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Takens' problem for systems of first order differential equations

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1. Introduction

In 1977, F. Takens [6] considered the following novel aspect of the classical Noether's theorem of calculus of variations: Let a local Lie group act on the space of independent and dependent variables, and let Γ be the Lie algebra of infinitesimal generators of the group action. Suppose that a system of differential equations is invariant under Γ and that each element in Γ generates a conservation law for the system. Does it then follow that the system arises from a variational principle? In his original paper Takens answered the question in a number of non-trivial cases. Subsequently, Takens' results on second order scalar equations and on systems of linear equations were substantially generalized by Anderson and Pohjanpelto [2], [3].

In this paper we consider Takens' question for systems of first order equations and for the Abelian Lie algebra $\Gamma=\mathbf{t}(n)$ of infinitesimal translations acting on the space of independent variables. This problem originally arose in connection with Takens' problem for vector field theories [4], and the results of this paper, in particular Theorem 8, will be directly applicable therein.

Let $\mathcal{T}(n)$ consist of all systems of n first order partial differential equations in n independent and n dependent variables which are invariant under $\mathbf{t}(n)$ and for which every element in $\mathbf{t}(n)$ generates a conservation law. Results in ref. [3], applied to $\mathbf{t}(n)$, imply that a system in $\mathcal{T}(n)$ whose components are polynomials in the dependent variables u^a and their first order derivatives u_i^a of degree at most n is variational. In contrast, here we consider the case when the components are allowed to be smooth functions in some open set in the space of the variables (u^a, u_i^a).

We begin by reviewing some basic definitions and results from the calculus of variations most relevant to our problem. In Section 3 we give an explicit description of systems belonging to $\mathcal{T}(n)$ in terms of the minors of the matrix (u_i^a), and we proceed in Section 4 to show that the subspace $\mathcal{V}(n)\subset\mathcal{T}(n)$ of variational systems is characterized by a simple algebraic condition. Surprisingly, these conditions turn out to be exactly the ones that guarantee a system in $\mathcal{T}(n)$ to be everywhere smooth in the derivative variables u_i^a. This result underscores the fact already apparent in refs. [2] and [3] that subtle smoothness properties of differential equations play a central role in Takens' problem. The algebraic conditions also allow us to construct a subspace $\mathcal{A}(n)\subset\mathcal{T}(n)$ complementary to the space $\mathcal{V}(n)$. Thus, in a sense, the subspace $\mathcal{A}(n)$ gives a measure of the degree to which Takens' question fails in the present problem.

Finally, in Section 5 we present some examples. The first two examples partly motivate our restricting the problem to first order systems in n independent and $m=n$ dependent variables. Specifically, these examples show that, in general, Takens' question for $\mathbf{t}(n)$ fails for everywhere smooth first order systems when $m>n$, and for everywhere smooth second order systems when $m=n$. When $m<n$ the results of refs. [2] and [3] can be applied to solve Takens' problem for $\mathbf{t}(n)$ in several particular instances. However, this case still remains to be studied in full generality.

2. Preliminaries

In this section we collect some definitions and results from calculus of variations needed in the sequel. For more details and proofs we refer to refs. [1] and [5].

Let $E=\mathbf{R}^n\times\mathbf{R}^m$ be the space of the independent and dependent variables with coordinates x^i, $i=1,2,\dots,n$ and u^a, $a=1,2,\dots,m$. We write $u^a_{i_1}, u^a_{i_1 i_2},\dots, u^a_{i_1 i_2\dots i_k},\dots$ for the first, second and higher order derivatives of the u^a. Let $I=(i_1,i_2,\dots,i_k)$ be a multi-index of integers $1\le i_j\le n$ of length $|I|=k$, and let $I^\#=(t_1,t_2,\dots,t_n)$ be the transpose of I, where t_j stands for the number of occurrences of the integer j amongst the entries $i_1,i_2,\dots,i_k$ of I. We write $I!=t_1!t_2!\dots t_n!$, and define the weighted partial derivative operators ∂^I_α by

$$\partial_a^I=\frac{I!}{|I|!}\frac{\partial}{\partial u_I^a},\quad a=1,2,\dots,m,\ |I|\ge 0,$$

and the total derivative operators D_i by

$$D_i=\frac{\partial}{\partial x^i}+\sum_{|I|\ge 0} u^a_{iI}\partial^I_a,\quad i=1,2,\dots,n.$$

We associate to a system of kth order differential equations

$$\Delta_a(x^i,u^b,u^b_{i_1},u^b_{i_1 i_2},\dots,u^b_{i_1 i_2\dots i_k})=0,\quad a=1,2,\dots,m,$$

the source form

$$\Delta = \Delta_a du^a \wedge \nu,$$

where $\nu = dx^1 dx^2 \dots dx^n$ is the volume form on $\mathbf{R}^n$. A source form $\Delta = \Delta_a du^a \wedge \nu$ is said to arise from a variational principle if there is a Lagrangian

$$L = L(x^i, u^a, u^a_{i_1}, u^a_{i_1 i_2}, \dots, u^a_{i_1 i_2 \dots i_k})$$

such that

$$\Delta_a = \mathrm{E}_a(L), \quad a = 1, 2, \dots, m,$$

where the Euler–Lagrange operators E_a are

$$\mathrm{E}_a(L) = \sum_{|I| \geq 0} (-D)_I \partial^I_a L. \tag{1}$$

Here the iterated total derivative $(-D)_I$, $I = (i_1, i_2, \dots, i_k)$, is given by

$$(-D)_I = (-D)_{i_1} (-D)_{i_2} \dots (-D)_{i_k}.$$

One can easily check that if a source form $\Delta = \Delta_a du^a \wedge \nu$ arises from a variational principle then the components of the Helmholtz operator of Δ given by

$$\mathcal{H}^I_{ab}(\Delta) = \partial^I_b \Delta_a - (-1)^{|I|} \mathrm{E}^I_a(\Delta_b), \quad a, b = 1, 2, \dots, n, \ |I| \geq 0, \tag{2}$$

vanish identically. In (2), the higher Euler operators E^I_a are given explicitly by

$$\mathrm{E}^I_a(f) = \sum_{|J| \geq 0} \binom{|I| + |J|}{|I|} (-D)_J \partial^{IJ}_a(f). \tag{3}$$

Conversely, if the Helmholtz conditions $\mathcal{H}^I_{ab}(\Delta) = 0$ are satisfied, then it can be shown that, at least locally, the source form Δ can be written as the Euler–Lagrange expression of some Lagrangian. Hence we will call a source form satisfying the Helmholtz conditions locally variational.

Suppose that the Lagrangian $L = L(u^a, u^a_{i_1}, u^a_{i_1 i_2}, \dots, u^a_{i_1 i_2 \dots i_k})$ is invariant under the infinitesimal transformation group $\mathbf{t}(n)$ of translations generated by the vector fields $\partial / \partial x^i$, $i = 1, 2, \dots, n$. Then it is well known that there are differential functions

$$\mathrm{V}^i_j = \mathrm{V}^i_j(u^a, u^a_{i_1}, u^a_{i_1 i_2}, \dots, u^a_{i_1 i_2 \dots i_k}), \quad i, j = 1, 2, \dots, n,$$

such that

$$u^a_j \mathrm{E}_a(L) = D_i \mathrm{V}^i_j.$$

Each vector field $(\mathrm{V}_j^1, \mathrm{V}_j^2, \dots, \mathrm{V}_j^n)$ is divergence free on the solutions of the system $\mathrm{E}_a(L)=0$ and therefore provides a conservation law for the system. Accordingly, we say that a source form $\Delta=\Delta_a du^a \wedge \nu$ admits $\mathbf{t}(n)$ conservation laws if there are differential functions V_j^i such that

$$u_j^a \Delta_a = D_i \mathrm{V}_j^i, \quad j=1,2,\dots,n. \tag{4}$$

It is well known that if (4) holds, then necessarily

$$E_b(u_j^a \Delta_a)=0, \quad b=1,2,\dots,m. \tag{5}$$

Conversely, if the Euler–Lagrange expressions in (5) vanish, then it can be shown that, at least locally, there exist differential functions V_j^i such that equations (4) are satisfied.

Suppose that Δ is locally variational. Then, by the classical Noether's theorem, Δ admits $\mathbf{t}(n)$ conservation laws if and only if Δ is $\mathbf{t}(n)$ invariant. In this paper we study the question whether the existence of $\mathbf{t}(n)$ symmetries and $\mathbf{t}(n)$ conservation laws implies that a source form is locally variational.

3. Conservation law conditions

Throughout this section and Section 4, $E=\mathbf{R}^n \times \mathbf{R}^n$. Given a source form $\Delta=\Delta_a du^a \wedge \nu$, we write

$$\Psi_i = u_i^a \Delta_a, \quad i=1,2,\dots,n.$$

Our first task is to transcribe the assumptions of Takens' problem for the infinitesimal transformation group $\mathbf{t}(n)$ into conditions for the functions Ψ_i.

The $\mathbf{t}(n)$ invariance of Δ simply means that the components Δ_a, and thus the functions Ψ_i, do not depend on the variables x^j.

Proposition 1. *Let $\Delta=\Delta_a du^a \wedge \nu$ be a first order source form on $E=\mathbf{R}^n \times \mathbf{R}^n$ invariant under $\mathbf{t}(n)$. Then Δ admits $\mathbf{t}(n)$ conservation laws if and only if the functions $\Psi_i = u_i^a \Delta_a$ satisfy the following equations*:

$$\partial_a \Psi_i - u_l^c \partial_c \partial_a^l \Psi_i = 0, \tag{6}$$

$$\partial_a^{(j} \partial_b^{k)} \Psi_i = 0, \tag{7}$$

for all $a, b, i, j, k=1,2,\dots,n$.

In (7), the round brackets indicate symmetrization on the enclosed indices.

Proof. Assume first that the source form Δ admits $\mathrm{t}(n)$ conservation laws. By equation (5) and by the definition of the functions Ψ_i we have that

$$\mathrm{E}_a(\Psi_i)=0, \quad a,i=1,2,\dots,n. \tag{8}$$

Recall that $\Psi_i=\Psi_i(u^a,u^a_i)$ does not depend upon x^j. Thus, when expanded using the expression (1) for the Euler–Lagrange operators, equation (8) becomes

$$0=\partial_a\Psi_i-D_j(\partial^j_a\Psi_i)=\partial_a\Psi_i-u^b_j\partial_b\partial^j_a\Psi_i-u^b_{jk}\partial^k_b\partial^j_a\Psi_i. \tag{9}$$

Now the coefficients of the second order terms u^b_{jk} in (9) must vanish, which immediately yields (7). The remaining terms in (9) yield (6).

Conversely, it is clear from (9) that equations (6) and (7) imply that (8) is satisfied, that is, the source form Δ admits $\mathrm{t}(n)$ conservation laws. □

In order to solve equations (6) and (7) we need to establish some notation. Let A and I be the multi-indices $A=(a_1,a_2,\dots,a_k)$ and $I=(i_1,i_2,\dots,i_k)$ of length $|A|=|I|=k$. If $|A|=|I|\le n$, we define the minor V^I_A of the matrix $(u^a_i)_{a,i=1,2,\dots,n}$ by

$$V^I_A=\frac{1}{(n-k)!}\varepsilon^{i_1\dots i_k i_{k+1}\dots i_n}\varepsilon_{a_1\dots a_k a_{k+1}\dots a_n}u^{a_{k+1}}_{i_{k+1}}\dots u^{a_n}_{i_n}, \tag{10.a}$$

where $\varepsilon^{i_1 i_2\dots i_n}$ and $\varepsilon_{a_1 a_2\dots a_n}$ are the permutation symbols. In particular, V stands for the determinant

$$V=\det(u^a_i).$$

We also let

$$V^I_A=0, \quad \text{if } |A|=|I|>n. \tag{10.b}$$

It is easy to see that the minors V^I_A satisfy the identities

$$\partial^j_b V^I_A=V^{Ij}_{Ab} \quad \text{and} \quad u^c_{i_k}V^{i_1\dots i_{k-1}i_k}_{a_1\dots a_{k-1}a_k}=k\delta^c_{[a_k}V^{i_1\dots i_{k-1}}_{a_1\dots a_{k-1}]}. \tag{11}$$

In (11) the square brackets indicate skew-symmetrization on the enclosed indices.

Lemma 2. *Let $\psi=\psi(u^a_i)$ be a function in the variables u^a_i, $a,i=1,2,\dots,n$, and assume that ψ satisfies the equations*

$$\partial^{(i}_a\partial^{j)}_b\psi=0, \quad a,b,i,j=1,2,\dots,n. \tag{12}$$

Then ψ is a constant linear combination of the minors V_I^A, $|A|=|I|=0,1,\dots,n$.

Proof. We first let $a=b$ in (12). Then the function ψ satisfies

$$\partial_a^i \partial_a^j \psi = 0,$$

for all $a,i,j=1,2,\dots,n$. It immediately follows from this that ψ is a linear combination of the monomials $M_{i_1 i_2 \dots i_k}^{a_1 a_2 \dots a_k} = u_{i_1}^{a_1} u_{i_2}^{a_2} \dots u_{i_k}^{a_k}$, where the indices $a_1, a_2, \dots, a_k$ are all distinct. Write ψ as a sum

$$\psi = \sigma_A^I M_I^A,$$

where the constants σ_A^I are invariant under simultaneous permutations of the entries in A and I. Then equations (12) for $a \neq b$ show that the σ_A^I must be skew symmetric in the indices I. Now the lemma follows. □

Theorem 3. *Let $\Delta = \Delta_a du^a \wedge \nu$ be a first order source form on $E = \mathbf{R}^n \times \mathbf{R}^n$. Then Δ is invariant under the infinitesimal group $\mathfrak{t}(n)$ of translations and Δ admits $\mathfrak{t}(n)$ conservation laws if and only if each component Δ_a of Δ can be expressed as a sum*

$$\Delta_a = V^{-1} \lambda_{i,I}^{\ A} V_a^i V_A^I, \quad a = 1,2,\dots,n, \tag{13}$$

where, if $|A|=|I|=n$, the coefficients $\lambda_{i,[I]}^{\ [A]} = \lambda_{i,I}^{\ A}$ are constant, and, if $|A|=|I|<n$, the coefficients $\lambda_{i,[I]}^{\ [A]} = \lambda_{i,I}^{\ A}$ are functions of the u^b,

$$\lambda_{i,I}^{\ A} = \lambda_{i,I}^{\ A}(u^b), \tag{14}$$

and satisfy

$$\partial_c \lambda_{i,i_1 i_2 \dots i_k}^{\ c a_2 \dots a_k} = 0. \tag{15}$$

Proof. First suppose that Δ is $\mathfrak{t}(n)$ invariant and that Δ admits $\mathfrak{t}(n)$ conservation laws. Then the functions $\Psi_i = u_i^a \Delta_a$ satisfy (6) and (7), and thus, by Lemma 2, we can write

$$\Psi_i = \lambda_{i,I}^{\ A} V_A^I, \tag{16}$$

where the coefficients $\lambda_{i,I}^{\ A} = \lambda_{i,I}^{\ A}(u^a)$ are smooth functions of the dependent variables u^a only. Note that equations (6) and (7) are homogeneous in the variables u_i^a. Thus we can assume that the summation in (16) extends only over multi-indices

I, A of some fixed length k. In the case $k=n$ the conclusion of the theorem is clear from (6). For $k<n$, we have, by (11), that

$$u_l^c \partial_a^l V_A^I = u_l^c V_{Aa}^{Il} = (k+1)\delta_{[a}^c V_{a_1 \dots a_k]}^{i_1 \dots i_k}.$$

Thus equation (6) implies that

$$\begin{aligned} 0 &= (\partial_a \lambda_{i,I}{}^{A}) V_A^I - (k+1)(\partial_c \lambda_{i,I}{}^{A}) \delta_{[a}^c V_{A]}^I \\ &= (\partial_a \lambda_{i,I}{}^{A}) V_A^I - (\partial_a \lambda_{i,I}{}^{A}) V_A^I + \sum_{s=1}^{k} \partial_{a_s} \lambda_{i,i_1 \dots i_k}{}^{a_1 \dots a_{s-1} a_s a_{s+1} \dots a_k} V_{a_1 \dots a_{s-1} a a_{s+1} \dots a_k}^{i_1 \dots i_k} \qquad (17) \\ &= k(\partial_{a_1} \lambda_{i,i_1 i_2 \dots i_k}{}^{a_1 a_2 \dots a_k}) V_{a\, a_2 \dots a_k}^{i_1 i_2 \dots i_k}. \end{aligned}$$

Hence the coefficient $\lambda_{i,I}{}^{A}$ satisfy the divergence condition (15).

Conversely, if (13), (14), and (15) hold, then it is immediate from Proposition 1 and the calculation leading to (17) that the source form Δ is $\mathbf{t}(n)$ invariant and that Δ admits $\mathbf{t}(n)$ conservation laws. ☐

Note that if (15) holds then, at least locally, there are functions $\xi_{i,I}^{a,A} = \xi_{i,I}^{a,A}(u^b)$ satisfying the skew symmetry condition $\xi_{i,[I]}^{[a,A]} = \xi_{i,I}^{a,A}$ such that

$$\lambda_{i,I}{}^{A} = \partial_a \xi_{i,I}^{a,A}.$$

4. Helmholtz conditions

Proposition 4. *Let $\Delta = \Delta_a du^a \wedge \nu$ be a first order source form on $E = \mathbf{R}^n \times \mathbf{R}^n$ invariant under $\mathbf{t}(n)$. Then Δ satisfies the Helmholtz conditions if and only if the functions $\Psi_i = u_i^a \Delta_a$ satisfy the following three equations:*

$$\partial_b \Psi_{[i} u_{j]}^b - u_p^b \partial_b \partial_c^p \Psi_{[i} u_{j]}^c = 0, \qquad (18)$$

$$\partial_b^i \Psi_{(j} u_{k)}^b - \delta_{(j}^i \Psi_{k)} = 0, \qquad (19)$$

$$\partial_a^{(i} \partial_b^{j)} \Psi_{[k} u_{l]}^b = 0, \qquad (20)$$

for all $a, i, j, k, l = 1, 2, \dots, n$.

Proof. Since the source form Δ is of first order and $\mathbf{t}(n)$ invariant the Helmholtz conditions (2) are satisfied provided that

$$2\partial_{[a} \Delta_{b]} - u_p^c \partial_c \partial_{[a}^p \Delta_{b]} = 0, \qquad (21)$$

$$\partial_{(a}^i \Delta_{b)} = 0, \qquad (22)$$

$$\partial_a^{(i} \partial_{[b}^{j)} \Delta_{c]} = 0. \qquad (23)$$

First assume that Δ satisfies the Helmholtz conditions (21), (22), (23). Multiply (21) by $u_i^a u_j^b$ and sum over a and b. An integration by parts shows that

$$u_i^a u_j^b \partial_a^p \Delta_b = u_i^a \partial_a^p (u_j^b \Delta_b) - \delta_j^p u_i^a \Delta_a = u_i^a \partial_a^p \Psi_j - \delta_j^p \Psi_i. \tag{24}$$

Thus (21) becomes

$$\begin{aligned} 0 &= 2u_{[i}^a u_{j]}^b \partial_a \Delta_b - u_p^c \partial_c (u_{[i}^a u_{j]}^b \partial_a^p \Delta_b) \\ &= 2\partial_a \Psi_{[j} u_{i]}^a - u_p^c \partial_c \partial_a^p \Psi_{[j} u_{i]}^a + u_p^c \partial_c \delta_{[j}^p \Psi_{i]} \\ &= -\partial_a \Psi_{[i} u_{j]}^a + u_p^c \partial_c \partial_a^p \Psi_{[i} u_{j]}^a, \end{aligned}$$

from which (18) follows. Equation (19) can be similarly derived from (22).

Finally multiply (23) by $u_k^b u_l^c$ and sum over b and c. An integration by parts results in

$$\partial_a^{(i} \partial_b^{j)} \Psi_{[k} u_{l]}^b + 2u_{[k}^b \delta_{l]}^{(i} \partial_{(a}^{l)} \Delta_{b)} = 0,$$

which, on account of (22), gives (20).

Now it is also a simple matter to check that (21), (22), (23) imply (18), (19), (20). ☐

Remark. Propositions 1 and 4 can also be derived by employing a moving frame for the contact bundle of the variational bicomplex (cf. ref. [1]) reflecting the transformation $\Psi_i = u_i^a \Delta_a$. However, for our purposes it is simply more expedient to give direct computational proofs of these results.

Proposition 5. *Let* $\Delta = \Delta_a du^a \wedge \nu$ *be a first order source form on* $E = \mathbf{R}^n \times \mathbf{R}^n$ *as in Theorem* 3. *Then* Δ *is locally variational if and only if the coefficient functions* $\lambda_{i,I}^{\ A}$ *are completely skew symmetric in the lower indices*:

$$\lambda_{[i,I]}^{\ A} = \lambda_{i,I}^{\ A}.$$

Proof. We first note that by assumption the functions $\Psi_i = u_i^a \Delta_a$ satisfy equations (6) and (7). Hence the Helmholtz conditions (18) and (20) are identically satisfied, and, consequently, Δ is locally variational if and only if (19) holds.

Calculations parallel to those leading to equation (17) in the proof of Theorem 3 yield

$$u_l^a \partial_a^i \Psi_j = \delta_l^i \lambda_{j,I}^{\ A} V_A^I - k \lambda_{j,l i_2 \dots i_k}^{\ a_1 a_2 \dots a_k} V_{a_1 a_2 \dots a_k}^{i i_2 \dots i_k}.$$

Hence (19) holds if and only if

$$\lambda_{(j,l) i_2 \dots i_k}^{\ a_1 a_2 \dots a_k} = 0,$$

that is, the coefficients $\lambda_{i,I}^{\ A}$ are completely skew symmetric in the lower indices. ☐

Lemma 6. *Let* V^I_A *be the minors* (10) *of the matrix* $(u^a_i)_{a,i=1,2,\dots,n}$. *Suppose that there are constants* $\sigma_{i,I}{}^{A}$, *where* $\sigma_{i,[I]}{}^{[A]} = \sigma_{i,I}{}^{A}$, *and polynomials* P_a *in the variables* u^a_i *such that*

$$\sigma_{i,I}{}^{A} V^I_A = u^a_i P_a, \quad i = 1, 2, \dots, n. \tag{25}$$

Then necessarily

$$\sigma_{[i,I]}{}^{A} = \sigma_{i,I}{}^{A}. \tag{26}$$

Conversely, suppose that the constants $\sigma_{i,I}{}^{A}$ *satisfy the skew symmetry condition* (26). *Then* (25) *holds for the polynomials* P_a *given by*

$$P_a = \frac{1}{|I|+1} \sigma_{i,I}{}^{A} V^{iI}_{aA}.$$

Proof. We first note that by expressing each polynomial P_a as a sum of its homogeneous components we can assume that in (25) both sides are homogeneous polynomials of degree $n-k$.

First suppose that equations (25) are satisfied for some constants $\sigma_{i,I}{}^{A}$ and for some polynomials P_a of degree $n-k-1$. Write

$$P_a = C_{a,a_{k+2}a_{k+3}\dots a_n}{}^{i_{k+2}i_{k+3}\dots i_n} u^{a_{k+2}}_{i_{k+2}} u^{a_{k+3}}_{i_{k+3}} \dots u^{a_n}_{i_n},$$

where the coefficients $C_{a,a_{k+2}a_{k+3}\dots a_n}{}^{i_{k+2}i_{k+3}\dots i_n}$ are invariant under simultaneous permutations of the indices $a_{k+2}, a_{k+3}, \dots, a_n$ and $i_{k+2}, i_{k+3}, \dots, i_n$. Now equations (25) become

$$\begin{aligned} \frac{1}{(n-k)!} \sigma_{i,i_1\dots i_k}{}^{a_1\dots a_k} \varepsilon^{i_1\dots i_k i_{k+1}\dots i_n} \varepsilon_{a_1\dots a_k a_{k+1}\dots a_n} u^{a_{k+1}}_{i_{k+1}} u^{a_{k+2}}_{i_{k+2}} \dots u^{a_n}_{i_n} & \\ = \delta^{i_{k+1}}_i C_{a_{k+1},a_{k+2}\dots a_n}{}^{i_{k+2}\dots i_n} u^{a_{k+1}}_{i_{k+1}} u^{a_{k+2}}_{i_{k+2}} \dots u^{a_n}_{i_n}. & \end{aligned} \tag{27}$$

Let the indices $b_{k+1}, b_{k+2}, \dots, b_n$ be distinct. Equating the coefficients of the monomial $u^{b_{k+1}}_{j_{k+1}} u^{b_{k+2}}_{j_{k+2}} \dots u^{b_n}_{j_n}$ on both sides of (27) we obtain the equation

$$\sigma_{i,j_1\dots j_k}{}^{b_1\dots b_k} \varepsilon^{j_1\dots j_k j_{k+1}\dots j_n} \varepsilon_{b_1\dots b_k b_{k+1}\dots b_n} = (n-k-1)! \sum_{p=k+1}^{n} \delta^{j_p}_i C_{b_p,b_{k+1}\dots b_{\hat p}\dots b_n}{}^{j_{k+1}\dots j_{\hat p}\dots j_n}.$$

Thus

$$\begin{aligned} \sigma_{i,j_1\dots j_k}{}^{b_1\dots b_k} &= \frac{1}{(n-k)!(n-k)} \varepsilon_{j_1\dots j_k j_{k+1}\dots j_n} \varepsilon^{b_1\dots b_k b_{k+1}\dots b_n} \sum_{p=k+1}^{n} \delta^{j_p}_i C_{b_p,b_{k+1}\dots b_{\hat p}\dots b_n}{}^{j_{k+1}\dots j_{\hat p}\dots j_n} \\ &= \frac{1}{(n-k)!(n-k)} \sum_{p=k+1}^{n} \varepsilon_{j_1\dots j_{p-1} i j_{p+1}\dots j_n} \varepsilon^{b_1\dots b_n} C_{b_p,b_{k+1}\dots b_{\hat p}\dots b_n}{}^{j_{k+1}\dots j_{\hat p}\dots j_n} \\ &= \frac{1}{(n-k)!} \varepsilon_{j_1\dots j_k i j_{k+2}\dots j_n} \varepsilon^{b_1\dots b_k b_{k+1} b_{k+2}\dots b_n} C_{b_{k+1},b_{k+2}\dots b_n}{}^{j_{k+2}\dots j_n}, \end{aligned}$$

which shows that

$$\sigma_{[i,j_1j_2\dots j_k]}^{\;b_1b_2\dots b_k} = \sigma_{i,j_1j_2\dots j_k}^{\;b_1b_2\dots b_k}.$$

Conversely, suppose that the constants $\sigma_{i,I}^{\;A}$ are skew symmetric in the lower indices. We compute

$$u^a_j\sigma_{i,I}^{\;A}V^{iI}_{aA} = \sigma_{i,I}^{\;A}u^a_jV^{iI}_{aA} = (|I|+1)\sigma_{i,I}^{\;A}\delta^{[i}_jV^{I]}_A = (|I|+1)\sigma_{i,I}^{\;A}\delta^i_jV^I_A = (|I|+1)\sigma_{j,I}^{\;A}V^I_A.$$

Hence (25) holds with

$$P_a = \frac{1}{|I|+1}\sigma_{i,I}^{\;A}V^{iI}_{aA}. \quad \square$$

Theorem 7. *Let $\Delta = \Delta_a du^a \wedge \nu$ be a first order source form on $E = \mathbf{R}^n \times \mathbf{R}^n$. Suppose that Δ is invariant under the infinitesimal transformation group $\mathbf{t}(n)$ of translations acting on the base $\mathbf{R}^n$ of E and that Δ admits $\mathbf{t}(n)$ conservation laws. Then Δ is locally variational if and only if the components Δ_a are smooth in the first order derivative variables u^a_i.*

Proof. First suppose that the source form Δ is locally variational, that is, Δ satisfies the Helmholtz conditions. By (22) and (23)

$$\partial^{(i}_a\partial^{j)}_b\Delta_c = \partial^{(i}_a\partial^{j)}_{(b}\Delta_{c)} + \partial^{(i}_a\partial^{j)}_{[b}\Delta_{c]} = 0,$$

for all $i, j, a, b, c = 1, 2, \dots, n$. We now proceed as in the proof of Lemma 2 to conclude that the components Δ_a are polynomials in the derivative variables u^a_i. In particular, each Δ_a is a smooth function in the variables u^a_i.

Conversely, suppose that the components Δ_a are smooth in the variables u^a_i. By the assumptions and by Theorem 3,

$$\Delta_a = V^{-1}\lambda_{i,I}^{\;A}V^i_aV^I_A,$$

where the coefficients $\lambda_{i,I}^{\;A}$ depend on u^a only. It follows that each Δ_a is smooth in the variables u^a_i only if $\lambda_{i,I}^{\;A}V^i_aV^I_A$ is divisible by V, that is, only if the components Δ_a are polynomials in the variables u^a_i. But then, an application of Lemma 6 shows that the coefficients $\lambda_{i,I}^{\;A}$ must satisfy the skew symmetry condition $\lambda_{[i,I]}^{\;A} = \lambda_{i,I}^{\;A}$, which, by Proposition 5, implies that $\Delta = \Delta_a du^a \wedge \nu$ is locally variational, as required. $\square$

Let the set $\mathcal{A}(n)$ consist of all source forms $\Delta = \Delta_a du^a \wedge \nu$ on $E = \mathbf{R}^n \times \mathbf{R}^n$ whose components Δ_a can be expressed as a sum

$$\Delta_a = V^{-1}\lambda_{i,I}^{\;A}V^i_aV^I_A, \quad |A| = |I| \geq 1, \tag{28}$$

where, if $|A| = |I| = n$, the coefficients $\lambda_{i,[I]}^{\;[A]} = \lambda_{i,I}^{\;A}$ are constant, and, if $|A| = |I| < n$, the coefficients $\lambda_{i,[I]}^{\;[A]} = \lambda_{i,I}^{\;A}$ are functions of the u^b, $\lambda_{i,I}^{\;A} = \lambda_{i,I}^{\;A}(u^b)$, and satisfy

$$\lambda_{[i,I]}^{\;A} = 0 \quad \text{and} \quad \partial_a\lambda_{i,i_1i_2\dots i_k}^{\;aa_2\dots a_k} = 0.$$

Theorem 8. *Consider the following three properties of a source form* Δ *on* $E=\mathbf{R}^n\times\mathbf{R}^n$:

(i) Δ *is* $\mathbf{t}(n)$ *invariant,*

(ii) Δ *admits* $\mathbf{t}(n)$ *conservation laws,*

(iii) Δ *is locally variational.*

Then every $\Delta\in\mathcal{A}(n)$ *satisfies* (i) *and* (ii) *but not* (iii). *Moreover, let* Δ *be a first order source form satisfying* (i) *and* (ii). *Then there is a unique source form* $\Delta_A\in\mathcal{A}(n)$ *and a unique source form* Δ_V *satisfying* (i), (ii), *and* (iii) *such that*

$$\Delta=\Delta_A+\Delta_V. \tag{29}$$

Proof. Let $\Delta\in\mathcal{A}(n)$. By Theorem 3 the source form Δ satisfies (i) and (ii). However, by Proposition 5, Δ satisfies (iii) only if $\lambda_{[i,I]}^{\ \ \ A}=\lambda_{i,I}^{\ \ A}$, that is, only if $\Delta=0$.

Next let Δ be a first order source form satisfying (i) and (ii). By Theorem 3 the components Δ_a of Δ are of the form

$$\Delta_a=V^{-1}\lambda_{i,I}^{\ \ A}V_a^iV_A^I. \tag{30}$$

Write

$$\Delta=\Delta_o+\Delta_1,$$

where Δ_o involves the terms in (30) with $|A|=|I|=0$ and Δ_1 involves the remaining terms. Then by Proposition 5 the source form Δ_o also satisfies (iii). Note that the coefficient functions $\lambda_{i,I}^{\ \ A}$ of Δ_1 can be uniquely expressed as a sum

$$\lambda_{i,I}^{\ \ A}=\sigma_{i,I}^{\ \ A}+\zeta_{i,I}^{\ \ A},$$

where the $\sigma_{i,I}^{\ \ A}$ are completely skew symmetric in the lower indices, $\sigma_{[i,I]}^{\ \ \ A}=\sigma_{i,I}^{\ \ A}$, and the $\zeta_{i,I}^{\ \ A}$ satisfy

$$\zeta_{i,[I]}^{\ \ \ A}=\zeta_{i,I}^{\ \ A}\quad\text{and}\quad\zeta_{[i,I]}^{\ \ \ A}=0.$$

In fact, we simply let

$$\sigma_{i,I}^{\ \ A}=\lambda_{[i,I]}^{\ \ \ A}\quad\text{and}\quad\zeta_{i,I}^{\ \ A}=\lambda_{i,I}^{\ \ A}-\lambda_{[i,I]}^{\ \ \ A}. \tag{31}$$

Let Δ_A and $\Delta_{\hat{V}}$ be the source forms with the components

$$\Delta_{A,a}=V^{-1}\zeta_{i,I}^{\ \ A}V_a^iV_A^I\quad\text{and}\quad\Delta_{\hat{V},a}=V^{-1}\sigma_{i,I}^{\ \ A}V_a^iV_A^I.$$

By Theorem 3 and by equation (31) the divergences

$$\partial_a\sigma_{i,i_1i_2\ldots i_k}^{\ \ \ aa_2\ldots a_k}=0\quad\text{and}\quad\partial_a\zeta_{i,i_1i_2\ldots i_k}^{\ \ \ aa_2\ldots a_k}=0$$

vanish. Write

$$\Delta_V = \Delta_o + \Delta_{\widehat{V}}.$$

Then

$$\Delta = \Delta_A + \Delta_V,$$

where $\Delta_A \in \mathcal{A}(n)$ and Δ_V satisfies (i), (ii), and (iii).

Finally, in order to prove the uniqueness of the decomposition (29) it suffices to show that if

$$\hat{\Delta}_A + \hat{\Delta}_V = 0 \tag{32}$$

for some $\hat{\Delta}_A \in \mathcal{A}(n)$ and some $\hat{\Delta}_V$ satisfying (i), (ii), and (iii), then both $\hat{\Delta}_A$ and $\hat{\Delta}_V$ vanish. But equation (32) implies that $\hat{\Delta}_A \in \mathcal{A}(n)$ is locally variational. Thus, by Proposition 5, $\hat{\Delta}_A$, and consequently $\hat{\Delta}_V$, must vanish, as required. □

5. Examples

We start with two examples showing that the various assumptions of Theorem 7 are also necessary.

Example 1

In this example we let $n<m$, i.e., the number of the independent variables is strictly less than the number of the dependent variables. Following a general construction in ref. [2] we let $\Delta = \Delta_a du^a \wedge \nu$ be a source form with the components

$$\Delta_a = \lambda^{b_1 \dots b_{m-n-1}} \varepsilon_{a b_1 \dots b_{m-n-1} c_1 \dots c_n} u_1^{c_1} \dots u_n^{c_n},$$

where $\lambda^{[b_1 \dots b_{m-n-1}]} = \lambda^{b_1 \dots b_{m-n-1}}$ are some functions depending on u^a. Clearly Δ is $\mathbf{t}(n)$ invariant. Moreover, one can easily check that

$$u_i^a \Delta_a = 0,$$

that is, every infinitesimal translation generates a trivial conservation law for Δ. It is a straightforward matter to check that Δ satisfies the first order Helmholtz conditions

$$\mathcal{H}_{ab}^i(\Delta) = \partial_b^i \Delta_a + \partial_a^i \Delta_b = 0.$$

The zeroth order Helmholtz conditions for Δ,

$$\mathcal{H}_{ab}(\Delta) = \partial_b \Delta_a - \partial_a \Delta_b + D_i \partial_a^i \Delta_b = 0,$$

reduce to

$$\mathcal{H}_{ab}(\Delta) = (m-n)\partial_{b_1}\lambda^{b_1 b_2 \dots b_{m-n-1}}\varepsilon_{abb_2\dots b_{m-n-1}c_1\dots c_n}u_1^{c_1}\dots u_n^{c_n} = 0.$$

Consequently, the source form Δ is locally variational if and only if the divergence $\partial_{a_1}\lambda^{a_1 a_2 \dots a_n}$ vanishes. Thus, in general, Theorem 7 fails if the number of independent variables is strictly less than the number of dependent variables.

Example 2

Let $\Delta = \Delta_a du^a \wedge \nu$ be a source form on $E = \mathbf{R}^n \times \mathbf{R}^n$ with the components

$$\Delta_a = P_b^i V_a^j u_{ij}^b + D_i(P_a^i V), \tag{33}$$

where $P_a^i = P_a^i(u^b, u_j^b)$ are functions in the dependent variables and their first order derivatives. Note that, in general, Δ is of second order. Clearly Δ is $\mathbf{t}(n)$ invariant. Also

$$u_k^a \Delta_a = u_k^a P_b^i V_a^j u_{ij}^b + u_k^a D_i(P_a^i V) = P_b^i V u_{ik}^b + u_k^a D_i(P_a^i V) = D_i(u_k^a P_a^i V).$$

Thus Δ admits $\mathbf{t}(n)$ conservation laws. However, in general Δ fails to be variational. For example, for a second order source form the Helmholtz condition $\mathcal{H}_{11}^1(\Delta)$ becomes

$$\mathcal{H}_{11}^1(\Delta) = 2\frac{\partial \Delta_1}{\partial u_1^1} - 2D_1\frac{\partial \Delta_1}{\partial u_{11}^1} - D_2\frac{\partial \Delta_1}{\partial u_{12}^1} - \dots - D_n\frac{\partial \Delta_1}{\partial u_{1n}^1} = 0.$$

After some long, though straightforward, calculations we find that for Δ as in (33),

$$\mathcal{H}_{11}^1(\Delta) = 2\frac{\partial P_a^i}{\partial u_1^1}V_1^j u_{ij}^a - 2\frac{\partial P_1^1}{\partial u_i^a}V_1^j u_{ij}^a + D_i\left(\frac{\partial P_1^i}{\partial u_1^1}V\right) - D_i\left(\frac{\partial P_1^1}{\partial u_i^1}V\right). \tag{34}$$

Thus, for example, with $P_2^1 = u_1^1$, $P_a^i = 0$ otherwise, Δ becomes a second order source form with polynomial components of degree $n+1$. But by (34), $\mathcal{H}_{11}^1(\Delta) = 2V_1^j u_{1j}^2 \neq 0$, and Δ is not locally variational. This shows that Theorem 7 fails for second order source forms.

Example 3

Here we explicitly write down all anomalous systems in $\mathcal{A}(2)$ as given in (28). We let u and v stand for the dependent variables. Then

$$V_1^1 = v_y, \quad V_2^1 = -u_y, \quad V_1^2 = -v_x, \quad V_2^2 = u_x, \quad \text{and} \quad V = u_x v_y - u_y v_x.$$

First, in the simpler case $|A|=|I|=2$, equation (28) gives

$$\Delta_1=(\lambda_1 v_y-\lambda_2 v_x)/V,$$
$$\Delta_2=(-\lambda_1 u_y+\lambda_2 u_x)/V,$$

where λ_1, λ_2 are some constants. For $|A|=|I|=1$ equation (28) gives

$$\Delta_1=\frac{\lambda^1_{1,1}(v_y)^2-\lambda^2_{1,1}u_yv_y-2\lambda^1_{1,2}v_xv_y+\lambda^2_{1,2}(u_yv_x+u_xv_y)+\lambda^1_{2,2}(v_x)^2-\lambda^2_{2,2}v_xv_y}{V},$$
$$\Delta_2=\frac{-\lambda^1_{1,1}u_yv_y+\lambda^2_{1,1}(u_y)^2+\lambda^1_{1,2}(u_xv_y+u_yv_x)-2\lambda^2_{1,2}u_xu_y-\lambda^1_{2,2}u_xv_x+\lambda^2_{2,2}(u_x)^2}{V},$$

where $\lambda^a_{1,1}$, $\lambda^a_{1,2}$, $\lambda^a_{2,2}$, $a=1,2$, are functions of u and v and satisfy the divergence condition

$$\frac{\partial\lambda^1_{i,j}}{\partial u}+\frac{\partial\lambda^2_{i,j}}{\partial v}=0.$$

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