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Commutators and interpolation methods

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1. Introduction

Recently R. Rochberg, G. Weiss, B. Jawerth, N. J. Kalton, M. Cwikel and M. Milman (cf. [RW], [JRW], [CJM], [CJMR] and [K]) have obtained interpolation theorems for commutators of bounded linear operators and certain operators Ω, generally unbounded and nonlinear, associated with an interpolation method for both the complex and the real case, with interesting applications to classical analysis.

In [RW] Rochberg and Weiss developed the study of these commutators for spaces obtained by complex interpolation. A similar analysis was carried out for the real method by Jawerth, Rochberg and Weiss in [JRW], where they noticed that, although there are strong analogies between the two cases, the details are very different.

The purpose of this paper is to set up a unified method of both theories. Our analysis leads to a simple approach to commutator theorems, giving the precise rôle that cancellation plays in the theory.

We set a general frame by considering pairs of interpolation methods with some nice "compatibility conditions" having in mind the two basic examples of [RW] and [JRW]:

In the complex case, the pair of interpolation methods is associated to the functionals δ_θ and δ'_θ (cf. [S] or [CC]) and the Ω-operator is defined by $\Omega a = h'_a(\theta)$, where h_a is "almost optimal" among all f such that $f(\theta)=a$.

Similarly, in the real J-method, the corresponding couple of functionals is

$$\int_0^\infty u(t)\,\frac{dt}{t} \quad \text{and} \quad \int_0^\infty (\log t) u(t)\,\frac{dt}{t},$$

and $\Omega a = \int_0^\infty (\log t) h_a(t)\,(dt/t)$, with $\int_0^\infty h_a(t)\,(dt/t) = a$.

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Our method can be also applied to obtain a unified approach to the higher order commutators of [R] and [M]. This will be the subject of the forthcoming paper [CCS].

The paper is organized as follows. In Section 2 we give a general construction of interpolation functors, that was first introduced by V. Williams in [W], and present some interesting examples of functors of this type.

In Section 3 we define the Ω-operator for functors constructed as in Section 2, we give a simple proof of the commutator theorem and characterize the spaces $\mathrm{Dom}(\Omega)$ and $\mathrm{Rang}(\Omega)$. These results answer Questions 1 and 8 in [CJMR].

Section 4 deals with the particular case of the complex method of Calderón, and Sections 5 and 6 with the J- and K-method, respectively.

Finally, in Section 7 we answer Question 6 in [CJMR] giving a precise description of the twisted direct sums.

For undefined notation and standard definitions we refer to [BL].

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2. The interpolators

The following definition should be compared with the one given in [W].

Definition 2.1. By an interpolator Φ over H, we mean a functor $H_\Phi=H$ from compatible couples $\bar{A}=(A_0,A_1)$ of Banach spaces to normed spaces $H(\bar{A})$, with the property that there exists a bounded linear operator

$$\Phi_{\bar{A}}\colon H(\bar{A})\to\Sigma(\bar{A})=A_0+A_1$$

for every couple $\bar{A}$, such that

$$T\circ\Phi_{\bar{A}}=\Phi_{\bar{B}}\circ H(T) \tag{1}$$

for every linear bounded $T\colon\bar{A}\to\bar{B}$.

We usually set $\bar{A}_\Phi=\Phi_{\bar{A}}(H(\bar{A}))$, with the norm

$$\|a\|_\Phi=\inf\{\|f\|_{H(\bar{A})}\,;\,\Phi_{\bar{A}}(f)=a\},$$

so that $\bar{A}_\Phi\hookrightarrow\Sigma(\bar{A})$, with norm $\le\|\Phi_{\bar{A}}\|$. If $H(\bar{A})$ is complete, $\bar{A}_\Phi$ is a Banach space.

If there exists a one to one bounded linear operator $\varphi\colon\Delta(\bar{A})\to H(\bar{A})$ such that

$$\Phi_{\bar{A}}\circ\varphi=\mathrm{id}_{\Delta(\bar{A})}, \tag{2}$$

then we have $\Delta(\bar{A})\hookrightarrow\bar{A}_\Phi$, with $\|a\|_\Phi\le\|\varphi(a)\|_{H(\bar{A})}\le\|\varphi\|\,\|a\|_\Delta$. Property (1) implies that $\bar{A}\to\bar{A}_\Phi$ is an interpolation method such that, for $T\colon\bar{A}\to\bar{B}$,

$$\|T\|_{\bar{A}_\Phi,\bar{B}_\Phi}\le\|H(T)\|_{H(\bar{A}),H(\bar{B})}.$$

Examples

(I) *First complex method.* This method is associated to the interpolator

$$\Phi_{\bar{A}}(f)=\delta_\theta(f)=f(\theta),$$

with $H(\bar{A})=\mathcal{F}(\bar{A})$, the Banach space of vector-valued analytic functions on the strip $\mathbf{S}$ considered by Calderón in [C], and $H(T)f=T\circ f$.

In this case, $\|H(T)\|\le\|T\|_{\bar{A},\bar{B}}$ (the norm as a bounded operator $T:\bar{A}\to\bar{B}$), and $\|\Phi_{\bar{A}}\|=1$. Moreover we have $\varphi:\Delta(\bar{A})\to H(\bar{A})$, defined by $\varphi(a)=\mathrm{e}^{(z-\theta)^2}a$, which satisfies (2), with $\|\varphi\|\le\mathrm{e}$.

If we change $\delta_\theta(f)$ by $\delta_\theta^{(n)}(f)=f^{(n)}(\theta)$ with the same spaces $H(\bar{A})=\mathcal{F}(\bar{A})$, we get the Lions–Schechter method of derivatives (see [S]).

(II) *The J-method.* Now we take

$$H(\bar{A})=\{u:\mathbf{R}^+\to\Delta(\bar{A})\text{ measurable};\,\Phi_{\theta,p}(J(t,u(t)))<\infty\},$$

where $J(t,a)=\max(\|a\|_{A_0},t\|a\|_{A_1})$, if $a\in\Delta(\bar{A})$ and

$$\Phi_{\theta,p}(\gamma(t))=\left(\int_0^\infty(t^{-\theta}\gamma(t))^p\,\frac{dt}{t}\right)^{1/p},$$

$0<\theta<1$, $1\le p\le\infty$. With the norm $\|u\|_{H(\bar{A})}=\Phi_{\theta,p}(J(t,u(t)))$, it is a Banach space.

For every bounded linear $T:\bar{A}\to\bar{B}$ we define $H(T)u=T\circ u$ and then $\|H(T)\|\le\|T\|_{\bar{A},\bar{B}}$. Now for $u\in H(\bar{A})$,

$$\Phi_{\bar{A}}(u)=\int_0^\infty u(t)\frac{dt}{t}\in\Sigma(\bar{A}),$$

and $\Phi_{\bar{A}}:H(\bar{A})\to\Sigma(\bar{A})$ is bounded:

$$\begin{aligned}\|\Phi_{\bar{A}}(u)\|_{\Sigma(\bar{A})}&\le\int_0^1\|u(t)\|_{A_0}\,\frac{dt}{t}+\int_1^\infty\|u(t)\|_{A_1}\,\frac{dt}{t}\\&\le\int_0^1 J(t,u(t))\,\frac{dt}{t}+\int_1^\infty t^{-1}J(t,u(t))\,\frac{dt}{t}\\&\le\left(\int_0^1\left(\frac{J(t,u(t))}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}\left(\int_0^1 t^{\theta p'}\,\frac{dt}{t}\right)^{1/p'}\\&\quad+\left(\int_1^\infty\left(\frac{J(t,u(t))}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}\left(\int_1^\infty\frac{1}{t^{(1-\theta)p'}}\,\frac{dt}{t}\right)^{1/p'}\\&\le C\|u\|_{H(\bar{A})}.\end{aligned}$$

Finally, if we set $\varphi\colon\Delta(\bar{A})\to H(\bar{A})$,

$$\varphi(a)(t)=\begin{cases} a, & t\in[1,e],\\ 0, & \text{otherwise},\end{cases}$$

then, $\Phi_{\bar{A}}(\varphi(a))=\int_1^e a\,(dt/t)=a$. Obviously $\bar{A}_\Phi=\bar{A}_{\theta,p;J}$.

(III) *The K-method.* Let $H(\bar{A})$ be the vector space of all measurable functions $(a_0,a_1)\colon\mathbf{R}^+\to A_0\times A_1$ such that $a_0(t)+a_1(t)$ is constant and

$$\|(a_0,a_1)\|_{H(\bar{A})}=\left(\int_0^\infty\left(\frac{\|a_0(t)\|_0+t\|a_1(t)\|_1}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}<\infty,$$

with $0<\theta<1$ and $1\le p\le\infty$. As before $H(\bar{A})$ is a Banach space, and if we define $H(T)(a_0,a_1)=(T\circ a_0,T\circ a_1)$, we obtain $H(T)\colon H(\bar{A})\to H(\bar{B})$, with $\|H(T)\|\le\|T\|_{\bar{A},\bar{B}}$. Now, if we consider $\Phi_{\bar{A}}(a_0,a_1)=a_0(t)+a_1(t)$, we obtain a linear operator $\Phi_{\bar{A}}\colon H(\bar{A})\to\Sigma(\bar{A})$ satisfying property (1):

$$(T\circ\Phi_{\bar{A}})(a_0,a_1)=T(a_0(t))+T(a_1(t))=\Phi_{\bar{B}}(T\circ a_0,T\circ a_1).$$

This operator is bounded:

$$\begin{aligned}\|\Phi_{\bar{A}}(a_0,a_1)\|_{\Sigma(\bar{A})}=\|a_0(t)+a_1(t)\|_{\Sigma(\bar{A})}&\le C\int_1^2(\|a_0(t)\|_0+\|a_1(t)\|_1)\,\frac{dt}{t}\\ &\le C'\int_1^2(\|a_0(t)\|_0+t\|a_1(t)\|_1)\,\frac{dt}{t}\le C'\|(a_0,a_1)\|_{H(\bar{A})}.\end{aligned}$$

For any $a\in\Delta(\bar{A})$ we define $\varphi(a)=(a_0(t),a_1(t))$, with $a_0(t)=\chi_{[1,\infty)}(t)a$ and $a_1(t)=\chi_{(0,1)}(t)a$. Then $\varphi\colon\Delta(\bar{A})\to H(\bar{A})$, it is one to one, $\|\varphi(a)\|_{H(\bar{A})}\le C\|a\|_{\Delta(\bar{A})}$ and $\Phi_{\bar{A}}(\varphi(a))=a$.

It is easily seen that $\bar{A}_\Phi=\bar{A}_{\theta,p;K}$, with equality of norms. Interpolation methods with function parameters (see [G]) are obtained in the same way.

(IV) *The minimal method.* For a given couple $\bar{Z}$ and a fixed intermediate space Z,

$$\Delta(\bar{Z})\hookrightarrow Z\hookrightarrow\Sigma(\bar{Z}),$$

the corresponding minimal method of Aronszajn–Gagliardo (see [AG] and [J]) is associated to the interpolator $\Phi_{\bar{A}}\colon H(\bar{A})\to\Sigma(\bar{A})$ defined by

$$\Phi_{\bar{A}}(\{z_S\}_{S\in\mathcal{U}})=\sum_{S\in\mathcal{U}}S(z_S),$$

over the Banach space

$$H(\bar{A})=l^1(Z;\mathcal{U}(\bar{A}))=\left\{\bar{z}=\{z_S\}_{S\in\mathcal{U}(\bar{A})}\,;z_S\in Z,\ \sum_S\|z_S\|_Z<\infty\right\},$$

with

$$\mathcal{U}(\bar{A})=\{S:\bar{Z}\to\bar{A}\text{ bounded and linear};\ \|S\|_{\bar{Z},\bar{A}}\le 1\}$$

and

$$\|\bar{z}\|_{H(\bar{A})}=\sum_S\|z_S\|_Z.$$

Obviously $\Phi_{\bar{A}}(\bar{z})$ is well defined for all $\bar{z}\in H(\bar{A})$ and $\Phi_{\bar{A}}$ is bounded:

$$\|\Phi_{\bar{A}}(\bar{z})\|_\Sigma\le\sum_S\|S\|_{Z,\Sigma}\|z_S\|_\Sigma\le C\|\bar{z}\|_{H(\bar{A})}.$$

We consider $\varphi:\Delta(\bar{A})\hookrightarrow H(\bar{A})$ such that $\varphi(a)=\{\delta^S_{\mathrm{id}}a\}_{S\in\mathcal{U}}$, where id: $\Delta(\bar{A})\hookrightarrow A$ is the embedding operator. Then

$$\|\varphi(a)\|_{H(\bar{A})}\le\|\,\mathrm{id}\,\|_{\Delta,A}\quad\text{and}\quad\Phi_{\bar{A}}(\varphi(a))=\mathrm{id}(a)=a.$$

3. The Ω-operator

Definition 3.1. Let (Φ,Ψ) be a pair of interpolators on the same spaces $H(\bar{A})$; i.e., such that $H_\Phi=H_\Psi=H$. We say that (Φ,Ψ) is compatible if

$$\Psi_{\bar{A}}(\operatorname{Ker}\Phi_{\bar{A}})=\operatorname{Im}\Phi_{\bar{A}},\tag{3}$$

for every couple $\bar{A}$, with equivalent norms, in the sense that there exists a constant $C=C(\bar{A})>0$ with the following properties:

(3a) If $g\in H(\bar{A})$ and $\Phi_{\bar{A}}(g)=0$, then $\Psi_{\bar{A}}(g)=\Phi_{\bar{A}}(f)$, for some $f\in H(\bar{A})$ such that $\|f\|_{H(\bar{A})}\le C\|g\|_{H(\bar{A})}$.

(3b) If $f\in H(\bar{A})$, then $\Phi_{\bar{A}}(f)=\Psi_{\bar{A}}(g)$, for some $g\in H(\bar{A})$ such that $\Phi_{\bar{A}}(g)=0$ and $\|g\|_{H(\bar{A})}\le C\|f\|_{H(\bar{A})}$.

Remark 3.2. Sometimes (3b) is not needed. If instead of condition (3) we only have $\Psi_{\bar{A}}(\operatorname{Ker}\Phi_{\bar{A}})\subset\operatorname{Im}\Phi_{\bar{A}}$, with property (3a), we say that (Φ,Ψ) is almost compatible.

Let $C>1$ be a fixed constant. We fix an almost optimal election for the interpolator Φ, which is a mapping

$$a\in\bar{A}_\Phi\mapsto h_a\in H(\bar{A}),$$

such that $\Phi_{\bar{A}}(h_a)=a$ and $\|h_a\|_{H(\bar{A})}\le C\|a\|_{\bar{A}_\Phi}$, for every couple $\bar{A}$. We can always assume that $h_{\lambda a}=\lambda h_a$.

Definition 3.3. Given (Φ,Ψ) a pair of interpolators, we define the Ω-operator

$$\Omega_{\bar{A}}: a\in \bar{A}_\Phi \to \Psi_{\bar{A}}(h_a)\in \bar{A}_\Psi,$$

with h_a as above.

Given any bounded linear $T:\bar{A}\to\overline{B}$, we define the commutator

$$[T,\Omega]=T\circ\Omega_{\bar{A}}-\Omega_{\overline{B}}\circ T:\bar{A}_\Phi\to\Sigma(\overline{B}).$$

Observe that $\Omega_{\bar{A}}$ need not be a linear operator on $\bar{A}_\Phi$. With these notations, for any pair (Φ,Ψ) of interpolators on the same spaces $H(\bar{A})$ we have that

$$[T,\Omega]:\bar{A}_\Phi\to\overline{B}_\Psi,$$

and it is a bounded operator, since

$$\|T\Omega_{\bar{A}}a\|_\Psi=\|T\Psi_{\bar{A}}(h_a)\|_\Psi=\|\Psi_{\overline{B}}H(T)h_a\|_\Psi\le C\|H(T)\|\,\|h_a\|_{H(\bar{A})}\le C'\|a\|_\Phi,$$

and

$$\|\Omega_{\overline{B}}Ta\|_\Psi=\|\Psi_{\overline{B}}(h_{Ta})\|_\Psi\le C\|h_{Ta}\|_{H(\bar{A})}\le C'\|Ta\|_\Phi\le C''\|a\|_\Phi.$$

Theorem 3.4. (Commutator theorem) *If (Φ,Ψ) is an almost compatible pair of interpolators, then $[T,\Omega]:\bar{A}_\Phi\to\overline{B}_\Phi$, and it is bounded.*

Proof. Let $a=\Phi_{\bar{A}}(h_a)\in\bar{A}_\Phi$. Then we get

$$T\Omega_{\bar{A}}a=T\Psi_{\bar{A}}h_a=\Psi_{\overline{B}}H(T)h_a,\ \Omega_{\overline{B}}Ta=\Psi_{\overline{B}}h_{Ta},\ \text{and}\ [T,\Omega]a=\Psi_{\overline{B}}(H(T)h_a-h_{Ta}),$$

with $\Phi_{\overline{B}}(H(T)h_a-h_{Ta})=Ta-Ta=0$. Now, by hypothesis (see Remark 3.2) we get $[T,\Omega]a=\Phi_{\overline{B}}(h)\in\overline{B}_\Phi$, with $\|h\|_{H(\overline{B})}\le C\|H(T)h_a-h_{Ta}\|_{H(\overline{B})}$, and hence

$$\|[T,\Omega]a\|_\Phi\le\|h\|_{H(\overline{B})}\le C(\|H(T)\|\,\|h_a\|_{H(\bar{A})}+c\|Ta\|_\Phi)\le C'\|a\|_\Phi.\quad\square$$

Corollary 3.5. *Let $\widetilde{\Omega}$ be the Ω-operator corresponding to a second almost optimal election $a\mapsto\tilde{h}_a$. We have:*

(a) *For any (Φ,Ψ) (on the same spaces $H(\bar{A})$), $\Omega_{\bar{A}}-\widetilde{\Omega}_{\bar{A}}:\bar{A}_\Phi\to\bar{A}_\Psi$ is bounded.*

(b) *If (Φ,Ψ) is almost compatible, then $\Omega_{\bar{A}}-\widetilde{\Omega}_{\bar{A}}:\bar{A}_\Phi\to\bar{A}_\Phi$ is bounded.*

Proof. (a) $\|(\Omega_{\bar{A}}-\widetilde{\Omega}_{\bar{A}})a\|_\Psi=\|\Psi(h_a-\tilde{h}_a)\|_\Psi\le\|h_a-\tilde{h}_a\|_{H(\bar{A})}\le 2C\|a\|_\Phi$.

(b) We have $\Phi(h_a-\tilde{h}_a)=0$, thus $(\Omega_{\bar{A}}-\widetilde{\Omega}_{\bar{A}})a=\Psi(h_a-\tilde{h}_a)=\Phi(g)\in\bar{A}_\Phi$, with

$$\|(\Omega_{\bar{A}}-\widetilde{\Omega}_{\bar{A}})a\|_\Phi\le\|g\|_{H(\bar{A})}\le C\|h_a-\tilde{h}_a\|_{H(\bar{A})}\le C'\|a\|_\Phi.\quad\square$$

Definition 3.6. On the set $\mathrm{Dom}(\Omega_{\bar{A}})=\{a\in\bar{A}_\Phi;\Omega_{\bar{A}}a\in\bar{A}_\Phi\}$, we define

$$\|a\|_D=\|a\|_\Phi+\|\Omega_{\bar{A}}a\|_\Phi.$$

Observe that $\|a\|_D>0$ if $a\neq 0$, and $\|\lambda a\|_D=|\lambda|\,\|a\|_D$.

Lemma 3.7. *If (Φ,Ψ) is almost compatible, then for $a,b\in\bar{A}_\Phi$,*

$$\Omega_{\bar{A}}(a+b)-\Omega_{\bar{A}}a-\Omega_{\bar{A}}b\in\bar{A}_\Phi,$$

and there is a constant $C=C_A$ such that

$$\|\Omega_{\bar{A}}(a+b)-\Omega_{\bar{A}}a-\Omega_{\bar{A}}b\|_\Phi\le C(\|a\|_\Phi+\|b\|_\Phi).$$

Proof. We have $\Omega_{\bar{A}}(a+b)-\Omega_{\bar{A}}a-\Omega_{\bar{A}}b=\Psi(h_{a+b}-h_a-h_b)$, with $\Phi(h_{a+b}-h_a-h_b)=0$. Hence, $\Psi(h_{a+b}-h_a-h_b)=\Phi(f)\in\bar{A}_\Phi$, and

$$\begin{aligned}\|\Omega_{\bar{A}}(a+b)-\Omega_{\bar{A}}a-\Omega_{\bar{A}}b\|_\Phi&\le\|f\|_{H(\bar{A})}\le C\|h_{a+b}-h_a-h_b\|_{H(\bar{A})}\\&\le C'(\|a\|_\Phi+\|b\|_\Phi).\quad\square\end{aligned}$$

Theorem 3.8. (a) *If (Φ,Ψ) is almost compatible, then $\mathrm{Dom}(\Omega_{\bar{A}})$ is a quasi-normed linear space and $\mathrm{Dom}(\Omega_{\bar{A}})=\mathrm{Dom}(\widetilde{\Omega}_{\bar{A}})$ (with equivalent quasi-norms), for a second almost optimal election (in fact $\mathrm{Dom}(\Omega_{\bar{A}})=\Phi_{\bar{A}}(\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi))$).*

Also, $\bar{A}\to\mathrm{Dom}(\Omega_{\bar{A}})$ is an interpolation method (i.e., for any $T:\bar{A}\to\bar{B}$, $T:\mathrm{Dom}(\Omega_{\bar{A}})\to\mathrm{Dom}(\Omega_{\bar{B}})$ is bounded).

(b) *If (Φ,Ψ) is compatible, then*

$$\mathrm{Dom}(\Omega_{\bar{A}})=\{\Phi_{\bar{A}}(f)\,;f\in H(\bar{A}),\Psi_{\bar{A}}(f)=0\}=\Phi_{\bar{A}}(\mathrm{Ker}\,\Psi_{\bar{A}}),$$

with $\|x\|_D\approx\inf\{\|f\|_{H(\bar{A})};x=\Phi_{\bar{A}}(f),\Psi_{\bar{A}}(f)=0\}$.

Proof. (a) If $a,b\in\mathrm{Dom}(\Omega_{\bar{A}})$, from the lemma we obtain:

$$\begin{aligned}\|a+b\|_D&=\|a+b\|_\Phi+\|\Omega_{\bar{A}}(a+b)\|_\Phi\\&\le\|a\|_\Phi+\|b\|_\Phi+\|\Omega_{\bar{A}}(a+b)-\Omega_{\bar{A}}a-\Omega_{\bar{A}}b\|_\Phi+\|\Omega_{\bar{A}}a\|_\Phi+\|\Omega_{\bar{A}}b\|_\Phi\\&\le C(\|a\|_\Phi+\|b\|_\Phi+\|a\|_D+\|b\|_D)\le 2C(\|a\|_D+\|b\|_D).\end{aligned}$$

To show that $\mathrm{Dom}(\Omega_{\bar{A}})=\Phi_{\bar{A}}(\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi))$, suppose that $a\in\mathrm{Dom}(\Omega_{\bar{A}})$; then there exists $h_a\in H(\bar{A})$ such that $\Phi_{\bar{A}}(h_a)=a$, $\|h_a\|_{H(\bar{A})}\le C\|a\|_{\bar{A}_\Phi}$ and $\Omega_{\bar{A}}a=\Psi_{\bar{A}}(h_a)$. Hence $h_a\in\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi)$, and $a\in\Phi_{\bar{A}}(\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi))$.

Conversely, if $a=\Phi_{\bar{A}}(h)$, $\Psi_{\bar{A}}(h)=\Phi_{\bar{A}}(h')$, $h,h'\in H(\bar{A})$, and $\Omega_{\bar{A}}a=\Psi_{\bar{A}}(h_a)$, with $\Phi_{\bar{A}}(h_a)=a$, $\|h_a\|_{H(\bar{A})}\leq C\|a\|_{\bar{A}_\Phi}$, then $\Phi_{\bar{A}}(h_a-h)=0$ and thus $\Psi_{\bar{A}}(h_a-h)=\Phi_{\bar{A}}(h'')\in \bar{A}_\Phi$. Hence, $\Omega_{\bar{A}}a=\Psi_{\bar{A}}(h)+\Phi_{\bar{A}}(h'')=\Phi_{\bar{A}}(h')+\Phi_{\bar{A}}(h'')\in\bar{A}_\Phi$.

Finally, given a bounded linear $T:\bar{A}\to\bar{B}$,

$$\|Ta\|_D=\|Ta\|_\Phi+\|\Omega_{\bar{B}}Ta\|_\Phi\leq C(\|a\|_\Phi+\|[T,\Omega]a\|_\Phi+\|T\Omega_{\bar{A}}a\|_\Phi)\leq C'\|a\|_D.$$

(b) Let now $B=\Phi_{\bar{A}}(\operatorname{Ker}\Psi_{\bar{A}})$, with

$$\|x\|_B=\inf\{\|f\|_{H(\bar{A})}\,;x=\Phi_{\bar{A}}(f),\Psi_{\bar{A}}(f)=0\}.$$

For any $x\in\operatorname{Dom}(\Omega_{\bar{A}})$ we have, $x=\Phi_{\bar{A}}(h_x)\in\bar{A}_\Phi$, $\Omega_{\bar{A}}x=\Psi_{\bar{A}}(h_x)=\Phi_{\bar{A}}(h)=\Psi_{\bar{A}}(g)$, with $\Phi_{\bar{A}}(g)=0$, $\|h\|_{H(\bar{A})}\leq(1+\varepsilon)\|\Omega_{\bar{A}}x\|_\Phi$, $\|g\|_{H(\bar{A})}\leq C\|h\|_{H(\bar{A})}$. Then $\Psi_{\bar{A}}(h_x-g)=0$, $x=\Phi_{\bar{A}}(h_x-g)$ and we have $x\in B$, with

$$\|x\|_B\leq\|h_x-g\|_{H(\bar{A})}\leq C(\|x\|_\Phi+(1+\varepsilon)\|\Omega_{\bar{A}}x\|_\Phi).$$

Hence, $\|x\|_B\leq C\|x\|_D$.

Conversely, if $x\in B$, $x=\Phi_{\bar{A}}(f)$, $\Psi_{\bar{A}}(f)=0$, with $\|f\|_{H(\bar{A})}\leq(1+\varepsilon)\|x\|_B$, then we get $\Omega_{\bar{A}}x=\Psi_{\bar{A}}(h_x)=\Psi_{\bar{A}}(h_x-f)=\Phi_{\bar{A}}(h)$, with $\|h\|_{H(\bar{A})}\leq C\|h_x-f\|_{H(\bar{A})}$ (observe that $\Phi_{\bar{A}}(h_x-f)=0$). Hence $\Omega_{\bar{A}}x\in\bar{A}_\Phi$, and

$$\begin{aligned}\|\Omega_{\bar{A}}x\|_\Phi&\leq C\|h_x-f\|_{H(\bar{A})}\leq C'(\|x\|_\Phi+(1+\varepsilon)\|x\|_B)\\&\leq C'(\|f\|_{H(\bar{A})}+(1+\varepsilon)\|x\|_B)\leq C''\|x\|_B.\end{aligned}$$

Finally,

$$\|x\|_D=\|x\|_\Phi+\|\Omega_{\bar{A}}x\|_\Phi\leq C(\|f\|_{H(\bar{A})}+\|\Omega_{\bar{A}}x\|_\Phi)\leq C'\|x\|_B.\quad\square$$

Observe that as a consequence of the theorem, a necessary and sufficient condition for $\operatorname{Dom}(\Omega_{\bar{A}})=\bar{A}_\Phi$ is that $H(\bar{A})=\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi)+\operatorname{Ker}\Phi_{\bar{A}}$. We can also give a converse result for (b):

Proposition 3.9. *(Φ,Ψ) is almost compatible, $\operatorname{Dom}(\Omega_{\bar{A}})=\Phi_{\bar{A}}(\operatorname{Ker}\Psi_{\bar{A}})$ and $\bar{A}_\Phi\hookrightarrow\bar{A}_\Psi$, if and only if (Φ,Ψ) is compatible.*

Proof. If $\operatorname{Dom}(\Omega_{\bar{A}})=\Phi_{\bar{A}}(\operatorname{Ker}\Psi_{\bar{A}})=\Phi_{\bar{A}}(\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi))$, then, for $h\in\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi)$, there exists $h'\in\operatorname{Ker}\Psi_{\bar{A}}$ such that $h-h'\in\operatorname{Ker}\Phi_{\bar{A}}$. Thus, $\Psi_{\bar{A}}^{-1}(\bar{A}_\Phi)\subset\operatorname{Ker}\Phi_{\bar{A}}+\operatorname{Ker}\Psi_{\bar{A}}$. Hence, if $a\in\bar{A}_\Phi$ and $h\in H(\bar{A})$, with $\Psi_{\bar{A}}(h)=a$, we have that $h=h^1+h^2$, $\Phi_{\bar{A}}(h^1)=\Psi_{\bar{A}}(h^2)=0$. Therefore, $a=\Psi_{\bar{A}}(h^1)\in\Psi_{\bar{A}}(\operatorname{Ker}\Phi_{\bar{A}})$. Conversely, if (Φ,Ψ) is compatible by Theorem 3.8 we need only show that $\bar{A}_\Phi\hookrightarrow\bar{A}_\Psi$. But if $a\in\bar{A}_\Phi$, then $a=\Phi_{\bar{A}}(h_a)=\Psi_{\bar{A}}(g)\in\bar{A}_\Psi$ and

$$\|g\|_{H(\bar{A})}\leq C\|h_a\|_{H(\bar{A})}\leq C'\|a\|_\Phi.\quad\square$$

Another important set related to the Ω-operator is the following:

Definition 3.10. $\mathrm{Rang}(\Omega_{\bar{A}})=\{\Omega_{\bar{A}}a; a\in \bar{A}_\Phi\}$ with

$$\|x\|_R=\inf\{\|a\|_\Phi\,;\Omega_{\bar{A}}a=x\}.$$

In general, $\mathrm{Rang}(\Omega_{\bar{A}})$ is not a linear space and it depends upon the choice made to define $\Omega_{\bar{A}}$. It is easy to check that $\lambda x\in\mathrm{Rang}(\Omega_{\bar{A}})$ if $x\in\mathrm{Rang}(\Omega_{\bar{A}})$, with $\|\lambda x\|_R=|\lambda|\,\|x\|_R$. We will also consider $\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})$ with

$$\|x\|_+=\inf\{\|a\|_\Phi+\|\Omega_{\bar{A}}b\|_R\,; x=a+b,\ a,\ b\in \bar{A}_\Phi\}.$$

From the definitions it follows that $\mathrm{Rang}(\Omega_{\bar{A}})\hookrightarrow \bar{A}_\Psi$ and $\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})\hookrightarrow \bar{A}_\Phi+\bar{A}_\Psi$, boundedly with constant C. In fact, for $x=\Omega_{\bar{A}}a$, with $\|a\|_\Phi\le(1+\varepsilon)\|x\|_R$ we have

$$\|x\|_\Psi=\|\Psi(h_a)\|_\Psi\le\|h_a\|_{H(\bar{A})}\le C(1+\varepsilon)\|x\|_R.$$

Theorem 3.11. (a) *If* (Φ,Ψ) *is almost compatible, then* $\bar{A}_\Psi\hookrightarrow\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})$, *and* $T\colon \bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})\to\bar{B}_\Phi+\mathrm{Rang}(\Omega_{\bar{B}})$ *is bounded, for any bounded linear operator* $T\colon\bar{A}\to\bar{B}$.

(b) *If* (Φ,Ψ) *is compatible, then* $\bar{A}_\Psi=\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})$, *with equivalent "norms".*

Proof. (a) Let us consider $x=\Psi_{\bar{A}}(f)\in\bar{A}_\Psi$, with $\|f\|_{H(\bar{A})}\le(1+\varepsilon)\|x\|_\Psi$, and let $a=\Phi_{\bar{A}}(f)\in\bar{A}_\Phi$. If $b=\Omega_{\bar{A}}a=\Psi_{\bar{A}}(h_a)\in\mathrm{Rang}(\Omega_{\bar{A}})$, then $x=(x-b)+b=\Psi_{\bar{A}}(f-h_a)+\Omega_{\bar{A}}a$, with $\Phi_{\bar{A}}(f-h_a)=0$, and it follows that

$$x=\Phi_{\bar{A}}(h)+\Omega_{\bar{A}}a\in\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}}),\quad \|h\|_{H(\bar{A})}\le C\|f-h_a\|_{H(\bar{A})},$$

and

$$\begin{aligned}\|x\|_+&\le\|h\|_{H(\bar{A})}+\|a\|_\Phi\le C\|f-h_a\|_{H(\bar{A})}+\|a\|_\Phi\\&\le C'(\|f\|_{H(\bar{A})}+\|a\|_\Phi)\le 2C'\|f\|_{H(\bar{A})}.\end{aligned}$$

Hence $\|x\|_+\le C\|x\|_\Psi$.

Now let $T\colon\bar{A}\to\bar{B}$. For any $x=a+\Omega_{\bar{A}}b\in\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})$, with $\|a\|_\Phi+\|\Omega_{\bar{A}}b\|_R\le(1+\varepsilon)\|x\|_+$, and $\|b\|_\Phi\le(1+\varepsilon)\|\Omega_{\bar{A}}b\|_R$ we have $\|a\|_\Phi+\|b\|_\Phi\le(1+\varepsilon)^2\|x\|_+$. It follows that

$$Tx=(Ta+[T,\Omega]b)+\Omega_{\bar{B}}Tb\in\bar{B}_\Phi+\mathrm{Rang}(\Omega_{\bar{B}}),$$

with

$$\|Tx\|_+\le(\|T\|_{\Phi,\Phi}+\|[T,\Omega]\|_{\Phi,\Phi})(\|a\|_\Phi+\|b\|_\Phi)\le(1+\varepsilon)^2(\|T\|_{\Phi,\Phi}+\|[T,\Omega]\|_{\Phi,\Phi})\|x\|_+.$$

Thus $\|T\|_{+,+}\le\|T\|_{\Phi,\Phi}+\|[T,\Omega]\|_{\Phi,\Phi}$.

(b) In this case we have $\bar{A}_\Phi\hookrightarrow\bar{A}_\Psi$ (see Proposition 3.9), and we have seen that

$$\bar{A}_\Psi\hookrightarrow\bar{A}_\Phi+\mathrm{Rang}(\Omega_{\bar{A}})\hookrightarrow\bar{A}_\Phi+\bar{A}_\Psi.\quad\square$$

4. The Ω-operator for the complex method

Let us now consider the pair $(\delta_\theta, \delta'_\theta)$ of interpolators, with $H(\bar{A})=\mathcal{F}(\bar{A})$ (see Example I).

Theorem 4.1. *The pair $(\delta_\theta, \delta'_\theta)$ is compatible.*

Proof. If $g\in\mathcal{F}(\bar{A})$ and $\delta_\theta(g)=g(\theta)=0$, then $\delta'_\theta(g)=g'(\theta)=f(\theta)$, with $f(z)=\varphi'(\theta)g(z)/\varphi(z)$, where φ is a conformal mapping from the strip $\mathbf{S}=\{z;0<\operatorname{Re} z<1\}$ onto the unit disk $\mathbf{D}=\{z;|z|<1\}$ such that $\varphi(\theta)=0$. We have $f\in\mathcal{F}(\bar{A})$, with $\|f\|_{\mathcal{F}(\bar{A})}=|\varphi'(\theta)|\,\|g\|_{\mathcal{F}(\bar{A})}$. Conversely, let now $f\in\mathcal{F}(\bar{A})$. Then

$$g(z)=\frac{\varphi(z)f(z)}{\varphi'(\theta)}\in\mathcal{F}(\bar{A})$$

with $\|g\|_{\mathcal{F}(\bar{A})}=\|f\|_{\mathcal{F}(\bar{A})}/|\varphi'(\theta)|$, $\delta_\theta(g)=g(\theta)=0$, and $\delta'_\theta(g)=g'(\theta)=f(\theta)=\delta_\theta(f)$. $\square$

In this example, for a given almost optimal election

$$a\in\bar{A}_{[\theta]}=\bar{A}_{\delta_\theta}\mapsto h_a\in\mathcal{F}(\bar{A}),$$

we have $\Omega_{\bar{A}}a=h'_a(\theta)$, and it follows from Theorem 3.11 that the interpolation method associated to $\operatorname{Rang}(\Omega_{\bar{A}})$ is Schechter's lower method (see [S]):

$$\bar{A}_{[\theta]}+\operatorname{Rang}(\Omega_{\bar{A}})=\bar{A}_{\delta'_\theta}.$$

In connection with the domain of $\Omega_{\bar{A}}$, we recall that Schechter's upper method is defined by

$$\begin{aligned}\bar{A}^{\delta'_\theta}=[A_0,A_1]^{\delta'_\theta}&=\{x\in\Sigma(\bar{A})\,;x\delta'_\theta=f\delta'_\theta,\ \text{for some } f\in\mathcal{F}\}\\ &=\{x\in\Sigma(\bar{A})\,;x=f(\theta),\ \text{for some } f\in\mathcal{F},\ f'(\theta)=0\}\end{aligned}$$

and we easily get as a consequence of Theorem 3.8 that (cf. [CJMR] and [CCMS])

$$\operatorname{Dom}(\Omega_{\bar{A}})=\bar{A}^{\delta'_\theta}.$$

5. The Ω-operator for the J-method

We assume $0<\theta<1$ and $1\le p\le\infty$, and let

$$H(\bar{A})=\{u:\mathbf{R}^+\to\Delta(\bar{A})\ \text{measurable}\,;\Phi_{\theta,p}(J(t,u(t)))<\infty\},$$

and

$$\Phi^J_{\bar{A}}(u)=\int_0^\infty u(t)\,\frac{dt}{t},$$

as in Example II, so that we obtain the J-interpolation method. A second functional $\Psi^J_{\bar{A}}$ is defined on $H(\bar{A})$ by

$$\Psi^J_{\bar{A}}(u)=\int_0^\infty(\log t)u(t)\,\frac{dt}{t}.$$

Theorem 5.1. *(Φ^J, Ψ^J) is a compatible pair of interpolators and*

$$\bar{A}_{\Psi,J}=\left\{a=\int_0^\infty v(t)\,\frac{dt}{t}\,; v\colon \mathbf{R}^+\to\Delta(\bar{A}) \text{ measurable, } \Phi_{\theta,p}\left(\frac{J(t,v(t))}{1+|\log t|}\right)<\infty\right\},$$

with

$$\|a\|_\Psi=\inf\Phi_{\theta,p}\left(\frac{J(t,v(t))}{1+|\log t|}\right),$$

the infimum being taken over all representations $a=\int_0^\infty v(t)\,(dt/t)$.

Proof. The functional Ψ^J is well defined and bounded from $H(\bar{A})$ to $\Sigma(\bar{A})$:

$$\begin{aligned}\|\Psi^J_{\bar{A}}(u)\|_\Sigma &\le \int_0^1 |\log t|\,\|u\|_0\,\frac{dt}{t}+\int_1^\infty (\log t)\|u\|_1\,\frac{dt}{t}\\ &\le \int_0^1 |\log t| J(t,u(t))\,\frac{dt}{t}+\int_1^\infty \frac{\log t}{t}J(t,u(t))\,\frac{dt}{t}\\ &\le \left[\left(\int_0^1 (|\log t|t^\theta)^{p'}\,\frac{dt}{t}\right)^{1/p'}+\left(\int_1^\infty\left(\frac{\log t}{t^{1-\theta}}\right)^{p'}\frac{dt}{t}\right)^{1/p'}\right]\Phi_{\theta,p}(J(t,u))\\ &=C\|u\|_{H(\bar{A})}.\end{aligned}$$

To see that Φ^J and Ψ^J are compatible, first assume that $\int_0^\infty u(t)\,(dt/t)=0$, (with $u\in H(\bar{A})$) and define

$$F(z)=\int_0^\infty t^z u(t)\,\frac{dt}{t},$$

on the strip $\{z\in\mathbf{C}; -\varepsilon<\operatorname{Re} z<\varepsilon\}$, with $\varepsilon>0$ such that $0<\theta-\varepsilon<\theta+\varepsilon<1$. It is easily seen that $F(\pm\varepsilon\pm ti)\in\bar{A}_{\theta\pm\varepsilon,p}$, with $\|F(\pm\varepsilon\pm ti)\|_{\theta\pm\varepsilon,p}\le C\|u\|_{H(\bar{A})}$, and, since $F(0)=0$, we have

$$F'(0)=\int_0^\infty (\log t)\,u(t)\,\frac{dt}{t}=\Psi^J_{\bar{A}}(u)\in[\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}]_0=\bar{A}_{\theta,p}=\bar{A}_{\Phi^J},$$

with $\|\Psi^J_{\bar{A}}(u)\|_{\Phi^J}\le C\|u\|_{H(\bar{A})}$.

For the converse inclusion $\operatorname{Im}\Phi^J_{\bar{A}}\hookrightarrow\Psi^J_{\bar{A}}(\operatorname{Ker}\Phi^J_{\bar{A}})$, for a given $u\in H(\bar{A})$ we consider $v(t)=u(t)-u(\mathrm{e}t)$ and we have $v\in H(\bar{A})$ such that $\Phi^J_{\bar{A}}(v)=\int_0^\infty v(t)\,(dt/t)=0$, and

$$\begin{aligned}\Psi^J_{\bar{A}}(v)&=\int_0^\infty(\log t)(u(t)-u(\mathrm{e}t))\,\frac{dt}{t}=\int_0^\infty(\log t)\,u(t)\,\frac{dt}{t}-\int_0^\infty\left(\log\frac{t}{\mathrm{e}}\right)u(t)\,\frac{dt}{t}\\ &=\int_0^\infty u(t)\,\frac{dt}{t}=\Phi^J_{\bar{A}}(u),\end{aligned}$$

with $\|v\|_{H(\bar{A})}\le C\|u\|_{H(\bar{A})}$.

For the last part of the theorem, let

$$B=\left\{a=\int_0^\infty v(t)\,\frac{dt}{t}\,;\left(\int_0^\infty\left(\frac{J(t,v(t))}{t^\theta(1+|\log t|)}\right)^p\frac{dt}{t}\right)^{1/p}<\infty\right\},$$

and $a\in\bar{A}_{\Psi^J}$, $a=\int_0^\infty(\log t)\,u(t)\,(dt/t)$, with

$$\left(\int_0^\infty\left(\frac{J(t,u(t))}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}\le\|a\|_{\Psi^J}+\varepsilon.$$

Then $a=\int_0^\infty v(t)\,(dt/t)$ with $v(t)=(\log t)\,u(t)$, and

$$\|a\|_B\le\Phi_{\theta,p}\left(\frac{J(t,v(t))}{1+|\log t|}\right)\le\Phi_{\theta,p}(J(t,u(t)))\le\|a\|_{\Psi^J}+\varepsilon.$$

To show that $B\hookrightarrow\bar{A}_{\Psi^J}$, for any $a\in B$ we have

$$a=\int_0^\infty v(t)\,\frac{dt}{t}=\int_0^\infty\frac{v(t)}{1+|\log t|}\,\frac{dt}{t}+\int_0^\infty\frac{|\log t|v(t)}{1+|\log t|}\,\frac{dt}{t}=b+c,$$

with

$$\Phi_{\theta,p}\left(\frac{J(t,v(t))}{1+|\log t|}\right)\le\|a\|_B+\varepsilon,$$

and $b\in\bar{A}_{\Phi^J}\hookrightarrow\bar{A}_{\Psi^J}$, with $\|b\|_\Psi\le C\|b\|_\Phi\le C(\|a\|_B+\varepsilon)$. On the other hand,

$$c=\int_0^\infty(\log t)\,w(t)\,\frac{dt}{t},$$

with $w(t)=\operatorname{sgn}(\log t)v(t)/(1+|\log t|)$, and $\Phi_{\theta,p}(t,w(t))\le\|a\|_B+\varepsilon$. It follows that $c\in\bar{A}_{\Psi^J}$ and $\|c\|_\Psi\le\|a\|_B+\varepsilon$. Hence $a\in\bar{A}_{\Psi^J}$ and $\|a\|_\Psi\le C\|a\|_B$. □

Let now $\Omega^J_{\bar{A}}$ be the Ω-operator associated to the pair (Φ^J,Ψ^J), and to a given almost optimal selection $a\mapsto h_a$ for Φ^J. Again, as an application of Theorem 3.11, we have

$$\bar{A}_{\theta,p;J}+\operatorname{Rang}(\Omega^J_{\bar{A}})=\bar{A}_{\Psi^J},$$

and

$$\operatorname{Dom}(\Omega^J_{\bar{A}})=\left\{a=\int_0^\infty u(t)\,\frac{dt}{t}\,;\int_0^\infty(\log t)\,u(t)\,\frac{dt}{t}=0,\ u\in H(\bar{A})\right\}\tag{4}$$

which has the following description (see [CJM] for another proof):

Theorem 5.2.

$$\operatorname{Dom}(\Omega_{\bar{A}}^J)=\left\{a=\int_0^\infty u(t)\,\frac{dt}{t}\,;u\in H(\bar{A}),\Phi_{\theta,p}((1+|\log t|)J(t,u(t)))<\infty\right\}.$$

Proof. Let $\mathcal{E}$ be the right hand side space, with the natural norm, and choose $a=\int_0^\infty u(t)\,(dt/t)\in\mathcal{E}$, with

$$\Phi_{\theta,p}=((1+|\log t|)J(t,u(t)))\le C\|a\|_{\mathcal{E}}.$$

Then $a=\int_0^\infty h_a(t)\,(dt/t)$, $\Omega_{\bar{A}}^J a=\int_0^\infty(\log t)h_a(t)\,(dt/t)$ and $\int_0^\infty(u(t)-h_a(t))\,(dt/t)=0$. Thus, from

$$b=\int_0^\infty(\log t)u(t)\,\frac{dt}{t}\in\bar{A}_{\Phi^J}\quad\text{and}\quad\Omega_{\bar{A}}^J a-b=\int_0^\infty(\log t)(h_a(t)-u(t))\,\frac{dt}{t}\in\bar{A}_{\Phi^J},$$

we obtain

$$\Omega_{\bar{A}}^J a=\int_0^\infty(\log t)h_a(t)\,\frac{dt}{t}\in\bar{A}_{\Phi^J},$$

and

$$\begin{aligned}\|a\|_D=\|a\|_\Phi+\|\Omega_{\bar{A}}^J a\|_\Phi&\le\|u\|_{H(\bar{A})}+\|\Omega_{\bar{A}}^J a-b\|_\Phi+\|b\|_\Phi\\&\le\|u\|_{H(\bar{A})}+C\|u-h_a\|_{H(\bar{A})}+C\|u\|_{H(\bar{A})}\le C\|a\|_{\mathcal{E}}.\end{aligned}$$

To show that $\operatorname{Dom}(\Omega_{\bar{A}}^J)\hookrightarrow\mathcal{E}$ we will use the following facts:

(i) $(\bar{A}_{\theta_0,q_0},\bar{A}_{\theta_1,q_1})$ is a partial retract of the couple $(l^{q_0}(2^{-n\theta_0}),l^{q_1}(2^{-n\theta_1}))$ (cf. [Cw] and [CJM]). Recall that $\bar{A}$ is a partial retract of $\bar{B}$, if for every $x\in\Sigma(\bar{A})$ there exists a pair of bounded linear operators, $F_x:\bar{A}\to\bar{B}$ and $P_x:\bar{B}\to\bar{A}$, such that $P_x\circ F_x x=x$ and $\sup_x\|F_x\|,\sup_x\|P_x\|<\infty$.

(ii) $[l^p(2^{-n\theta_0}),l^p(2^{-n\theta_1})]^{\delta'_\mu}=l^p((1+|n|)2^{-n\theta})$, with $\theta=(1-\mu)\theta_0+\mu\theta_1$ (see reference [CC2]).

(iii) $(\bar{A}_{\theta_0,q_0},\bar{A}_{\theta_1,q_1})_{\varphi_\mu,p}=\bar{A}_{\varphi_\theta,p}$, with $\varphi_\lambda(x)=(1+|\log x|)x^{-\lambda}$ (cf. [G]).

Let now $a\in\operatorname{Dom}(\Omega_{\bar{A}}^J)$, and $u\in H(\bar{A})$ such that (by (4))

$$a=\int_0^\infty u(t)\,\frac{dt}{t},\quad\int_0^\infty(\log t)u(t)\,\frac{dt}{t}=0,\quad\Phi_{\theta,p}(J(t,u(t)))<\infty.$$

Then

$$F(z)=\int_0^\infty t^z u(t)\,\frac{dt}{t}\in\mathcal{F}(\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}),$$

on the strip $\theta-\varepsilon<\operatorname{Re} z<\theta+\varepsilon$, $F(0)=a$ and $F'(0)=0$. Hence

$$a\in[\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}]^{\delta'_0} \quad\text{with}\quad \|a\|_{[\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}]^{\delta'_0}}\le\|F\|_{\mathcal{F}}\le\|u\|_{H(\bar{A})}.$$

With the notation of (i), let $F\colon(\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p})\to(l^p(2^{-n(\theta-\varepsilon)}),l^p(2^{-n(\theta+\varepsilon)})$, and let $P\colon(l^p(2^{-n(\theta-\varepsilon)}),l^p(2^{-n(\theta+\varepsilon)}))\to(\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p})$ be a pair of bounded linear mappings such that $PFa=a$. Then, by interpolation (we use (ii) and (iii)):

$$F\colon[\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}]^{\delta'_0}\to l^p((1+|n|)2^{-n\theta}),$$

and

$$P\colon l^p((1+|n|)2^{-n\theta})\to\bar{A}_{\varphi_\theta,p}.$$

Hence,

$$\|a\|_{\varphi_\theta,p}=\|PFa\|_{\varphi_\theta,p}\le C\|Fa\|_{l^p((1+|n|)2^{-n\theta})}\le C\|a\|_{[\bar{A}_{\theta-\varepsilon,p},\bar{A}_{\theta+\varepsilon,p}]^{\delta'_0}},$$

we have $a\in\bar{A}_{\varphi_\theta,p}$, and there exists $v\in H(\bar{A})$ such that

$$a=\int_0^\infty v(t)\,\frac{dt}{t}\quad\text{and}\quad \Phi_{\theta,p}((1+|\log t|)J(t,v(t)))<\infty.$$

Thus $a\in\mathcal{E}$. $\square$

Remark 5.3. From (i)–(iii) we obtain the reiteration property

$$[\bar{A}_{\theta_0,q_0},\bar{A}_{\theta_1,q_1}]^{\delta'_\mu}=\bar{A}_{\varphi_\theta,q},$$

with $1/q=(1-\theta)/q_0+\theta/q_1$.

6. The Ω-operator for the K-method

Let now

$$\Phi^K_{\bar{A}}(a_0,a_1)=a_0(t)+a_1(t)\quad\text{and}\quad \Psi^K_{\bar{A}}(a_0,a_1)=\int_0^1 a_0(t)\,\frac{dt}{t}-\int_1^\infty a_1(t)\,\frac{dt}{t}$$

be defined over the Banach space $H^K(\bar{A})$ of all pairs of measurable functions

$$(a_0,a_1)\colon\mathbf{R}^+\to A_0\times A_1$$

such that $a_0(t)+a_1(t)$ is constant and

$$\|(a_0,a_1)\|_H=\Phi_{\theta,p}(\|a_0(t)\|_0+t\|a_1(t)\|_1)<\infty,$$

$0<\theta<1$, $1\le p\le\infty$, as in Example III, and $H^K(T)(a_0,a_1)=(T(a_0),T(a_1))$, for any bounded linear operator $T\colon\bar{A}\to\bar{B}$. We observe that

$$\begin{aligned}\|\Psi^K_{\bar{A}}(a_0,a_1)\|_\Sigma&\le\left\|\int_0^1 a_0(t)\,\frac{dt}{t}\right\|_0+\left\|\int_1^\infty a_1(t)\,\frac{dt}{t}\right\|_1\\&\le C\left[\left(\int_0^1\left(\frac{\|a_0(t)\|_0}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}+\left(\int_1^\infty\left(\frac{t\|a_1(t)\|_1}{t^\theta}\right)^p\frac{dt}{t}\right)^{1/p}\right]\\&\le C\|(a_0,a_1)\|_H.\end{aligned}$$

Theorem 6.1. *(Φ^K,Ψ^K) is a compatible pair of interpolators.*

Proof. Let $\Phi^K_{\bar{A}}(a_0,a_1)=0=a_0(t)+a_1(t)$, with $(a_0,a_1)\in H^K(\bar{A})$. Then,

$$\Psi^K_{\bar{A}}(a_0,a_1)=\int_0^\infty a_0(t)\,\frac{dt}{t},$$

and, since $\Phi_{\theta,p}(J(t,a_0(t)))\le\Phi_{\theta,p}(\|a_0(t)\|_0+t\|a_1(t)\|_1)=\|(a_0,a_1)\|_H$, we have

$$\Psi^K_{\bar{A}}(a_0,a_1)\in\bar{A}_{\theta,p;J}=\bar{A}_{\theta,p;K}\quad\text{and}\quad\|\Psi^K_{\bar{A}}(a_0,a_1)\|_\Phi\le\|(a_0,a_1)\|_H.$$

Let now $a\in\bar{A}_\Phi$. We have to find $(b_0,b_1)\in H(\bar{A})$ such that

$$\Phi^K_{\bar{A}}(b_0,b_1)=b_0(t)+b_1(t)\equiv0\quad\text{and}\quad\Psi^K_{\bar{A}}(b_0,b_1)=\int_0^\infty b_0(t)\,\frac{dt}{t}=a.$$

This follows from the fundamental lemma of interpolation theory (cf. [BL]):

If we discretize, we have to show that there exists a sequence $(b_0^n,b_1^n)\in A_0\times A_1$ such that

$$b_0^n+b_1^n=0\quad\text{and}\quad\sum_{n=-\infty}^{\infty}b_0^n=a.$$

Let $a=a_0^n+a_1^n$, with $\|a_0^n\|_0+2^n\|a_1^n\|_1\le(1+\varepsilon)K(2^n,a)$. We have

$$\lim_{n\to-\infty}\|a_0^n\|_0=0\quad\text{and}\quad\lim_{n\to\infty}\|a_1^n\|_1=0.$$

Write $b_n^0=a_0^n-a_0^{n-1}$ and $b_n^1=a_1^n-a_1^{n-1}$. Then $b_n^0+b_n^1=0$ and

$$K\left(1,a-\sum_{-N}^{M}b_n^0\right)=K(1,a_0^{-N-1}+a_1^M)\to0,\quad\text{as }N,M\to\infty.$$

Hence $\sum_n b_n^0=a$. □

7. Twisted direct sums

Let us define, as in [CJMR], the twisted direct sum of $\bar{A}_\Phi$, $\bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi$, the set of all pairs $(a,b)\in(A_0+A_1)\times(A_0+A_1)$ such that

$$\|a\|_{\bar{A}_\Phi}+\|\Omega a-b\|_{\bar{A}_\Phi}<+\infty.$$

We first have the following result.

Proposition 7.1. *Let $T\colon \bar{A}_\Phi\to\bar{B}_\Phi$ be a bounded linear operator. Then the operator $[T,\Omega]\colon \bar{A}_\Phi\to\bar{B}_\Phi$ is bounded if and only if, $\widetilde{T}\colon \bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi\to\bar{B}_\Phi \bigoplus_\Omega \bar{B}_\Phi$ is bounded, where $\widetilde{T}(a,b)=(Ta,Tb)$.*

Proof. Let $(a,b)\in\bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi$. Then

$$\begin{aligned}\|\widetilde{T}(a,b)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi} &= \|Ta\|_{\bar{B}_\Phi}+\|\Omega Ta-Tb\|_{\bar{B}_\Phi}\\ &\le C\|a\|_{\bar{A}_\Phi}+\|\Omega Ta-T\Omega a\|_{\bar{B}_\Phi}+\|T(\Omega a-b)\|_{\bar{B}_\Phi}\\ &\le C(\|a\|_{\bar{A}_\Phi}+\|\Omega a-b\|_{\bar{A}_\Phi})=C\|(a,b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}.\end{aligned}$$

Conversely, let $a\in\bar{A}_\Phi$, then

$$\begin{aligned}\|[T,\Omega]a\|_{\bar{B}_\Phi} &= \|\Omega Ta-T\Omega a\|_{\bar{B}_\Phi}\le\|(Ta,T\Omega a)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi}\\ &=\|\widetilde{T}(a,\Omega a)\|_{\bar{B}_\Phi \oplus_\Omega \bar{B}_\Phi}\le C\|(a,\Omega a)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}=C\|a\|_{\bar{A}_\Phi}.\quad\square\end{aligned}$$

Proposition 7.2. *If (Φ,Ψ) is compatible, then*

$$\bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi=\{(a,b)\,;a=\Phi(f),\ b=\Psi(f),\ f\in H(\bar{A})\}.$$

Proof. Let $\mathcal{E}$ be the right hand side space, with the usual norm, and choose $(a,b)\in\bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi$. Then $\Omega a=\Psi(h_a)$ with $\Phi(h_a)=a$, $\|h_a\|_{H(\bar{A})}\le C\|a\|_{\bar{A}_\Phi}$, $b-\Omega a=\Phi(g)=\Psi(h)$ with $\Phi(h)=0$ and $\|h\|_{H(\bar{A})}\le C\|b-\Omega a\|_{\bar{A}_\Phi}$.

Therefore, $a=\Phi(h_a+h)$, $b=b-\Omega a+\Omega a=\Psi(h_a+h)$; that is, $(a,b)\in\mathcal{E}$ and

$$\|(a,b)\|_{\mathcal{E}}\le\|h_a+h\|_{H(\bar{A})}\le C(\|a\|_{\bar{A}_\Phi}+\|b-\Omega a\|_{\bar{A}_\Phi})=C\|(a,b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi}.$$

Let now $(a,b)\in\mathcal{E}$ and set $a=\Phi(h)$, $b=\Psi(h)$ and $\|h\|_{H(\bar{A})}\le C\|(a,b)\|_{\mathcal{E}}$. Then $a\in\bar{A}_\Phi$, $\Omega a=\Psi(h_a)$ and $\Omega a-b=\Psi(h_a-h)$, with $\Phi(h_a-h)=0$. Therefore, $\Omega a-b\in\bar{A}_\Phi$ and

$$\begin{aligned}\|(a,b)\|_{\bar{A}_\Phi \oplus_\Omega \bar{A}_\Phi} &= \|a\|_{\bar{A}_\Phi}+\|\Omega a-b\|_{\bar{A}_\Phi}\le C(\|h\|_{H(\bar{A})}+\|h_a-h\|_{H(\bar{A})})\\ &\le C\|h\|_{H(\bar{A})}\le C\|(a,b)\|_{\mathcal{E}}.\quad\square\end{aligned}$$

Remark 7.3. In particular, since the J-method satisfies the hypothesis of Proposition 7.2, we have that if $\bar{A}_\Phi = (A_0, A_1)_{\theta,p;J}$,

$$\bar{A}_\Phi \bigoplus_\Omega \bar{A}_\Phi = \left\{ (a,b)\, ; a = \int_0^\infty u(t)\, \frac{dt}{t},\ b = \int_0^\infty (\log t)\, u(t)\, \frac{dt}{t} \right\},$$

where $\phi_{\theta,p}(J(t,u(t))) < \infty$ (see Example II). This answers Question 6 in [CJMR].

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