}(\bar{x}) + \lambda; \lambda \geq 0; j = 0, \dots, n-1$. The lower envelope of these rays and the locus $y = \theta_2^{(s)}(x)$ defines on the interval $[\ln(\frac{1}{2}\sqrt{3}), 0)$ a piecewise continuously differentiable function $\theta_2^{(t)}(x)$, which determines a decomposition of the interval into a finite number of subintervals on which the locus $y = \theta_2^{(t)}(x)$ has slope -1 or 0 . We define $\theta_2^{(t)}(x)$ in the interval $(\ln \rho(\epsilon), -\ln \rho(\epsilon))$ by

$$\theta_2^{(t)} = \begin{cases} \pi, & \ln \rho(\epsilon) < x \leq \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}) - \pi + \theta_2^{(s)}, \\ \theta_2^{(s)} - x + \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}), & \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}) - \pi + \theta_2^{(s)} < x < \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}), \\ \theta_2^{(t)}(x), & \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}) \leq x < 0, \\ \frac{2}{3}\pi + \lambda x^2, & 0 \leq x < \sqrt{\pi/3\lambda}, \\ \pi, & \sqrt{\pi/3\lambda} \leq x < -\ln \rho(\epsilon), \end{cases}$$

where λ is a positive parameter to be determined later. The domain determined by

$$-\theta_1(x) < y < \theta_2^{(t)}(x); \quad \ln \rho(\epsilon) < x < -\ln \rho(\epsilon),$$

becomes a quadrangle $Q_{\rho(\epsilon)}^{(1)}$. The part of $Q_{\rho(\epsilon)}^{(1)}$ below the x -axis is the same as for $S_{\rho(\epsilon)}^{(1)}$ and the part above the x -axis is as in Figure 2 1b.

If we let $\tilde{\Gamma}'$ be the family of curves $Q_{\rho(\epsilon)}^{(1)}$ joining the pair of sides complementary to the two vertical sides, we have that

$$M(\tilde{\Gamma}, S_{\rho(\epsilon)}^{(1)}) \leq M(\tilde{\Gamma}', Q_{\rho(\epsilon)}^{(1)})$$

Thus it is enough to obtain an upper bound for $M(\tilde{\Gamma}', Q_{\varrho(\varepsilon)}^{(1)})$. It is known that an upper bound for this module is given by the Dirichlet integral of any piecewise continuously differentiable function in $Q_{\varrho(\varepsilon)}^{(1)}$ taking the value 0 on the side given by $y = -\theta_1(x)$, and the value 1 on the side given by $y = \theta_2^{(t)}(x)$. A function like this is given by

$$u(x, y) = \frac{y + \theta_1(x)}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x) = \theta_1(x) + \theta_2^{(t)}(x)$. To estimate the Dirichlet integral of $u(x, y)$ we subdivide the domain $Q_{\varrho(\varepsilon)}^{(1)}$ into five pieces each corresponding to one of the following intervals in the x axis:

$$\begin{aligned} I &= (\ln \varrho(\varepsilon), \ln(\frac{1}{2}\sqrt{3})(1-n^{-1}) - \pi + \theta_2^{(s)}]; \\ II &= (\ln(\frac{1}{2}\sqrt{3})(1-n^{-1}) - \pi + \theta_2^{(s)}, \ln(\frac{1}{2}\sqrt{3})(1-n^{-1})); \\ III &= [\ln(\frac{1}{2}\sqrt{3})(1-n^{-1}), 0); \\ IV &= [0, \sqrt{\pi/3\lambda}); \\ V &= [\sqrt{\pi/3\lambda} - \ln \varrho(\varepsilon)] \end{aligned}$$

On the two pieces of the Dirichlet integral corresponding to the intervals I and V , the function $u(x, y) = (y + \pi)/2\pi$, and since when we take the limit as the number of subdivisions $n \rightarrow \infty$ then $\theta_2^{(s)} \rightarrow \frac{5}{6}\pi$, we have that

$$\begin{aligned} \iint_I + \iint_V |\nabla u(x, y)|^2 dx dy &= \frac{1}{2\pi} \ln\left(\frac{1}{\varrho(\varepsilon)}\right) + \frac{1}{2\pi} \left[\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right] \\ &\quad + \frac{1}{2\pi} \ln\left(\frac{1}{\varrho(\varepsilon)}\right) - \frac{1}{2\pi} \sqrt{\frac{\pi}{3\lambda}} \end{aligned}$$

It is not difficult to see that the Dirichlet integral corresponding to II after we let $n \rightarrow \infty$ tends to

$$\begin{aligned} &\int_{\ln(\sqrt{3}/2) - \pi/6}^{\ln(\sqrt{3}/2)} \int_{-\pi}^{5\pi/6 - (x - \ln(\sqrt{3}/2))} \left| \nabla \left(\frac{y + \pi}{11\pi/6 - (x - \ln(\sqrt{3}/2))} \right) \right|^2 dy dx \\ &= \int_{\ln(\sqrt{3}/2) - \pi/6}^{\ln(\sqrt{3}/2)} \int_{-\pi}^{5\pi/6 - x + \ln(\sqrt{3}/2)} \left[\frac{1}{(11\pi/6 - x + \ln(\sqrt{3}/2))^2} \right. \\ &\quad \left. + \frac{(y + \pi)^2}{(11\pi/6 - x + \ln(\sqrt{3}/2))^4} \right] dy dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{\ln(\sqrt{3}/2) - \pi/6}^{\ln(\sqrt{3}/2)} \frac{4}{3} \left[\frac{1}{11\pi/6 - x + \ln(\sqrt{3}/2)} \right] dx \\
 &= \frac{4}{3} \left[-\ln \left(\frac{11\pi}{6} - x + \ln \frac{\sqrt{3}}{2} \right) \right]_{\ln(\sqrt{3}/2) - \pi/6}^{\ln(\sqrt{3}/2)} \\
 &= \frac{4}{3} \ln \frac{12}{11}
 \end{aligned}$$

As for the piece corresponding to *IV*, we have that after some calculations

$$\iint_{IV} \left| \nabla \left(\frac{y + \pi}{5\pi/3 + \lambda x^2} \right) \right|^2 dx dy = \left[\sqrt{\frac{3}{5\pi}} \frac{1}{\sqrt{\lambda}} - \frac{4}{3} \sqrt{\frac{5\pi}{3}} \sqrt{\lambda} \right] \arctan \frac{1}{5} + \frac{4}{3} \sqrt{\frac{\pi}{3}} \sqrt{\lambda}$$

The estimate corresponding to *III* is more delicate, and we will treat it carefully

$$\begin{aligned}
 \iint_{III} |\nabla u(x, y)|^2 dx dy &= \int_{\ln(\sqrt{3}/2)}^0 \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3} \int_{\ln(\sqrt{3}/2)}^0 \frac{(\theta_1^{(t)})^2 - \theta_2^{(t)'} \theta_1' + (\theta_2^{(t)'})^2}{\theta^{(t)}(x)} dx \\
 &= \int_{\ln(\sqrt{3}/2)}^0 \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3} \sum_{j=0}^{n-1} \int_{\Omega_j} \frac{1}{\theta^{(t)}(x)} dx \\
 &= (i) + (ii),
 \end{aligned}$$

where Ω_j is the subinterval of Δ_j over which $\theta_2^{(t)'}(x)$ is equal to -1 . We proceed to estimate these two integrals (i) and (ii)

$$(i) = \int_{\ln(\sqrt{3}/2)}^0 \frac{dx}{\theta^{(t)}(x)} \leq -\frac{3}{5\pi} \ln \frac{\sqrt{3}}{2}$$

We estimate (ii) as follows,

$$\begin{aligned}
 (ii) &= \frac{1}{3} \sum_{j=0}^{n-1} \int_{\Omega_j} \frac{dx}{\theta^{(t)}(x)} = \frac{1}{3} \sum_{j=0}^{n-1} \int_{\Omega_j} \frac{1}{\pi + \theta_2^{(s)}(x) + \ln(\sqrt{3}/2)(1 - (j+1)/n) - x} dx \\
 &= \frac{1}{3} \sum_{j=0}^{n-1} \int_{\Omega_j} \frac{1}{\pi + \theta_2^{(s)}_j + \ln(\sqrt{3}/2)(1 - (j+1)/n) - x} dx \\
 &= \sum_{j=0}^{n-1} \frac{-1}{3} \left[\ln \left(\pi + \theta_2^{(s)}_j + \ln \left(\frac{\sqrt{3}}{2} \right) \left(1 - \frac{j+1}{n} \right) - x \right) \right]_{x_j^{(l)}}^{x_j^{(r)}},
 \end{aligned}$$

where $x_j^{(r)}$ is the right endpoint of the interval Ω_j and $x_j^{(l)}$ is the left endpoint
Hence, for n large enough,

$$\theta_{2j}^{(s)} + \ln\left(\frac{\sqrt{3}}{2}\right)\left(1 - \frac{j+1}{n}\right) - x_j^{(r)} = \theta_{2j+1}^{(s)} + \ln\left(\frac{\sqrt{3}}{2}\right)\left(1 - \frac{j+2}{n}\right) - x_{j+1}^{(l)}$$

for any $j=0, \dots, n-2$, thus

$$\begin{aligned} \sum_{j=0}^{n-1} \frac{-1}{3} \left[\ln\left(\pi + \theta_{2j}^{(s)} + \ln\left(\frac{\sqrt{3}}{2}\right)\left(1 - \frac{j+1}{n}\right) - x\right) \right]_{x_j^{(l)}}^{x_j^{(r)}} \\ = \frac{1}{3} \left[\ln\left(\pi + \theta_{2 \cdot 0}^{(s)} + \ln\left(\frac{\sqrt{3}}{2}\right)\left(1 - \frac{1}{n}\right) - x_0^{(l)}\right) - \ln\left(\pi + \theta_{2n}^{(s)} - x_n^{(r)}\right) \right], \end{aligned}$$

since $x_0^{(l)} = \ln\left(\frac{1}{2}\sqrt{3}\right)$, $\lim_{n \rightarrow \infty} \theta_{2 \cdot 0}^{(s)} = \frac{5}{6}\pi$, $x_n^{(r)} = 0$, and $\theta_{2n}^{(s)} = \frac{2}{3}\pi$, letting n go to ∞ , we obtain that

$$(ii) \rightarrow \frac{1}{3} \ln \frac{11}{10}$$

This completes all our estimates, putting all of them together, we obtain that

$$\begin{aligned} M(\tilde{\Gamma}', Q_{\rho(\varepsilon)}^{(1)}) \leq \frac{1}{2\pi} \ln\left(\frac{1}{\rho^2(\varepsilon)}\right) + \frac{1}{2\pi} \left[\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right] - \frac{1}{2\pi} \sqrt{\frac{\pi}{3\lambda}} + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} \\ + \left[\sqrt{\frac{3}{5\pi}} \frac{1}{\sqrt{\lambda}} - \frac{4}{3} \sqrt{\frac{5\pi}{3}} \sqrt{\lambda} \right] \arctan \frac{1}{5} + \frac{4}{3} \sqrt{\frac{\pi}{3}} \sqrt{\lambda} + \frac{1}{200} + \frac{1}{3} \ln \frac{11}{10} \end{aligned}$$

Let us call $G(\lambda)$ the expression on the right hand side of the above inequality involving the positive parameter λ ,

$$G(\lambda) = \left[\sqrt{\frac{3}{5\pi}} \frac{1}{\sqrt{\lambda}} - \frac{4}{3} \sqrt{\frac{5\pi}{3}} \sqrt{\lambda} \right] \arctan \frac{1}{5} + \frac{4}{3} \sqrt{\frac{\pi}{3}} \sqrt{\lambda} - \frac{1}{2\pi} \sqrt{\frac{\pi}{3}} \frac{1}{\sqrt{\lambda}},$$

and solve the equation $G(\lambda)=0$, hence

$$\left[\frac{4}{3} \sqrt{\frac{\pi}{3}} - \frac{4}{3} \sqrt{\frac{5\pi}{3}} \arctan \frac{1}{5} \right] \sqrt{\lambda} = \left[\frac{1}{2\pi} \sqrt{\frac{\pi}{3}} - \sqrt{\frac{3}{5\pi}} \arctan \frac{1}{5} \right] \frac{1}{\sqrt{\lambda}},$$

thus,

$$\lambda = \left(\frac{1}{2\pi} \sqrt{\frac{\pi}{3}} - \sqrt{\frac{3}{5\pi}} \arctan \frac{1}{5} \right) / \left(\frac{4}{3} \sqrt{\frac{\pi}{3}} - \frac{4}{3} \sqrt{\frac{5\pi}{3}} \arctan \frac{1}{5} \right) = 0.10050259$$

Choosing λ to be this value the expression on the right hand side of the inequality involving λ is equal to zero, therefore

$$M(\tilde{\Gamma}', Q_{\varrho(\varepsilon)}^{(1)}) \leq \frac{1}{2\pi} \ln \left(\frac{1}{\varrho^2(\varepsilon)} \right) + \frac{1}{2\pi} \left(\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + \frac{1}{200} + \frac{1}{3} \ln \frac{11}{10} + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2}$$

Putting together the two estimates of $M(\Gamma, \Theta_{\varrho(\varepsilon)}^{(1)})$ from above and from below, we obtain

$$\frac{2\pi}{\ln((1-\varepsilon)/\varrho(\varepsilon)(\varrho(\varepsilon)+\varepsilon)|F_1'(1))} \geq \left(\frac{1}{2\pi} \ln \left(\frac{1}{\varrho^2(\varepsilon)} \right) + \frac{1}{2\pi} \left(\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + \frac{1}{200} + \frac{1}{3} \ln \frac{11}{10} + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} \right)^{-1}$$

Taking inverses and exponentiating both sides, we obtain

$$\frac{1-\varepsilon}{(\varrho(\varepsilon)+\varepsilon)\varrho(\varepsilon)|F_1'(1)} \leq \frac{1}{\varrho^2(\varepsilon)} \exp \left\{ \left(\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + 2\pi \left(\frac{1}{200} + \frac{1}{3} \ln \frac{11}{10} + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} \right) \right\}$$

It is not difficult to see that choosing ε conveniently and letting $\varepsilon \rightarrow 0$ we get that

$$\gamma = |F_1'(1)| \geq \exp \left\{ - \left(\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) - 2\pi \left(\frac{1}{200} + \frac{1}{3} \ln \frac{11}{10} + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} \right) \right\} = \alpha$$

Hence,

$$\gamma^{1/2} = |F_1'(1)|^{1/2} \geq \alpha^{1/2} \geq 0.79249$$

2.2 Estimating $\beta = |F_2'(a)|$

Let ϱ be a small positive number, and consider the discs $D_\varrho^{(2)} = \{z: |z-a| < \varrho\}$, and $D_{1/\varrho}^{(2)} = \{z: |z-a| < 1/\varrho\}$. Let $\Theta_\varrho^{(2)}$ be the doubly connected domain

$$\Theta_\varrho^{(2)} = ((\Theta \cap D_{1/\varrho}^{(2)}) \setminus \bar{D}_\varrho^{(2)})$$

Let $H_\varrho^{(2)}$ be the image under $F_2(z)$ of $\Theta_\varrho^{(2)}$, by the same reason as in the first estimate 2.1, for any positive ε , there exists a small $\varrho(\varepsilon)$ positive such that;

$$\{z: |z| < |F_2'(a)|(\varrho(\varepsilon) - \varepsilon)\} \cap H \subset F_2(D_{\varrho(\varepsilon)}^{(2)}) \subset \{z: |z| < |F_2'(a)|(\varrho(\varepsilon) + \varepsilon)\} \cap H$$

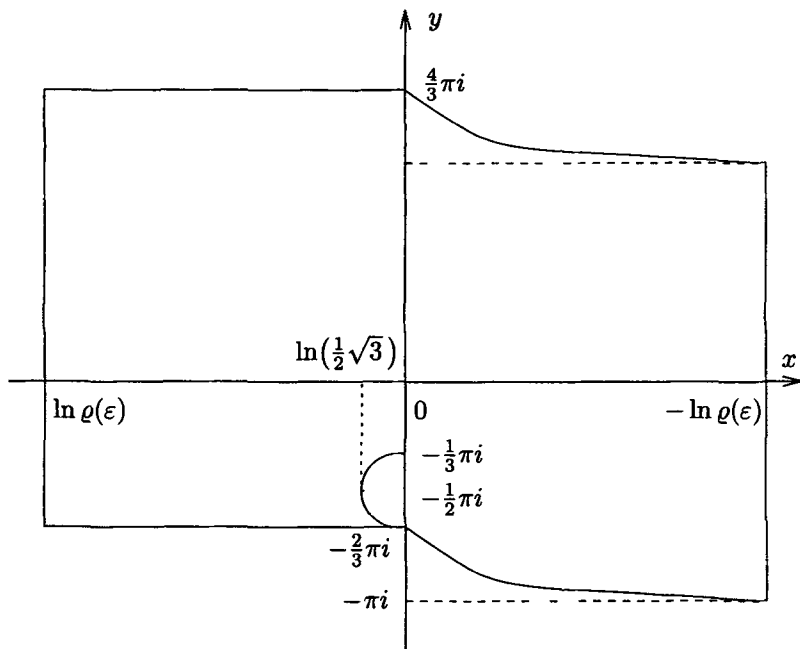


Figure 2 2a

and

$$\{z: |z| < (1-\varepsilon)/\rho(\varepsilon)\} \cap H \subset F_2(D_{1/\rho(\varepsilon)}^{(2)}) \subset \{z: |z| < (1+\varepsilon)/\rho(\varepsilon)\} \cap H$$

Considering now the module problem for the family of curves Γ joining $\partial D_{\rho(\varepsilon)}^{(2)}$ with $\partial D_{1/\rho(\varepsilon)}^{(2)}$ in $\Theta_{\rho(\varepsilon)}^{(2)}$, we have that

$$M(\Gamma, \Theta_{\rho(\varepsilon)}^{(2)}) \leq \frac{2\pi}{\ln((1-\varepsilon)/\rho(\varepsilon)(\rho(\varepsilon)+\varepsilon)|F_2'(a)|)}$$

Our goal is to obtain a lower bound for the above module. For this we consider the conformal mapping $\Psi(z) = \ln(z-a)$,

$$\Psi(z): \Theta_{\rho(\varepsilon)}^{(2)} \rightarrow S_{\rho(\varepsilon)}^{(2)},$$

where $S_{\rho(\varepsilon)}^{(2)}$ is again a quadrangle as in Figure 2 2a above

Let $\tilde{\Gamma}$ be the family of curves in $S_{\rho(\varepsilon)}^{(2)}$ joining the pair of sides opposite to the vertical sides of $S_{\rho(\varepsilon)}^{(2)}$. By the conformal invariance of the module we have that,

$$M(\Gamma, \Theta_{\rho(\varepsilon)}^{(2)}) = M(\tilde{\Gamma}, S_{\rho(\varepsilon)}^{(2)}),$$

where $\bar{\Gamma}$ is the family of curves in $S_{\varrho(\varepsilon)}^{(2)}$ joining the pair of vertical sides. Since the families of curves $\bar{\Gamma}$ and $\tilde{\Gamma}$ are conjugate in $S_{\varrho(\varepsilon)}^{(2)}$, we have that

$$M(\bar{\Gamma}, S_{\varrho(\varepsilon)}^{(2)}) = 1/M(\tilde{\Gamma}, S_{\varrho(\varepsilon)}^{(2)}),$$

thus, to obtain a lower bound for $M(\Gamma, \Theta_{\varrho(\varepsilon)}^{(2)})$, all we need is an upper bound for the module $M(\tilde{\Gamma}, S_{\varrho(\varepsilon)}^{(2)})$. To obtain this bound we proceed as in case 2.1. Our function $\theta_2(x)$ in this case is given by,

$$\theta_2(x) = \begin{cases} \frac{4}{3}\pi, & \text{if } \ln \varrho(\varepsilon) < x \leq 0, \\ \pi + \arctan \sqrt{3/(4e^{2x}-3)}, & \text{if } 0 \leq x < -\ln \varrho(\varepsilon) \end{cases}$$

We modify the function $\theta_1(x)$ in the same way we did with $\theta_2(x)$ in case 2.1 for values of x satisfying $\ln \varrho(\varepsilon) < x < 0$, and for values of x in the interval $[0, -\ln \varrho(\varepsilon)]$ we are going to modify $\theta_1(x)$ as follows;

$$\theta_1^{(t)}(x) = \begin{cases} \frac{1}{3}\pi + \delta x, & \text{if } 0 \leq x < \lambda, \\ \theta_1(x), & \text{if } \lambda \leq x < -\ln \varrho(\varepsilon) \end{cases}$$

Where $\delta > 0$ is a free parameter and λ is implicitly defined by the equation

$$\delta \lambda + \arctan \sqrt{\frac{3}{4e^{2\lambda}-3}} = \frac{2\pi}{3}$$

The domain determined by

$$-\theta_1^{(t)}(x) < y < \theta_2(x); \quad \ln \varrho(\varepsilon) < x < -\ln \varrho(\varepsilon),$$

becomes a quadrangle $Q_{\varrho(\varepsilon)}^{(2)}$ on assigning, as a pair of opposite sides, the segments

$$-\theta_1^{(t)}(\ln \varrho(\varepsilon)) < y < \theta_2(\ln \varrho(\varepsilon))$$

and

$$-\theta_1^{(t)}(-\ln \varrho(\varepsilon)) < y < \theta_2(-\ln \varrho(\varepsilon))$$

The part of $Q_{\varrho(\varepsilon)}^{(2)}$ above the x -axis is the same as for $S_{\varrho(\varepsilon)}^{(2)}$ and the part below the x -axis is as in Figure 2b

As in the case 2.1 we have that

$$M(\tilde{\Gamma}, S_{\varrho(\varepsilon)}^{(2)}) \leq M(\tilde{\Gamma}', Q_{\varrho(\varepsilon)}^{(2)}),$$

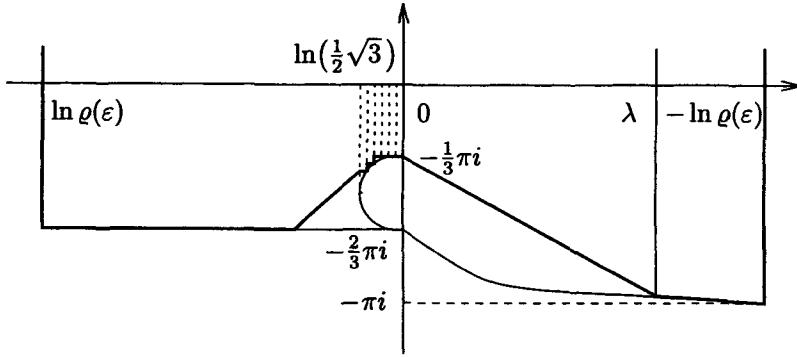


Figure 2 2b

where $\tilde{\Gamma}'$ is the family of curves in $Q_{\rho(\varepsilon)}^{(2)}$ joining the pair of sides complementary to the vertical sides. Thus, it is enough to obtain an upper bound for the module $M(\tilde{\Gamma}', Q_{\rho(\varepsilon)}^{(2)})$. An upper bound is given by the Dirichlet integral of the piecewise continuously differentiable function in $Q_{\rho(\varepsilon)}^{(2)}$

$$u(x, y) = \frac{\theta_2(x) - y}{\theta^{(t)}(x)},$$

where $\theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2(x)$. Hence,

$$\begin{aligned} \iint_{Q_{\rho(\varepsilon)}^{(2)}} |\nabla u(x, y)|^2 dx dy &= \iint_{Q_{\rho(\varepsilon)}^{(2)} \cap \{\text{Re}\{z\} \leq 0\}} |\nabla u(x, y)|^2 dx dy + \iint_{Q_{\rho(\varepsilon)}^{(2)} \cap \{\text{Re}\{z\} > 0\}} |\nabla u(x, y)|^2 dx dy \\ &= I + II \end{aligned}$$

The estimate of the integral I is the same as in case 2.1 because if we look at the left hand sides of the domains $Q_{\rho(\varepsilon)}^{(1)}$ and $Q_{\rho(\varepsilon)}^{(2)}$, they are the same up to a symmetry and a vertical translation. Thus,

$$I \leq \frac{1}{2\pi} \ln\left(\frac{1}{\rho(\varepsilon)}\right) + \frac{1}{3} \ln \frac{11}{10} + \frac{1}{200} + \frac{1}{2\pi} \left[\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right] + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2}$$

We pass to estimate the second integral II ,

$$\begin{aligned} II &= \iint_{Q_{\rho(\varepsilon)}^{(2)} \cap \{\text{Re}\{z\} > 0\}} |\nabla u(x, y)|^2 dx dy \\ &= \int_0^{-\ln \rho(\varepsilon)} \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3} \int_0^{-\ln \rho(\varepsilon)} \frac{\theta_1^{(t)'}(x)^2 - \theta_1^{(t)'}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)} dx, \end{aligned}$$

where

$$\theta_2(x) = \pi + \arctan \sqrt{\frac{3}{4e^{2x} - 3}} \quad \text{for } 0 < x < -\ln \rho(\varepsilon),$$

and

$$\theta^{(t)}(x) = \theta_1^{(t)}(x) + \theta_2(x) = 2\pi$$

for values of x such that $\lambda \leq x < -\ln \rho(\varepsilon)$ Thus II is equal to

$$\begin{aligned} II &= \int_0^\lambda \frac{1}{\theta^{(t)}(x)} dx + \frac{1}{3} \int_0^\lambda \frac{\theta_1^{(t)'}(x)^2 - \theta_1^{(t)'}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)} dx \\ &\quad + \frac{1}{2\pi} \ln \frac{1}{\rho(\varepsilon)} - \frac{\lambda}{2\pi} + \frac{1}{6\pi} \int_\lambda^{-\ln \rho(\varepsilon)} 3(\theta_2'(x))^2 dx \end{aligned}$$

Let us compute the last integral in the above equality,

$$\int_\lambda^{-\ln \rho(\varepsilon)} (\theta_2'(x))^2 dx = 3 \int_\lambda^{-\ln \rho(\varepsilon)} \frac{dx}{4e^{2x} - 3} = \frac{1}{2} \ln \left(\frac{4 - 3\rho^2(\varepsilon)}{4 - 3e^{-2\lambda}} \right)$$

It remains to estimate

$$\begin{aligned} &\int_0^\lambda \frac{dx}{\theta^{(t)}(x)} + \frac{1}{3} \int_0^\lambda \frac{\theta_1^{(t)'}(x)^2 - \theta_1^{(t)'}(x)\theta_2'(x) + \theta_2'(x)^2}{\theta^{(t)}(x)} dx \\ &= \int_0^\lambda \frac{1 + \frac{1}{3}\delta^2 + 1/(4e^{2x} - 3) + \frac{1}{3}\delta \sqrt{3/(4e^{2x} - 3)}}{\frac{4}{3}\pi + \delta x + \arctan \sqrt{3/(4e^{2x} - 3)}} dx \\ &= \int_0^\lambda \frac{1 + \frac{2}{3}\delta^2 + 1/(4e^{2x} - 3)}{\frac{4}{3}\pi + \delta x + \arctan \sqrt{3/(4e^{2x} - 3)}} dx \\ &\quad - \frac{1}{3}\delta \left[\ln \left(\delta x + \frac{4}{3}\pi + \arctan \sqrt{3/(4e^{2x} - 3)} \right) \right]_0^\lambda \end{aligned}$$

The second term in the formula above is equal to $\frac{1}{3}\delta \ln \frac{6}{5}$ Thus, to complete our estimate, our final goal is to find a suitable bound for the following integral

$$\int_0^\lambda \left[\frac{1 + \frac{2}{3}\delta^2 + 1/(4e^{2x} - 3)}{\frac{4}{3}\pi + \delta x + \arctan \sqrt{3/(4e^{2x} - 3)}} - \frac{1}{2\pi} \right] dx,$$

where $\delta\lambda + \arctan \sqrt{3/(4e^{2\lambda} - 3)} = \frac{2}{3}\pi$ Our first observation is that

$$\frac{4\pi}{3} + \delta x + \arctan \sqrt{\frac{3}{(4e^{2x} - 3)}} \geq 2\pi - \arctan \sqrt{\frac{3}{(4e^{2x} - 3)}}$$

for $0 < x < \lambda$ Therefore it is enough to estimate the integral

$$\int_0^\lambda \left[\frac{1 + 1/(4e^{2x} - 3)}{2\pi - \arctan \sqrt{3/(4e^{2x} - 3)}} - \frac{1}{2\pi} \right] dx$$

$$\leq \int_0^\infty \frac{2\pi/(4e^{2x} - 3) + \arctan \sqrt{3/(4e^{2x} - 3)}}{2\pi(2\pi - \arctan \sqrt{3/(4e^{2x} - 3)})} dx$$

In the above integral we have dropped the term $\frac{2}{3}\delta^2$ in the numerator, since when λ goes to ∞ then δ goes to 0 as $2\pi/3\lambda$, thus

$$\int_0^\lambda \frac{\frac{2}{3}\delta^2}{\frac{4}{3}\pi + \delta x + \arctan \sqrt{3/(4e^{2x} - 3)}} dx \leq \frac{\delta^2 \lambda}{2\pi},$$

and this goes to $\frac{1}{3}\delta$ as λ goes to ∞ , hence the term in the integral corresponding to $\frac{2}{3}\delta^2$ can be made as small as we please Using the change of variable $u = \arctan \sqrt{3/(4e^{2x} - 3)}$, the above integral becomes

$$- \int_{\pi/3}^0 \frac{\frac{2}{3}\pi(\tan u)^2 + u}{2\pi(2\pi - u)} \frac{du}{\tan u} = \frac{1}{2\pi} \int_0^{\pi/3} \frac{2\pi \tan u}{3(2\pi - u)} du + \frac{1}{2\pi} \int_0^{\pi/3} \frac{u}{(2\pi - u)} \cot u du$$

$$= A + B$$

Standard numerical integration methods give us the following estimates from above for the two integrals A and B :

$$A \leq \frac{1}{3}(0.126) = 0.042$$

and

$$B \leq \frac{1}{2\pi}(0.158) \leq 0.0252$$

Putting all these estimates together and letting δ go to 0, we obtain that

$$M(\tilde{\Gamma}', Q_{\varrho(\varepsilon)}^{(2)}) \leq \frac{1}{2\pi} \ln \frac{1}{\varrho^2(\varepsilon)} + \frac{1}{3} \ln \frac{11}{10} + \frac{1}{200} + \frac{1}{2\pi} \left[\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right]$$

$$+ \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} + \frac{1}{4\pi} \ln \left(\frac{4 - 3\varrho^2(\varepsilon)}{4} \right) + 0.042 + 0.0252$$

Putting together the estimates of $M(\Gamma, \Theta_{\varrho(\varepsilon)}^{(1)})$ from above and from below, taking inverses and exponentiating, as we did in case 2.1, and letting ε go to 0, we have that

$$\beta = |F_2'(a)|$$

$$\geq \exp \left(-2\pi \left\{ \frac{1}{3} \ln \frac{11}{10} + \frac{1}{200} + \frac{1}{2\pi} \left(\ln \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + \frac{4}{3} \ln \frac{12}{11} - \frac{3}{5\pi} \ln \frac{\sqrt{3}}{2} + 0.0672 \right\} \right)$$

$$= \eta$$

Hence,

$$\beta^{1/2} = |F'_2(a)|^{1/2} \geq \eta^{1/2} \geq 0.6403$$

Therefore putting together the two estimates 2.1 and 2.2, we have that

$$|F'_2(a)|^{1/2} + |F'_1(1)|^{1/2} = \beta^{1/2} + \gamma^{1/2} > 0.79249 + 0.6403 = 1.43279 > \sqrt{2},$$

and this proves the theorem \square

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