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Linear analysis of quadrature domains

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1. Introduction

A recent study of the L-problem of moments in two real variables [16] has revealed that its extremal solutions coincide with the characteristic functions of all bounded quadrature domains in the complex plane. The main technical tool in obtaining this result was the theory of linear bounded Hilbert space operators with rank-one self-commutator. The aim of the present paper is to isolate and present in more detail the relationship between quadrature domains and this class of operators naturally attached to them, without any explicit reference to the original moment problem.

The identities which relate these two categories of objects are rather simple and constructive. They resemble very much some one variable formulae in the spectral theory of self-adjoint operators. Although this paper is not intended to be related to applied mathematics, we have the feeling, partially based on this comparison, that the basic formulae of this paper will be accessible in the future to a numerical approach, with benefits both for operators with rank-one self-commutator as well as for quadrature domains.

First, without entering into technical details, a few definitions and general remarks are in order. Let Ω be a bounded domain of the complex plane bounded by finitely many piece-wise smooth boundaries. The domain Ω is said to be of *quadrature* for the class $L^1_a(\Omega)$ of all integrable analytic functions in Ω, if there is a distribution u with finite support in Ω which satisfies:

$$\int_\Omega f(z)\,dA(z)=u(f), \quad f\in L^1{}_a(\Omega). \tag{1}$$

Here and throughout this paper dA stands for the planar Lebesgue measure.

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The simplest example of a quadrature domain is a disk D in $\mathbf{C}$, in which case the distribution u is a multiple of the Dirac measure carried by the centre of D.

A quadrature domain Ω has in fact real algebraic boundary of a very special form, see [1], [10], [11]. A characteristic property of Ω and its boundary $\partial\Omega$ is the existence of a meromorphic function S in Ω, continuous up to $\overline{\Omega}$ and satisfying:

$$(2) \qquad S(z)=\bar{z}, \quad z\in\partial\Omega.$$

This function is called the *Schwarz function* of the domain Ω and it encodes in a finite form the structure of Ω. As the name suggests, the meromorphic function which satisfies identity (2) is related to the Schwarz reflection in portions of the boundary of Ω. The monograph [9] by Phillip Davis is devoted to various aspects of the theory of the Schwarz function.

The last two decades have witnessed a renewed interest and constant progress in the theory of quadrature domains. Questions such as the constructions and parametrization of quadrature domains with prescribed distribution u, the algebraic structure of the boundary or various functional analytic characterizations of quadrature domains have been succesfully investigated by using essentially two categories of methods: the theory of compact Riemann surfaces and variational methods for partial differential equations. The recent monograph [19] by Harold Shapiro contains a general overview and ample bibliographical remarks of this interesting theory.

We turn now to the second class of objects appearing in the present note. Approximately in the same period of time, the structure of Hilbert space operators with rank-one self-commutator was studied and understood. Let H be a separable, complex infinite dimensional Hilbert space and let T be a bounded linear operator acting on H. We will be interested in the commutation relation:

$$(3) \qquad [T^*,T]=\xi\otimes\xi,$$

where by definition the rank-one self-adjoint operator $\xi\otimes\xi$ acts on a vector $\eta\in H$ as follows $(\xi\otimes\xi)(\eta)=\langle\eta,\xi\rangle\xi$.

Assuming that the operator T is pure, that is it has no normal direct summand, the unitary equivalence class of T is parametrized by a "spectral parameter" which is a function $g\colon\mathbf{C}\to[0,1]$ with compact support, called the *principal function* of T. A possible way of relating the operator T to its principal function g is the following remarkable trace formula:

$$(4) \qquad \operatorname{Tr}[p(T,T^*),q(T,T^*)]=\frac{1}{\pi}\int_{\mathbf{C}}(\bar{\partial}p\partial q-\partial p\bar{\partial}q)g\,dA,$$

where p, q are polynomials in two variables, $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial \bar{z}$. We put by convention in a polynomial $r(T, T^*)$ all powers of T^* to the right of the powers of T. In fact formula (4) is valid for an ordered functional calculus with smooth functions in T and T^*.

An equivalent form of the trace formula (4) is the following expression for the standard multiplicative commutator resolvent of T:

$$\det((T-w)^{*-1}(T-z)(T-w)^*(T-z)^{-1}) = \exp\left(\frac{-1}{\pi}\int_{\mathbf{C}} \frac{g(\zeta)\,dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\right), \tag{5}$$
$$|z|, |w| > \|T\|.$$

Note that the above determinant is not equal to one, as it may appear to be, because in the infinite dimensional setting the factors $T-z$, etc. do not belong to the determinant class, and hence $\det(T-z)$ does not exist.

One of the most important features of this spectral parameter is its freeness. More precisely, there is a bijection between the unitary equivalence classes of operators with rank-one self-commutator and all elements $g \in L^1{}_{\mathrm{comp}}(\mathbf{C}), 0 \le g \le 1$, a.e.

The principal function has appeared for the first time in a landmark paper by Pincus [14]; the trace formula (4) was obtained thanks to the independent efforts of Helton–Howe [12] and Carey–Pincus [4]. Further details on the theory of the principal function can be found in the monographs [5], [13], [20].

Let $g: \mathbf{C} \to [0,1]$ be a measurable function with compact support and let T be a Hilbert space operator with rank-one self-commutator (3) and principal function equal almost everywhere to g. It was Kevin Clancey who, in a series of papers [6], [7], [8], has investigated the properties of the exponential kernel which appears in formula (5) and its relevance for the spectral theory of the operator T. To simplify notation we denote:

$$E_g(z, w) = \exp\left(\frac{-1}{\pi}\int_{\mathbf{C}} \frac{g(\zeta)\,dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{w})}\right), \quad z, w \in \mathbf{C}. \tag{6}$$

If the integral is infinite for certain values $z=w$ we take the exponential to be zero by definition.

One remarks that, for a fixed $z \in \mathbf{C}$, the equation $(T^* - \bar{z})x = \xi$ has a unique solution of minimal norm. We denote by $x = (T^* - \bar{z})^{-1}\xi$ this solution. An important result of [6] asserts that the function E_g is separately continuous and can be expressed everywhere by the formula:

$$E_g(z, w) = 1 - \langle (T^* - \bar{w})^{-1}\xi, (T^* - \bar{z})^{-1}\xi \rangle. \tag{7}$$

The relation between quadrature domains and the above operators appears in [16] in the following form. The function $E_g(z, w)$ is rational at infinity with a denominator of the form $P(z)\overline{P(w)}$, where P is a polynomial, if and only if the function g coincides almost everywhere with the characteristic function of a quadrature domain. Moreover, this happens if and only if the linear span K of the vectors $T^{*k}\xi$ ($k\in\mathbf{N}$) is finite dimensional.

Thus we have three objects in bijective correspondence: the quadrature domain Ω, the operator T with the preceding finiteness condition, and the kernel E_g with the rationality condition described before. The investigation of some constructive relations among them forms the body of this paper. Next we enumerate a few of these relations.

For instance, $\overline{\Omega}$ is the spectrum of T and $\partial\Omega$ is the essential spectrum of T (the Fredholm index inside Ω being -1). The operator T^* can be realized as the multiplication with the variable $\bar{z}$ on a Sobolev space of first order, with reproducing kernel E_g. On the other hand, the defining equation of Ω can be written as:

$$\Omega=\{z\in\mathbf{C}\,;\,\|(T^*\,|\,K-\bar{z})^{-1}\xi\|>1\}.$$

The spectrum of the finite dimensional operator $(T^*|K)^*$ coincides up to multiplicities with the support of the quadrature distribution u (appearing in (1)). The operator T and *a fortiori* the domain Ω is determined up to unitary equivalence by the finite dimensional matrix $(T^*|K,\xi)$, while the distribution u is determined by the rational function $\langle (T^*|K-\bar{z})^{-1}\xi,\xi\rangle$.

As a matter of fact, the matricial structure of T in terms of the finite dimensional data $(T^*|K,\xi)$ is described in Section 4 of this paper. As a corollary we obtain a characterization of all $n\times(n+1)$ complex matrices which appear as $(T^*|K,\xi)$ above and hence we have, at least theoretically, a "moduli space" of all quadrature domains of order $n=\dim(K)$.

Let us finally mention a formula which expresses the Schwarz function S of the domain Ω in terms of some generalized resolvents of the operator T^*:

$$S(z)=\bar{z}+\langle\xi,(T^*-\bar{z})^{-1}\xi-(T^*\,|\,K-\bar{z})^{-1}\xi\rangle,\quad z\in\Omega.$$

Although the preceding relations seem quite natural and not accidental, at this moment only a very little part of the known results about quadrature domains can be explained in this novel framework. We are convinced that some future work will diminish this gap.

It is also interesting to remark a recent work of Bell [2] which computes explicitly, in simple terms, the Bergman and Szegő kernels for a part of the above mentioned quadrature domains. From this perspective there is a strong similarity between the properties of the Bergman, Szegő and the exponential kernel considered below.

Due to the inhomogeneous background necessary in the present paper, the preliminary part (included in Section 2) is expository. Section 3 deals with the proofs of the relations between quadrature domains and their associated Hilbert space operators, while Section 4 is devoted to a canonical matricial decomposition of these operators.

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2. Preliminaries

The technical aspects discussed but not proved in this section can be found in the paper [10] of Gustafsson and in the monograph [13].

2.1. Operators with rank-one self-commutator

Let H be a separable, complex, infinite dimensional Hilbert space and let $T\in L(H)$ be a linear bounded operator acting on H. The commutation relation (3) (or more generally $[T^*,T]\geq 0$, in which case T is called *hyponormal*) has strong and rather unexpected consequences.

Throughout this section we assume that the operator T is *pure*, that is H is generated by the vectors $T^kT^{*l}\xi$, $k,l\in\mathbf{N}$. This is equivalent to saying that T has not non-trivial normal direct summands. If the commutation relation (3) is fulfilled, then some deep inequalitites originating in the works of T. Kato and C. R. Putnam show that the self-adjoint operators $\operatorname{Re} T$ and $\operatorname{Im} T$ (the real and respectively imaginary part of T) have spectral measures which are absolutely continuous with respect to the linear Lebesgue measure. Moreover, in our case the spectral multiplicities of both self-adjoint operators are equal to one.

Thus either $\operatorname{Re} T$ or $\operatorname{Im} T$ can be diagonalized and represented as the multiplication with the variable $x\in\mathbf{R}$ on $L^2(\sigma, dx)$, where σ is a compact subset of the real axis. Suppose that we diagonalize $\operatorname{Re} T$. A simple argument shows that in this representation the other self-adjoint operator, $\operatorname{Im} T$, is a combination of multiplications with essentially bounded functions and the Hilbert transform. This (Cartesian) singular integral model for T led Pincus to the discovery of the principal function, as a phase shift (in the sense of M. G. Krein) of the parameters in the integral representation of $\operatorname{Im} T$. The fact that the principal function has several invariance and naturality properties and it satisfies the trace formula (4) is remarkable, non-trivial and was proved much later. One of the important trends of the principal function lies in the fact that it is a free variable in the parametrization of all pure operators T which satisfy the commutation relation (3). See for references [4], [5], [12], [13], [14], [20].

Suppose, as before that the pure hyponormal operator T has rank one self-commutator $[T^*,T]=\xi\otimes\xi$, and fix a point $z\in\mathbf{C}$. Then the inequality

$$(T^*-\bar{z})(T-z)\geq[T^*-\bar{z},T-z]=\xi\otimes\xi \tag{8}$$

shows that the vector ξ belongs to the range of the operator $T^*-\bar{z}$. Let $(T^*-\bar{z})^{-1}\xi$ denote the unique vector $x\in H$ which satisfies the conditions:

$$(T^*-\bar{z})x=\xi, \quad x\perp \operatorname{Ker}(T^*-\bar{z}). \tag{9}$$

The inequality (8) also shows that

$$\|(T^*-\bar{z})^{-1}\xi\|\leq 1, \quad z\in\mathbf{C}.$$

Moreover, it turns out that the function $z\mapsto(T^*-\bar{z})^{-1}\xi$ is weakly continuous on the whole complex plane, see [6] or [13, Chapter XI].

Simple Hilbert space arguments prove that the germ at infinity of the real analytic function $z\mapsto\|(T^*-\bar{z})^{-1}\xi\|$ is a complete unitary invariant for the original operator T. This is equivalent to the fact that the infinite "covariance matrix"

$$N_T(k,l)=\langle T^{*k}\xi,T^{*l}\xi\rangle, \quad k,l\in\mathbf{N}, \tag{10}$$

determines the unitary equivalence class of T.

The obvious positivity condition, plus a system of non-linear recurrent algebraic relations, characterize the matrices N_T, see [13, Proposition 4.1].

The relation between these two different unitary invariants of T, the principal function g and the covarianve matrix N_T, is established by Clancey's remarkable formula:

$$1-\|(T^*-\bar{z})^{-1}\xi\|^2=\exp\left(\frac{-1}{\pi}\int_{\mathbf{C}}\frac{g(\zeta)\,dA(\zeta)}{(\zeta-z)(\bar{\zeta}-\bar{z})}\right), \quad z\in\mathbf{C}. \tag{11}$$

This is a non-trivial extension of identity (5) across the spectrum of T. For a proof, see [6], [8] or [13, Chapter XI].

We mention without giving precise details that the properties of the generalized resolvent $(T^*-\bar{z})^{-1}\xi$ govern the spectral behaviour of T. See for details [13, Chapter XI].

2.2. The exponential kernel

Let $g\colon \mathbf{C}\to[0,1]$ be a measurable function with compact support. The kernel $E_g(z,w)$ defined by relation (6) represents the polarized version of the kernel appearing in (11). As proved in [13, Proposition XI.2.4], the following conditions characterize the kernel $E_g(z,w)$:

(a) $E_g(z,w)$ *is separately continuous in* $z,w\in\mathbf{C}$;

(b) $\lim_{|z|+|w|\to\infty}|E_g(z,w)|=0$;

(c) $1-E_g(z,w)$ *is non-negative definite on* $\mathbf{C}^2$;

(d) *the equation*

$$(\bar{w}-\bar{z})\bar{\partial}_z E_g(z,w)=g(z)(1-E_g(z,w))$$

holds in the sense of distributions.

According to (11), along the diagonal $z=w$ the kernel $E_g(z,z)$ satisfies the inequalities:

$$0\le E_g(z,z)<1,\quad z\in\mathbf{C}.$$

Moreover, for a point z in the spectrum of T but not in its essential spectrum, we have $E_g(z,z)=0$, while $E_g(z,z)>0$ for z in the resolvent set of T (which coincides with the closed support of g).

Although we do not need the following two-variable singular integral model for T derived from the kernel $E_g(z,w)$, we mention it for completeness. Let W be the Hilbert space Hausdorff completion of the space of smooth test functions $\mathcal{D}(\mathbf{C})$ with respect to the hermitian seminorm:

$$\|\phi\|^2=\int_{\mathbf{C}^2} E_g(z,w)\bar{\partial}\phi(z)\partial\phi(w)\,dA(z)\,dA(w),\quad \phi\in\mathcal{D}(\mathbf{C}).$$

Then the following formulae realize the operator T on W:

$$(T\phi)(z)=z\phi(z)-\frac{1}{\pi}\int_{\mathbf{C}}\frac{g(\zeta)\phi(\zeta)\,dA(\zeta)}{\bar{\zeta}-\bar{z}},$$
$$(T^*\phi)(z)=\bar{z}\phi(z).$$

Above $\phi\in\mathcal{D}(\mathbf{C})$ and then the expressions are extended to W by continuity.

Thus the cycle $T\mapsto g\mapsto E_g\mapsto T$ is complete.

2.3. Quadrature domains

Let Ω be a bounded quadrature domain which satisfies the quadrature identity (1) with a distribution u with finite support, say $\{a_1,\dots,a_m\}$ in Ω. Thus we can express u as follows:

$$u(f)=\sum_{i=1}^{m}\sum_{j=0}^{n_i-1} c_{ij} f^{(j)}(a_i), \quad f\in L_a^1(\Omega). \tag{12}$$

Assuming that the highest order coefficients c_{i,n_i-1} are non-zero for $i=1,\dots,m$, we define the *order* n of the quadrature domain Ω by:

$$n=n_1+\dots+n_m.$$

It is worth remarking that in the above representation, u is regarded as an analytic functional with finite support rather than a distribution. With this caution it is obvious that u is unique.

As we have mentioned in the introduction, the boundary of the quadrature domain Ω is real algebraic. Next we recall from Gustafsson [10] a procedure of finding the defining equation of $\partial\Omega$. Let X be the Schottky double of Ω (that is Ω glued together with a reversed copy of it). By taking into account the global reflection formula (2), one extends meromorphically to X the identity function on Ω to a function f which has the poles in the reversed copy of Ω in X. The mirror reflection of f is another meromorphic function h which is algebraically dependent of f. More precisely, there exists a polynomial $P(z,w)$ of degree exactly equal to the order of the quadrature domain Ω which satisfies $P(f,h)=0$. Since for a point $z\in\partial\Omega$ we have $f(z)=z$ and $h(z)=\bar z=S(z)$, the polynomial P is self-conjugate (that is $\overline{P(z,w)}=P(\bar z,\bar w)$) and the identity

$$P(z,S(z))=0$$

holds for $z\in\Omega$. In addition one proves that P is irreducible and

$$\partial\Omega=\{z\in\mathbf{C}\,;P(z,\bar z)=0\}\setminus V,$$

where V is a finite set. See for details [10] and [11].

After a scalar normalization the defining polynomial P is unique. According to [10, Theorem 10], the coefficients of $z^k w^l$, $0\le k\le n$, $l=n-1,n$, determine the analytic functional u.

The polynomial P appears also in a classification due to Gustafsson of the possible singular points in the boundary of Ω , see [11].

3. The basic formulae

This section contains the relations, expressed in simple formulae, among quadrature domains and the corresponding hyponormal operators.

Let Ω be a bounded quadrature domain with real algebraic boundary and with the analytic functional u defined by (12). Let n denote the order of Ω and let $P(z,w)$ be the self-conjugate polynomial of degree n in each variable, which defines the boundary of Ω. We normalize by convention P such that the coefficient of $z^n w^n$ is equal to one. Let S be the Schwarz function of Ω (characterized by relation (2)).

Let T be the pure hyponormal operator with rank-one self-commutator, i.e. $[T^*,T]=\xi\otimes\xi$, and with principal function g equal to the characteristic function of the quadrature domain Ω. With these assumptions we know from [16, Propositions 2.1 and 4.2], that the space:

$$K=\bigvee_{m=0}^{\infty} T^{*m}\xi$$

is finite dimensional. Obviously K is invariant to T^*. We denote:

$$U=(T^*\,|\,K)^*,$$

and we regard U as a linear endomorphism of K.

Since the spectrum of T coincides with $\bar{\Omega}$, Clancey's formula (11) can be read outside $\bar{\Omega}$ as:

$$1-\exp\left(\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{|\zeta-z|^2}\right)=\|(U^*-\bar{z})^{-1}\xi\|^2,\quad z\in\mathbf{C}\setminus\bar{\Omega}. \tag{13}$$

Let p denote the monic minimal polynomial of U. Relation (13) shows that the spectrum of U and hence the zeroes of p lie in Ω. Putting together these observations we obtain the following result.

Proposition 3.1. *With the preceding notation we have:*
(a) $p(z)=\prod_{i=1}^m (z-a_i)^{n_i}$;
(b) $P(z,\bar{w})=p(z)\overline{p(w)}-\langle\overline{p(w)}(U^*-\bar{w})^{-1}\xi,\overline{p(z)}(U^*-\bar{z})^{-1}\xi\rangle$;
(c) $u(f)=\pi\langle f(U)\xi,\xi\rangle,\ f\in L^1_a(\Omega)$.

Proof. First we notice that, because p is the minimal polynomial of U, the vector valued function

$$p(z)(U-z)^{-1}\xi=(p(z)-p(U))(U-z)^{-1}\xi$$

is polynomial in z. Second, the functional calculus $f(U)$ in formula (c) is by convention the natural one, for instance the extension of the polynomial functional calculus given by the Jordan form of U.

We start with formula (c), which is a consequence of the trace identity (4). Let h be an analytic function defined in a neighbourhood of $\overline{\Omega}$ and let $h^*(z)=\overline{h(\bar{z})}$. By virtue of the trace formula (4) we obtain:

$$\begin{aligned}\overline{u(h)} &= \int_\Omega h^*(\bar{z})\,dA(z) = \pi\operatorname{Tr}[T^*, Th^*(T^*)] \\ &= \pi\operatorname{Tr}([T^*,T]h^*(T^*)) = \pi\langle h^*(T^*)\xi,\xi\rangle \\ &= \pi\langle h^*(U^*)\xi,\xi\rangle = \pi\langle \xi, h(U)\xi\rangle = \pi\overline{\langle h(U)\xi,\xi\rangle}.\end{aligned}$$

Since the space of analytic functions in neighbourhoods of $\overline{\Omega}$ is dense in $L^1_a(\Omega)$ because of the regularity of $\partial\Omega$, formula (c) is proved.

In particular we infer from (c) that the zeroes of the minimal polynomial p of U are contained, including their multiplicities, in the support of u. On the other hand, formula (13) shows that the function

$$|p(z)|^2 - \|p(z)(U^*-\bar{z})^{-1}\xi\|^2 = |p(z)|^2 \exp\left(\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{|\zeta-z|^2}\right)$$

is polynomial and it vanishes when z approaches $\partial\Omega$ from outside Ω. By remarking that $\overline{p(z)}(U^*-\bar{z})^{-1}\xi$ has degree at most $n-1$ in $\bar{z}$, the uniqueness of the defining polynomial P implies identity (b).

Finally, let us remark that $\deg(p)=\dim(K)$ because the vector ξ is U^*-cyclic. On the other hand $n=\deg(P)=\deg(p)$, thus the multiplicity of each zero $z=a_i$ of p is necessarily equal to n_i, $i=1,\dots,m$.

This finishes the proof of Proposition 3.1.

A direct consequence of Proposition 3.1 is the following description of the domain Ω in terms of the matrix U and the distinguished vector ξ.

Corollary 3.2. *With the above notation we have:*

$$\Omega \approx \{z\in\mathbf{C}; \|(U^*-\bar{z})^{-1}\xi\|>1\},$$
$$\partial\Omega \approx \{z\in\mathbf{C}; \|(U^*-\bar{z})^{-1}\xi\|=1\},$$

where "$\approx$" means equal up to a finite set.

So far we have established the relations between Ω, n, P and the linear data U,ξ. Next we represent the Schwarz function S in terms of T and U.

Proposition 3.3. *Let Ω be a bounded quadrature domain and let T, U be its associated linear operators. Then:*

$$S(z)=\bar{z}+\langle\xi,((T^*-\bar{z})^{-1}\xi-(U^*-\bar{z})^{-1}\xi)\rangle, \quad z\in\Omega.$$

Proof. By taking residues at infinity in the variable $\bar{z}$, Clancey's formula (11) yields:

$$\langle\xi,(T^*-\bar{z})^{-1}\xi\rangle=\frac{1}{\pi}\int_\Omega\frac{dA(\zeta)}{\zeta-z}, \quad z\in\mathbf{C}.$$

Consequently the Cauchy–Pompeiu formula can be written as:

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\bar{\zeta}d\zeta}{\zeta-z}=\begin{cases}\bar{z}+\langle\xi,(T^*-\bar{z})^{-1}\xi\rangle, & z\in\Omega,\\ \langle\xi,(T^*-\bar{z})^{-1}\xi\rangle, & z\in\mathbf{C}\setminus\bar{\Omega}.\end{cases}$$

On the other hand,

$$\langle\xi,(T^*-\bar{z})^{-1}\xi\rangle=\langle\xi,(U^*-\bar{z})^{-1}\xi\rangle, \quad z\in\mathbf{C}\setminus\bar{\Omega},$$

and the right hand side member of this identity extends meromorphically inside Ω.

Thus the function

$$\bar{z}+\langle\xi,(T^*-\bar{z})^{-1}\xi\rangle-\langle\xi,(U^*-\bar{z})^{-1}\xi\rangle$$

is meromorphic in Ω, continuous on $\bar{\Omega}$ and it coincides with $\bar{z}$ on $\partial\Omega$. Therefore this is necessarily the Schwarz function of the quadrature domain Ω.

The proof of Proposition 3.3 also identifies the polar parts of the Schwarz function S and the compressed resolvent $-\langle(U^*-\bar{z})^{-1}\xi,\xi\rangle$.

Indeed, let $z\in\Omega$ and consider the operator $T-z$. Since T is pure it follows that $T-z$ is one to one, with closed range of codimension one in H. Thus the self-adjoint operator $(T^*-\bar{z})(T-z)$ is invertible and

$$(T^*-\bar{z})^{-1}\xi=(T-z)[(T^*-\bar{z})(T-z)]^{-1}\xi, \tag{14}$$

because both sides of (14) satisfy the equation $(T^*-\bar{z})x=\xi$ and they are both orthogonal to the kernel of $T^*-\bar{z}$. Moreover, we know that $\|(T^*-\bar{z})^{-1}\xi\|=1$, whence:

$$\begin{aligned}1&=\|(T-z)[(T^*-\bar{z})(T-z)]^{-1}\xi\|^2\\ &=\langle(T^*-\bar{z})(T-z)[(T^*-\bar{z})(T-z)]^{-1}\xi,[(T^*-\bar{z})(T-z)]^{-1}\xi\rangle\\ &=\langle\xi,[(T^*-\bar{z})(T-z)]^{-1}\xi\rangle.\end{aligned}$$

These equalities imply the formula:

$$\bar{z}+\langle\xi,(T^*-\bar{z})^{-1}\xi\rangle=\langle\xi,[(T^*-\bar{z})(T-z)]^{-1}\xi\rangle.$$

The last term in the preceding equation is obviously real analytic in $\mathrm{Re}(z),\mathrm{Im}(z)$. Thus we can state the following result.

Corollary 3.4. *The Schwarz function S of the quadrature domain Ω satisfies*

$$S(z) = -\langle (U-z)^{-1}\xi, \xi\rangle + H(z), \tag{15}$$

where H is an analytic function in Ω.

Proof. It remains to remark that, in view of Proposition 3.3,

$$H(z) = \bar{z} + \langle \xi, (T^* - \bar{z})^{-1}\xi\rangle$$

is an analytic function in $\Omega \setminus \{a_i; i=1, \dots, m\}$. The isolated singularities a_i are removable because the function H is real analytic everywhere in Ω.

Thus for functions $f \in A(\Omega)$ which are analytic in Ω and continuous on $\bar{\Omega}$ we have a second proof of assertion (c) in Proposition 3.1:

$$\begin{aligned} u(f) &= \int_\Omega f(\zeta)\, dA(\zeta) = \frac{1}{2i}\int_{\partial\Omega} f(\zeta)\bar{\zeta}\, d\zeta \\ &= \frac{1}{2i}\int_{\partial\Omega} f(\zeta) S(\zeta)\, d\zeta = \left\langle \frac{1}{2i}\int_{\partial\Omega} f(\zeta)(\zeta - U)^{-1}\xi\, d\zeta, \xi \right\rangle = \pi\langle f(U)\xi, \xi\rangle. \end{aligned}$$

Summing up, we have also proved the following fact.

Corollary 3.5. *Let Ω be a bounded quadrature domain with the associated linear data U and ξ. Then there is a bijection between the analytic functional u and the rational function $\langle (U-z)^{-1}\xi, \xi\rangle$.*

More precisely we have the following relation:

$$u(f) = \frac{1}{2i}\int_{\partial\Omega} f(\zeta)\langle (U-z)^{-1}\xi, \xi\rangle\, d\zeta, \quad f \in A(\Omega). \tag{16}$$

In addition we know from the proofs of Propositions 3.1 and 3.3 that there is a unique irreducible representation

$$\langle (U-z)^{-1}\xi, \xi\rangle = \frac{q(z)}{p(z)},$$

where p is given by Proposition 3.1.(a) and q is a polynomial of degree $n-1$ determined by the identity (16).

Finally, we point out two algebraic identities which can be derived from the previous considerations.

Proposition 3.6. *Let Ω be a bounded quadrature domain of order n with Schwarz function $S(z)$ with denominator $p(z)$, and defining polynomial $P(z,w)$. Then there are analytic functions $a_1(z),\dots,a_{n-1}(z)$ defined in a neighbourhood of $\overline{\Omega}$ such that the identity*

$$P(z,w)=(w-S(z))p(z)(w^{n-1}+a_1(z)w^{n-2}+\dots+a_{n-1}(z)),$$

holds for $z\in\Omega$ and $w\in\mathbf{C}$.

Proof. According to Proposition 3.1.(b), the identity

$$1-\langle (U^*-\bar{w})^{-1}\xi,(U^*-\bar{z})^{-1}\xi\rangle=\frac{P(z,\bar{w})}{p(z)\overline{p(w)}}$$

is valid everywhere, in the sense of meromorphic functions. In particular, for a point z in the boundary of Ω, we have:

$$\frac{P(z,\bar{w})}{p(z)\overline{p(w)}}=\|(U^*-\bar{z})^{-1}\xi\|^2-\langle (U^*-\bar{w})^{-1}\xi,(U^*-\bar{z})^{-1}\xi\rangle,$$

and by the resolvent equation we can write

$$(U^*-\bar{z})^{-1}\xi-(U^*-\bar{w})^{-1}\xi=(\bar{z}-\bar{w})(U^*-\bar{w})^{-1}(U^*-\bar{z})^{-1}\xi.$$

Therefore, by substituting $\bar{z}=S(z)$ and multiplying both sides by $p(z)\overline{p(w)}$ we obtain an identity of the form:

$$P(z,\bar{w})=(S(z)-\bar{w})m(z,\bar{w}),$$

where $m(z,\bar{w})$ is a polynomial in $\bar{w}$ of degree at most $n-1$ and with coefficients meromorphic functions in z, defined in a neighbourhood of $\overline{\Omega}$, namely in the region where the Schwarz function is defined. By analytic continuation, the identity

$$m(z,\bar{w})=\frac{P(z,\bar{w})}{S(z)-\bar{w}}$$

holds for all $z\in\Omega$ and $w\in\mathbf{C}$.

The preceding resolvent equation shows that the only possible poles of the meromorphic function m in Ω are at the points $\lambda\in\Omega$ with the property that $S(\lambda)$ is in the spectrum of U^*. But it is clear from the definition of the function m that these singularities are removable. On the other hand, since $S(z)$ has a pole at every zero of the polynomial p, with exactly the same multiplicity, the coefficients of the polynomial in $\bar{w}$, $m(z,\bar{w})$ are each divisible by p. Finally, the coefficient of $\bar{w}^n$ in $P(z,\bar{w})$ is exactly $p(z)$, and, after changing $\bar{w}$ into w, this finishes the proof of Proposition 3.6.

A similar, but less constructive conclusion can be derived from the compact Riemann surface approach to quadrature domains. The implications of an identity as above in the statement of Proposition 3.6 are amply discussed in [10] and [11].

Exactly as in the proof of Proposition 3.6, the analytic continuation of the identity $\|(U^*-\bar{z})^{-1}\xi\|=1$, $z\in\partial\Omega$, gives the following result.

Lemma 3.7. *With the above notations, the identity*

$$\langle (U^*-S(z))^{-1}\xi, (U^*-\bar{z})^{-1}\xi\rangle = 1,$$

holds for every point $z\in\Omega$.

In fact the identity in Lemma 3.7 extends in the exterior of the domain Ω, as far as the Schwarz function extends. Let $\lambda\in\mathbf{C}\setminus\Omega$ be such a point. According to Corollary 3.2 we have $\|(U^*-\bar{\lambda})^{-1}\xi\|<1$ and hence, by Lemma 3.7, $\|(U^*-S(z))^{-1}\xi\|>1$, that is $\overline{S(z)}\in\Omega$, and so on. We shall resume this type of analysis elsewhere.

4. Linear data associated to quadrature domains

Let $T\in L(H)$ be a pure hyponormal operator with rank-one self-commutator, $[T^*,T]=\xi\otimes\xi$, and satisfying the condition $\dim(K)<\infty$, where $K=\bigvee_{k=0}^{\infty} T^{*k}\xi$. We know from [16] and the preceding sections that the set of unitary classes of these operators T is in bijective correspondence with the bounded planar quadrature domains. Moreover, we know that the finite matrix (U,ξ), where $U=(T^*|K)^*$, characterizes up to unitary equivalence the operator T. The purpose of this last section is to describe a block-matricial form for T which contains only U and simple operations on U, and secondly to characterize the matrices (U,ξ) which can arise in this construction.

We begin with a few elementary computations which will gradually lead to the matricial form of T. Let

$$T=\begin{pmatrix} U & 0 \\ V & W \end{pmatrix}$$

be the matrix of T with respect to the orthogonal sum decomposition $H=K\oplus K^{\perp}$. Then

$$[T^*,T]=\begin{pmatrix} [U^*,U]+V^*V & V^*W-UV^* \\ W^*V-VU^* & [W^*,W]-VV^* \end{pmatrix}=\begin{pmatrix} \xi\otimes\xi & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the following relations hold:

$$[U^*,U]+VV^*=\xi\otimes\xi, \tag{17}$$

$$W^*V=VU^*, \tag{18}$$

$$[W^*,W]=VV^*. \tag{19}$$

According to (18), the operator W^* leaves invariant the range of V. Let us denote $K_1=VK\subset K^\perp$. It is clear that $\dim(K_1)\le\dim(K)$.

Now corresponding to the decomposition $K^\perp=K_1\oplus(K^\perp\ominus K_1)$ the operator W has the form:

$$W=\begin{pmatrix} U_1 & 0 \\ V_1 & W_1 \end{pmatrix}.$$

Exactly as before, by taking the self-commutator of W we find:

$$[U_1^*,U_1]+V_1^*V_1=VV^*, \tag{20}$$

$$W_1^*V_1=V_1U_1^*, \tag{21}$$

$$[W_1^*,W_1]=V_1V_1^*. \tag{22}$$

Notice also that $T(K)\subset K\oplus K_1$.

Thus the whole process can be repeated and we end with a block-matrix

$$\begin{pmatrix} U & 0 & 0 & 0 & \dots \\ V & U_1 & 0 & 0 & \dots \\ 0 & V_1 & U_2 & 0 & \dots \\ 0 & 0 & V_2 & U_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{23}$$

which corresponds to a subspace L of H given by:

$$L=K\oplus K_1\oplus K_2\oplus\dots\,.$$

We remark the inequalities:

$$\dim(K)\ge\dim(K_1)\ge\dim(K_2)\ge\dots\,.$$

The elements of the above infinite matrix satisfy the relations:

$$[U_k^*,U_k]=V_{k-1}V_{k-1}^*-V_k^*V_k,\quad k\ge 2, \tag{24}$$

$$V_k^*U_{k+1}=U_kV_k^*,\qquad k\ge 2. \tag{25}$$

We also remark that

$$T^kK\subset K\oplus K_1\oplus\dots\oplus K_k,\quad k\ge 1. \tag{26}$$

Lemma 4.1. *With the above notation* $L=H$ *(hence the matrix* (23) *represents the operator* T*) and* $\dim(K_k)=\dim(K)$ *for all* $k\geq 1$.

Proof. Since the operator T is pure hyponormal, the space H is generated by the vectors $T^kT^{*l}\xi$, $k,l\in\mathbf{N}$. Thus inclusion (26) proves that $L=H$.

Suppose that $\dim(K_k)<\dim(K)$ for some integer $k>0$. Then the matrix

$$\begin{pmatrix} U_k & 0 & 0 & \dots \\ V_k & U_{k+1} & 0 & \dots \\ 0 & V_{k+1} & U_{k+2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

represents an operator S which is a finite-rank perturbation of T. Moreover, relations (24) and (25) prove that the self-commutator of S is diagonal and has the form

$$[S^*,S]=\operatorname{diag}(V_{k-1}V_{k-1}^*;0;0;\dots).$$

Therefore the principal functions of T and S coincide (see [13, Chapter X]) and consequently the identity

$$\operatorname{Tr}[T^*,T(T^*-\bar{z})^{-1}]=\operatorname{Tr}[S^*,S(S^*-\bar{z})^{-1}]$$

holds for large values of $|z|$. But,

$$\begin{aligned}\operatorname{Tr}[S^*,S(S^*-\bar{z})^{-1}]&=\operatorname{Tr}([S^*,S](S^*-\bar{z})^{-1})\\ &=\operatorname{Tr}(V_{k-1}V_{k-1}^*(S^*-\bar{z})^{-1})\\ &=\operatorname{Tr}(V_{k-1}^*(S^*|K_k-\bar{z})^{-1}V_{k-1}).\end{aligned}$$

Hence $\operatorname{Tr}[S^*,S(S^*-\bar{z})^{-1}]$ is a rational function with at most $\dim(K_k)$ poles, counting them together with their multiplicities. On the other hand Corollary 3.4 shows that the function

$$\operatorname{Tr}[T^*,T(T^*-\bar{z})^{-1}]=\langle (T^*-\bar{z})^{-1}\xi,\xi\rangle$$

has exactly $n=\dim(K)$ poles.

In conclusion $\dim(K_k)=\dim(K)$ for every $k>0$.

Starting with Lemma 4.1, the structure of the matrix (23) can be further simplified. To simplify notation we put $V_0=V$. Let us denote by $V_k=Y_kA_k$ the polar decomposition of the operator $V_k\colon K_k\to K_{k+1}$, where $A_k\in L(K_k)$ is self-adjoint and $Y_k\colon K_k\to K_{k+1}$ is an isometry, $k\geq 0$. Because $\dim(K_{k+1})=\dim(K_k)$ and the operator V_k is onto by the definition of the space K_{k+1}, the operator A_k turns out to be invertible and Y_k is unitary.

We consider the following block-diagonal unitary matrix:

$$Y = I \oplus Y_0 \oplus Y_1 Y_0 \oplus Y_2 Y_1 Y_0 \oplus \ldots$$

which identifies the space $l^2(\mathbf{N}, K) = K \oplus K \oplus K \oplus \ldots$ with $H = K \oplus K_1 \oplus K_2 \oplus \ldots$. A simple computation shows that the operator $Y^{-1}TY$ has a similar matrix decomposition to (23) and the sub-diagonal entries V_k are replaced by their self-adjoint moduli A_k, $k \geq 0$.

Summing up we have proved the following result.

Theorem 4.2. *Let T be a pure hyponormal operator with rank-one self-commutator $[T^*, T] = \xi \otimes \xi$ and satisfying $\dim(K) < \infty$ where $K = \bigvee_{k=0}^{\infty} T^{*k}\xi$.*

Then T is unitarily equivalent to the following block-matrix acting on $l^2(\mathbf{N}, K)$:

$$M = \begin{pmatrix} U_0 & 0 & 0 & 0 & \ldots \\ A_0 & U_1 & 0 & 0 & \ldots \\ 0 & A_1 & U_2 & 0 & \ldots \\ 0 & 0 & A_2 & U_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{27}$$

The entries of M satisfy $A_k > 0$, $k \geq 0$, and they are recurrently defined by:

$$U_0 = (T^*|K)^*, \qquad A_0{}^2 = \xi \otimes \xi - [U_0^*, U_0], \tag{28}$$

$$U_{k+1} = A_k{}^{-1} U_k A_k, \quad A_{k+1}{}^2 = A_k{}^2 - [U_{k+1}^*, U_{k+1}], \quad k \geq 0. \tag{29}$$

At this point it is evident that, starting with an $n \times (n+1)$ matrix (U_0, ξ), where $U_0 \in L(K)$ and $\xi \in K$, the formulae (28) and (29) generate a hyponormal operator as in Theorem 4.2 if and only if, at every step of the inductive construction the condition:

$$A_{k-1}{}^2 - [U_k^*, U_k] > 0, \quad A_{-1} = \xi \otimes \xi, \ k \geq 0 \tag{30}$$

is satisfied and moreover,

$$\sup_{k \in \mathbf{N}} (\|U_k\| + \|A_k\|) < \infty. \tag{31}$$

The latter condition is obviously equivalent to the boundedness of the matrix (27).

Although these conditions seem to be far from being practical, they desribe at least theoretically the "moduli space" of all quadrature domains.

Corollary 4.3. *There exists a bijective correspondence between the bounded quadrature domains of order n and the unitary equivalence classes of pairs* (U_0,ξ), $U_0\in L(\mathbf{C}^n)$, $\xi\in\mathbf{C}^n$, *which satisfy conditions* (30), (31) *defined by the recurrent relations* (28), (29).

In the preceding statement, a unitary equivalent pair to (U_0,ξ) has by definition the form $(YU_0Y^{-1},Y\xi)$, with Y a unitary transformation of $\mathbf{C}^n$.

To count parameters in the orbit space of matrices described by Corollary 4.3 is at this stage near to impossible. However we do not exclude some possible further simplifications and a better picture of this orbit space. Next we restrict ourselves to derive a few simple algebraic restrictions imposed by conditions (30) and (31) on the matrix (U_0,ξ).

Corollary 4.4. *Let* (U_0,ξ) *be an* $n\times(n+1)$ *matrix,* $n\geq2$, *as in Corollary* 4.3. *Then:*

(a) *the vector* ξ *is* U_0^**-cyclic;*

(b) *the self-adjoint matrix* $[U_0^*,U_0]$ *is invertible and has signature* $+--\dots-$;

(c) U_0 *cannot be decomposed into a non-trivial orthogonal sum of linear transformations.*

Proof. Condition (a) follows from the identification $U_0^*=T^*|K$ and the definition of the space K.

Let $[U_0^*,U_0]=\alpha_+-\alpha_-$ be an orthogonal decomposition corresponding to $K=K_+\oplus K_-$, where $\alpha_+|K_+\geq0$ and $\alpha_-|K_->0$. By assumption, $\xi\otimes\xi>[U_0^*,U_0]$, whence $\xi\otimes\xi>\alpha_+-\alpha_-$. Therefore, for a non-zero vector $x\in K_+$ we obtain

$$\langle\alpha_+x,x\rangle<|\langle x,\xi\rangle|^2. \tag{32}$$

Thus $\operatorname{rank}(\alpha_+)\leq1$. If $\alpha_+=0$, then $[U_0^*,U_0]\leq0$. By a well known result (see for instance [13, Corollary III.1.6]) it follows that the operator U_0 is normal. But in that case the difference $\xi\otimes\xi-[U_0^*,U_0]$ cannot be invertible. Therefore $\operatorname{rank}(\alpha_+)=1$ and $\dim(K_+)=1$, otherwise the inequality (32) would be violated by a non-zero vector $x\in K_+$ which is orthogonal to ξ.

Finally, in order to prove assertion (c), let us suppose that $U_0=P\oplus Q$. Then according to (b), at least one of the self-adjoint commutators $[P^*,P],[Q^*,Q]$ is negative definite. By the same argument ([13, Corollary III.1.6]) this commutator turns out to be identically equal to zero. But this contradicts the invertibility of $[U_0^*,U_0]$ and the proof of Corollary 4.4 is complete.

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