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Weak type estimates for some maximal operators on the weighted Hardy spaces

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Abstract. Weighted weak type estimates are proved for some maximal operators on the weighted Hardy spaces H^p_w ($0<p<1$, $w\in A_1$); in particular, weighted weak type endpoint estimates are obtained for the maximal operators arising from the Bochner–Riesz means and the spherical means on H^p_w.

1. Introduction

Let

$$S^\delta_R(f)(x)=\int(1-R^{-2}|\xi|^2)^\delta_+\hat{f}(\xi)e^{2\pi ix\xi}\,d\xi$$

be the Bochner–Riesz means of order δ on $\mathbf{R}^n$; put

$$S^\delta_*(f)(x)=\sup_{R>0}|S^\delta_R(f)(x)|.$$

We also consider the spherical means; let

$$N^\alpha_r(f)(x)=\frac{r^n}{\Gamma(\alpha)}\int f(x-y)(1-r^2|y|^2)^{\alpha-1}_+\,dy$$

and put

$$N^\alpha_*(f)(x)=\sup_{r>0}|N^\alpha_r(f)(x)|.$$

In [8] it was proved that $S^{\delta(p)}_*$ and $N^{\alpha(p)}_*$ are bounded from the Hardy space H^p to the weak L^p space if $\delta(p)=n/p-(n+1)/2$ and $\alpha(p)=1+n(1/p-1)$, $0<p<1$, respectively (see [1], [2], [3], [5], [6], [7] and [11] for results when $p\geq1$). In this note we shall prove A_1-weighted versions of these theorems; in fact, we shall prove more general results. Here and in the sequel, A_p denotes the Muckenhoupt class of weight functions.

Let $\varphi\in S(\mathbf{R}^n)$ (the Schwartz space) satisfy $\int\varphi=1$. Let $0<p<1$, $w\in A_1$. We say that a tempered distribution f belongs to the weighted Hardy space H^p_w if

$$\|f\|_{H^p_w}=\left(\int\sup_{\varepsilon>0}|\varphi_\varepsilon*f(x)|^p w(x)\,dx\right)^{1/p}<\infty,$$

where $\varphi_\varepsilon(x)=\varepsilon^n\varphi(\varepsilon x)$.

Now we define a dense subspace of H^p_w.

Definition 1. Let $f\in S(\mathbf{R}^n)$. We say that $f\in S_0$ if $\hat{f}$ (the Fourier transform) is compactly supported and vanishes in a neighborhood of the origin.

If $0<p<1$ and $w\in A_1$, the space S_0 is dense in H^p_w. (See [9].) To state our theorem, we introduce a function space.

Definition 2. Let $\mathcal{P}_m$ denote the class of polynomials of degree less than or equal to m. Let $\sigma\geq 0$. For a locally integrable function f, put

$$|f|_{m,\sigma}=\sup_{z\in\mathbf{R}^n,s>0}\ \inf_{Q\in\mathcal{P}_m} s^{-\sigma-n}\int_{B(z,s)}|f(y)-Q(y)|\,dy.$$

(See [4, Chap. III], [10].) Let $\theta\geq 0$. Let $f\in L^1$. We say that $f\in\mathcal{F}(m,\sigma,\theta)$ if we can write $f=\sum_{k=0}^{\infty}2^{-\theta k}g_k$ for a sequence $\{g_k\}_{k=0}^{\infty}$ of integrable functions such that

(a) $\sup_{k\geq 0}|g_k|_{m,\sigma}<\infty$;

(b) g_0 is supported in $\{|x|\leq 4\}$ and each g_k is supported in $\{2^{k-2}\leq|x|\leq 2^{k+2}\}$ for $k\geq 1$.

Put $\sigma(p)=n(1/p-1)$ and $\mathcal{F}_p=\mathcal{F}([\sigma(p)],\sigma(p),\sigma(p)+n)$, where $[r]$ denotes the greatest integer not exceeding r.

We shall prove the following.

Theorem. *Let $0<p<1$, $w\in A_1$. Let $K\in\mathcal{F}_p$. Define*

$$T_r(f)(x)=\int f(x-y)r^nK(ry)\,dy \quad \text{and} \quad T^*(f)(x)=\sup_{r>0}|T_r(f)(x)|$$

for $f\in S_0$. Then, there exists a unique sublinear extension of T^* to H^p_w, which we also denote by T^*, such that

$$\sup_{\lambda>0}\lambda^p w(\{x\in\mathbf{R}^n:T^*(f)(x)>\lambda\})\leq c_{p,w}\|f\|^p_{H^p_w}. \tag{1.1}$$

Corollary. *Let* $0<p<1$, $w\in A_1$. *Both* $S_*^{\delta(p)}$ *and* $N_*^{\alpha(p)}$ *initially defined on* S_0 *extend to* H_w^p *(as in the theorem) and satisfy*

$$\text{(a)} \qquad \sup_{\lambda>0} \lambda^p w(\{x\in\mathbf{R}^n : S_*^{\delta(p)}(f)(x)>\lambda\}) \le c_{p,w}\|f\|_{H_w^p}^p,$$

$$\text{(b)} \qquad \sup_{\lambda>0} \lambda^p w(\{x\in\mathbf{R}^n : N_*^{\alpha(p)}(f)(x)>\lambda\}) \le c_{p,w}\|f\|_{H_w^p}^p.$$

We assume that $0<p<1$ and $w\in A_1$ in the sequel.

Proposition. *Let* $K\in L^1$ *be such that*

$$\text{(1.2)} \qquad \sup_{r>0} \inf_{P\in\mathcal{P}_{[\sigma(p)]}} \int_{|y|<1} |r^n K(r(x-y))-P(y)|\,dy \le c(1+|x|)^{-n/p}.$$

Define T^* *on* S_0 *by* K *as in the theorem. Then, as in the theorem,* T^* *extends to* H_w^p *and the estimate* (1.1) *holds.*

By the proposition and the following lemma we obtain the theorem.

Lemma 1. *Let* $K\in\mathcal{F}_p$. *Then* K *satisfies the condition* (1.2).

We shall prove the proposition in §2 and Lemma 1 in §3. Finally we shall prove the corollary in §4.

2. Proof of the proposition

Let N be a non-negative integer and $B(x_0,t)$ be the closed ball of $\mathbf{R}^n$ with center x_0 and radius t. Then a function a on $\mathbf{R}^n$ is called a (p,N,w)-atom if

$$\text{(2.1)} \qquad a \text{ is supported in } B(x_0,t) \text{ for some } x_0 \text{ and } t;$$

$$\text{(2.2)} \qquad \|a\|_\infty \le w(B(x_0,t))^{-1/p}, \quad \text{where } w(E)=\int_E w;$$

$$\text{(2.3)} \qquad \int a(x)x^\alpha\,dx=0 \text{ for all } |\alpha|\le N, \text{ where } \alpha=(\alpha_1,\dots,\alpha_n),\ x^\alpha=x_1^{\alpha_1},\dots,x_n^{\alpha_n}.$$

First we prove estimates for atoms.

Lemma 2. *Let* a *be a* $(p,[\sigma(p)],w)$*-atom supported in* $B(x_0,s)$. *Then*

$$w(\{x\in\mathbf{R}^n : T^*(a)(x)>\lambda\}) \le c\lambda^{-p}.$$

Lemma 3. *Let a be as in Lemma 2. Then*

$$T^*(a)(x) \le c(|B(x_0,s)|/w(B(x_0,s)))^{1/p}(s+|x-x_0|)^{-n/p}.$$

Proof of Lemma 3. It is sufficient to prove the case $w=1$. (See [8, (2.9)].) By (2.1), (2.2), (2.3) with $N=[\sigma(p)]$ and (1.2), we see that

$$\begin{aligned}
|T_r(a)(x)| &= \left|\int a(y) r^n K(r(x-y))\,dy\right| \\
&= \inf_{P\in\mathcal{P}_{[\sigma(p)]}} \left|\int_{B(x_0,s)} a(y)(r^n K(r(x-y)) - P(y))\,dy\right| \\
&\le \|a\|_\infty \inf_{P\in\mathcal{P}_{[\sigma(p)]}} \int_{B(x_0,s)} |r^n K(r(x-y)) - P(y)|\,dy \\
&= \|a\|_\infty \inf_{P\in\mathcal{P}_{[\sigma(p)]}} \int_{|y|<1} |(rs)^n K(rs(s^{-1}x - s^{-1}x_0 - y)) - P(y)|\,dy \\
&\le c\|a\|_\infty\, s^{n/p}(s+|x-x_0|)^{-n/p} \\
&\le c(s+|x-x_0|)^{-n/p}.
\end{aligned}$$

Thus we have $T^*(a)(s) \le c(s+|x-x_0|)^{-n/p}$. This completes the proof.

Proof of Lemma 2. Since $w\in A_1$ and $s^n(s+|x-x_0|)^{-n} \approx M(\chi_{B(x_0,s)})(x)$, where M is the Hardy–Littlewood maximal operator, by Lemma 3 we see that

$$\begin{aligned}
w(\{x : T^*(a)(x) > \lambda\}) &\le w(\{x : (s+|x-x_0|)^{-n/p} > c\lambda s^{-n/p} w(B(x_0,s))^{1/p}\}) \\
&= w(\{x : s^n(s+|x-x_0|)^{-n} > c\lambda^p w(B(x_0,s))\}) \\
&\le w(\{x : M(\chi_{B(x_0,s)})(x) > c\lambda^p w(B(x_0,s))\}) \\
&\le c\lambda^{-p}.
\end{aligned}$$

This completes the proof.

We need a weighted version of [8, Lemma (1.8)].

Lemma 4. *Let $0<p<1$ and let $\{f_k\}$ be a sequence of measurable functions on $\mathbf{R}^n$ such that*

$$\sup_{\lambda>0} \lambda^p w(\{x : |f_k(x)| > \lambda\}) \le 1 \quad \textit{for all } k.$$

Then, if $\sum |c_k|^p \le 1$, we have

$$\sup_{\lambda>0} \lambda^p w\left(\left\{x : \sum |c_k f_k(x)| > \lambda\right\}\right) \le \frac{2-p}{1-p}.$$

Let $f \in S_0$ and let $f = \sum \lambda_k a_k$ be its $(p, [\sigma(p)], w)$-atomic decomposition such that $\sum \lambda_k^p \le c\|f\|_{H_w^p}^p$, $\sum \lambda_k a_k = f$ a.e. and $\sum \lambda_k |a_k| \le cf^*$, where f^* denotes the grand maximal function (see [9]). Then, since f^* is bounded, we have $T_r(f) = \sum \lambda_k T_r(a_k)$ a.e. by the dominated convergence theorem. Thus by Lemmas 2 and 4 we see that

$$\sup_{\lambda>0} \lambda^p w(\{x \in \mathbf{R}^n : T^*(f)(x) > \lambda\}) \le c \sum \lambda_k^p \le c\|f\|_{H_w^p}^p.$$

Since S_0 is dense in H_w^p, by a standard argument, using this estimate, we can find the unique sublinear extension of T^* to H_w^p. This completes the proof of the proposition.

3. Proof of Lemma 1

Let $K \in \mathcal{F}_p$. We can write $K = \sum_{k=0}^{\infty} 2^{-kn/p} g_k$, where $\{g_k\}$ is as in the definition of the space $\mathcal{F}_p$. To prove (1.2), we have to show that for $r > 0$ and $x \in \mathbf{R}^n$, there is a polynomial $P_{r,x} \in \mathcal{P}_{[\sigma(p)]}$ such that

$$\int_{|y|<1} |r^n K(r(x-y)) - P_{r,x}(y)|\, dy \le c(1+|x|)^{-n/p}. \tag{3.1}$$

Suppose that $|x| < 2$. Then we can take $P_{r,x} = 0$ since $K \in L^1$. Next, suppose that $|x| \ge 2$. If $2^m \le |rx| \le 2^{m+1}$ for $m \ge 5$ and $|y| < 1$, then

$$K(r(x-y)) = \sum_{k=m-3}^{m+4} 2^{-kn/p} g_k(r(x-y)).$$

Since $\sup_k |g_k|_{[\sigma(p)],\sigma(p)} < \infty$, there is a polynomial $P \in \mathcal{P}_{[\sigma(p)]}$ such that

$$\int_{|y|<1} |r^n K(r(x-y)) - P(y)|\, dy \le c 2^{-mn/p} r^{n/p} \le c|x|^{-n/p}. \tag{3.2}$$

If $r|x| \le 32$ and $|y| < 1$, we have $K(r(x-y)) = \sum_{k=0}^{8} 2^{-kn/p} g_k(r(x-y))$. Since, as before, $\sup_k |g_k|_{[\sigma(p)],\sigma(p)} < \infty$, we can find a polynomial $P \in \mathcal{P}_{[\sigma(p)]}$ such that

$$\int_{|y|<1} |r^n K(r(x-y)) - P(y)|\, dy \le c r^{n/p} \le c|x|^{-n/p}. \tag{3.3}$$

By (3.2) and (3.3) we have (3.1) in the case $|x| \ge 2$. This completes the proof.

4. Proof of the corollary

Since the function $(1-|y|^2)_+^{n(1/p-1)}$ clearly belongs to $\mathcal{F}_p$ (see [8, p. 93]), by the theorem we have corollary (b). To prove corollary (a), the following lemmas are needed.

Lemma 5. *Let N be a positive integer and h an N times continuously differentiable function. Suppose that*

$$(4.1)\qquad |(\partial/\partial x)^\alpha h(x)| \le c_\alpha (1+|x|)^{-n/p} \quad \text{for } |\alpha| \le N,$$

where $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$, $\alpha=(\alpha_1, \dots, \alpha_n)$. Then, for all $\sigma \in [0, N]$, $h \in \mathcal{F}(N-1, \sigma, n/p)$.

Lemma 6. *Let $\delta=\delta(p)=n/p-(n+1)/2$ and put*

$$K(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(n/2+\delta)} J_{n/2+\delta}(2\pi|x|),$$

where J_ν is the Bessel function. Then K satisfies the condition (4.1) *of Lemma* 5 *for all N.*

Proof of Lemma 5. We consider a C^∞-partition of unity on $\mathbf{R}^n$: $\sum_{k=0}^\infty \varphi_k(x)=1$, where φ_0 is supported in $\{|x|\le 4\}$ and $\varphi_k(x)=\varphi(2^{-k}x)$ for $k\ge 1$, with φ supported in $\{2^{-2}\le|x|\le 2^2\}$. Set $g_k(x)=2^{kn/p}\varphi_k(x)h(x)$. Then $h(x)=\sum_{k=0}^\infty 2^{-kn/p} g_k(x)$.

To see that $h \in \mathcal{F}(N-1, \sigma, n/p)$, we have to examine $|g_k|_{N-1,\sigma}$. By Taylor's formula we have

$$(4.2)\quad g_k(v+w) = \sum_{|\alpha|<N} \frac{1}{\alpha!} w^\alpha g_k^{(\alpha)}(v) + \sum_{|\beta|=N} \frac{1}{\beta!} w^\beta \int_0^1 N(1-t)^{N-1} g_k^{(\beta)}(v+tw)\, dt,$$

where $g_k^{(\alpha)}(x)=(\partial/\partial x)^\alpha g_k(x)$ and $\alpha! = \alpha_1! \dots \alpha_n!$. Put

$$Q_{k,v}(w) = \sum_{|\alpha|<N} \frac{1}{\alpha!} w^\alpha g_k^{(\alpha)}(v).$$

We note that

$$(4.3)\qquad \sup_{k\ge 0, x\in\mathbf{R}^n} |g_k^{(\alpha)}(x)| \le c \quad \text{for } |\alpha| \le N.$$

Suppose that $|w|\le s$ and $0<s\le 1$. Then by (4.2) and (4.3) it is readily seen that

$$|g_k(v+w) - Q_{k,v}(w)| \le cs^N \le cs^\sigma \quad \text{for } \sigma \in [0, N].$$

Thus, for $\sigma\in[0,N]$, it follows that

$$\sup_{v\in\mathbf{R}^n, s\in(0,1]} s^{-\sigma-n}\int_{|w|<s}|g_k(v+w)-Q_{k,v}(w)|\,dw\le c \quad \text{uniformly in } k. \tag{4.4}$$

Furthermore, since $\sup_k\|g_k\|_\infty<\infty$, for $\sigma\ge 0$ we have

$$\sup_{v\in\mathbf{R}^n, s\in[1,\infty)} s^{-\sigma-n}\int_{|w|<s}|g_k(v+w)|\,dw\le c \quad \text{uniformly in } k. \tag{4.5}$$

By (4.4) and (4.5) we see that $h\in\mathcal{F}(N-1,\sigma,n/p)$ for all $\sigma\in[0,N]$. This completes the proof.

Proof of Lemma 6. Recall that

$$K(x)=\int(1-|\xi|^2)_+^\delta e^{2\pi i x\xi}\,d\xi.$$

Thus K is infinitely differentiable. Therefore to prove the estimate (4.1), we may assume that $|x|>1$. Put $\eta(t)=\pi^{-\delta}\Gamma(\delta+1)t^{-(n/2+\delta)}J_{n/2+\delta}(2\pi t)$. Then we have (see, e.g., [8])

$$\left|\frac{d^k}{dt^k}\eta(t)\right|\le c_{k,\delta}(1+t)^{-n/p}\quad \text{for } t>0.$$

It is easy to see that this implies the estimate (4.1) for $|x|>1$. This completes the proof.

Let K be as in Lemma 6. Since $\sigma(p)<[\sigma(p)]+1$, we see that $K\in\mathcal{F}_p$ by Lemma 5 (with $N=[\sigma(p)]+1$) and Lemma 6. (If $\sigma(p)$ is a positive integer, similarly, we see that $K\in\mathcal{F}(\sigma(p)-1,\sigma(p),n/p)$.) Since $S_R^{\delta(p)}(f)(x)=\int R^nK(R(x-y))f(y)\,dy$, by the theorem we have corollary (a).

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