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Random recursive construction of Salem sets

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Abstract. We introduce a random recursive method for constructing random Salem sets in $\mathbf{R}^d$. The method is inspired by Salem's construction [13] of certain singular monotonic functions.

1. Introduction

Let $K\subset\mathbf{R}^d$ be a compact set. For $\alpha\in[0,d]$ the α-dimensional *Hausdorff measure* of K is defined by

$$H^\alpha(K):=\sup_{\delta>0}\inf\left\{\sum_{k=1}^\infty \operatorname{diam}(U_k)^\alpha \,\middle|\, U_k\subseteq\mathbf{R}^d,\ K\subseteq\bigcup_{k=1}^\infty U_k,\ \operatorname{diam}(U_k)\le\delta\right\}.$$

Then the *Hausdorff dimension* of K is defined by

$$\dim_H(K):=\sup\{\alpha\ge 0\mid H^\alpha(K)=+\infty\}=\inf\{\alpha\ge 0\mid H^\alpha(K)=0\}.$$

Frostman's theory [4] implies that the Hausdorff dimension of K is equal to its *capacitarian dimension*. Therefore, if we write $M_1^+(K)$ for the set of probability measures with support in K, we have the following equality:

$$\dim_H(K)=\sup\left\{\alpha\ge 0\,\middle|\,\exists\mu\in M_1^+(K):\iint|x-y|^{-\alpha}\mu(dx)\,\mu(dy)<\infty\right\}. \tag{1}$$

In the sequel we write

$$\hat\mu(x):=\int e^{i\langle x,y\rangle}\mu(dy)\quad(x\in\mathbf{R}^d)$$

([1]) This work contains parts of the author's forthcoming doctoral thesis [2] which were presented at the *Conference on Harmonic Analysis from the Pichorides Viewpoint* in Anogia Academic Village on Crete in July 1995.

for the Fourier transform of measures $\mu\in M_1^+(\mathbf{R}^d)$. It is well known (cf. [8, Chapter 10]) that for $0<\alpha<d$ there exists a constant $c=c(d,\alpha)>0$ with

$$\iint |x-y|^{-\alpha}\mu(dx)\,\mu(dy)=c\int|\hat{\mu}(x)|^2|x|^{\alpha-d}dx. \tag{2}$$

If K has Lebesgue measure zero then by (1) and (2) its *Fourier dimension*

$$\dim_F(K):=\sup\{\alpha\geq 0\mid \exists\mu\in M_1^+(K):\hat{\mu}(x)=O(|x|^{-\alpha/2})\ (|x|\to\infty)\}$$

is majorized by the Hausdorff dimension of K. A compact set $K\subset\mathbf{R}^d$ is called a *Salem set*, if $\dim_F(K)=\dim_H(K)$ (cf. [8, Chapter 17]). A *random Salem set with dimension α in $\mathbf{R}^d$* is a random compact set in $\mathbf{R}^d$ (cf. [10]), whose Fourier and Hausdorff dimensions are almost surely the same and have (almost surely) the value α.

In 1950 Salem [13] constructed for given $\alpha\in\,]0,1[$ a random Cantor set, which is a random Salem set of dimension α in $\mathbf{R}$. His construction solved the *existence problem for Salem sets* in $\mathbf{R}$.

Later on in 1985 Kahane [8] treated the existence problem for Salem sets in $\mathbf{R}^d$, $d\in\mathbf{N}$. He showed that under a certain condition the image of a compact set in $\mathbf{R}^n$ under (n,d)-fractional Brownian motion is a random Salem set in $\mathbf{R}^d$.

Salem's construction [13] rests on a rather delicate dissection method based on the Steinhaus parametrization (cf. [14]). His dissection method uses step by step an increasing number of contractions with randomized Lipschitz factors and fixed translation vectors (cf. [1] for an appropriate technical setting). It seems to be difficult to find a (direct) generalization of Salem's construction for $\mathbf{R}^d$ with $d>1$.

In this paper we introduce a Salem-like construction of random Salem sets in $\mathbf{R}^d$, but we use a completely different random mechanism by fixing the Lipschitz factors and randomizing the translation vectors. That results in a method for constructing random Salem sets with given dimension in $\mathbf{R}^d$. Moreover it is possible to push the topological dimension of the resulting sets down to zero. This leads to a proof for the existence of sets with given topological, Fourier and Hausdorff dimension in $\mathbf{R}^d$.

2. The random recursive construction method

Fix $\alpha\in\,]0,d[$ and let $(N^{(k)})_{k\in\mathbf{N}}$ be a sequence of positive integers with

$$2^d\leq N^{(1)}<N^{(2)}<\ldots<N^{(k)}<N^{(k+1)}\quad (k\in\mathbf{N}).$$

We refer to $N^{(k)}$ as the number of contractions in the kth step of a dissection ($k\in\mathbf{N}$). The Lipschitz factor $\varrho^{(k)}$ for the kth step of a dissection will be defined by

$$(\varrho^{(k)})^{\alpha}N^{(k)}=1 \quad (k\in\mathbf{N}). \tag{3}$$

For convenience we set additionally $N^{(0)}=\varrho^{(0)}:=1$.

Moreover we choose a sequence of *independent* random variables

$$X_j^{(k)}:(\Omega,\mathcal{A},P)\to(\mathbf{R}^d,\mathcal{B}^d) \quad (k\in\mathbf{N},\ j=1,\dots,N^{(k)}),$$

defined on an appropriate probability space $(\Omega,\mathcal{A},P)$, where $\mathcal{B}^d$ means the Borel-σ-field in $\mathbf{R}^d$. Furthermore we assume that the random variables $X_j^{(k)}$ ($k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$) are *uniformly bounded.* This means that there exists a compact set $X\subset\mathbf{R}^d$ with $X_j^{(k)}(\omega)\in X$ for all $\omega\in\Omega$, $k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$.

To describe our construction we need the following code spaces ($m\in\mathbf{N}$):

$$D_m:=\prod_{k=1}^{m}\{1,\dots,N^{(k)}\},\quad D_\infty:=\prod_{k=1}^{\infty}\{1,\dots,N^{(k)}\},\quad \text{and}\quad D:=\bigcup_{m=1}^{\infty}D_m.$$

Here $\prod A_k$ denotes the cartesian product of A_k. For $\sigma=(\sigma(1),\sigma(2),\dots,\sigma(m))\in D_m$ we define a random variable

$$X_\sigma:(\Omega,\mathcal{A},P)\to(\mathbf{R}^d,\mathcal{B}^d),\quad \omega\mapsto\sum_{k=1}^{m}\varrho^{(0)}\dots\varrho^{(k-1)}X_{\sigma(k)}^{(k)}(\omega).$$

The following estimation shows the absolute convergence of the sum on the right hand side above for $m\to\infty$ ($\sigma\in D_\infty$):

$$\begin{aligned}\sum_{k=1}^{\infty}\varrho^{(0)}\dots\varrho^{(k-1)}|X_{\sigma(k)}^{(k)}(\omega)| &\le\sum_{k=1}^{\infty}(N^{(0)}\dots N^{(k-1)})^{-1/\alpha}|X_{\sigma(k)}^{(k)}(\omega)|\\ &\le\sup_{x\in X}|x|\sum_{k=0}^{\infty}2^{-kd/\alpha}\le 2\sup_{x\in X}|x|<\infty.\end{aligned}$$

Therefore we have for every $\sigma=(\sigma(1),\sigma(2),\dots)\in D_\infty$ a random variable

$$X_\sigma:(\Omega,\mathcal{A},P)\to(\mathbf{R}^d,\mathcal{B}^d),\quad \omega\mapsto\sum_{k=1}^{\infty}\varrho^{(0)}\dots\varrho^{(k-1)}X_{\sigma(k)}^{(k)}(\omega).$$

Now we define a random set K in $\mathbf{R}^d$ through

$$K:\omega\mapsto K(\omega):=\{X_\sigma(\omega)\mid\sigma\in D_\infty\}.$$

The next theorem shows that $K(\omega)$ can be interpreted as the limit set of a random recursive construction in the sense of Graf [5] and Mauldin & Williams [11] (cf. [1] for a general framework). As usual we write $\varrho M:=\{\varrho x|x\in M\}$ for $\varrho\in\mathbf{R}$ and $M\subseteq\mathbf{R}^d$.

Proposition 2.1. *There exists a compact set $M \subset \mathbf{R}^d$ such that*

$$K(\omega)=\bigcap_{m=1}^{\infty}\bigcup_{\sigma\in D_m}(X_\sigma(\omega)+\varrho^{(0)}\dots\varrho^{(m)}M)\quad\forall\omega\in\Omega. \tag{4}$$

Proof. By assumption on the variables $X_j^{(k)}$ there exists a compact set $X\subset\mathbf{R}^d$ with $X_j^{(k)}(\omega)\in X$ for all $\omega\in\Omega$, $k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$. We define the set M as

$$M:=\left\{x\in\mathbf{R}^d \mid |x|\le 2\sup_{x\in X}|x|\right\}.$$

Because $\varrho^{(k)}<\frac{1}{2}$ $(k\in\mathbf{N})$ we conclude that the mappings

$$S_j^{(k)}(\omega): M\to M,\quad x\mapsto X_j^{(k)}(\omega)+\varrho^{(k)}x\quad(\omega\in\Omega,\ k\in\mathbf{N},\ j=1,\dots,N^{(k)})$$

are random contractions *in* M. Then the equality

$$S_{\sigma(1)}^{(1)}(\omega)\circ\dots\circ S_{\sigma(m)}^{(m)}(\omega)(M)=X_\sigma(\omega)+\varrho^{(0)}\dots\varrho^{(m)}M\quad(\omega\in\Omega,\ \sigma\in D_m)$$

implies immediately the assertion of the theorem. □

Corollary 2.2. *For every $\omega\in\Omega$ the set $K(\omega)$ is compact in $\mathbf{R}^d$.*

Proof. The assertion of the corollary follows from the representation (4) of $K(\omega)$ $(\omega\in\Omega)$ given in Proposition 2.1. □

Remark 2.3. It is obvious that the mapping $K:\omega\mapsto K(\omega)$ is a random compact set in the sense of Matheron [10].

3. An upper bound for the Hausdorff dimension

Theorem 3.1. *For every $\omega\in\Omega$ we have $H^\alpha(K(\omega))<+\infty$ and therefore $\dim_H(K(\omega))\le\alpha$.*

Proof. Fix $\omega\in\Omega$. From the representation (4) of $K(\omega)$ given in Theorem 2.1 we see that for $\delta>0$ and $m=m(\delta)\in\mathbf{N}$ large enough, the sets

$$X_\sigma(\omega)+\varrho^{(0)}\dots\varrho^{(m)}M\quad(\sigma\in D_m)$$

can be used as a δ-covering of $K(\omega)$. This leads to an upper estimate of $H^\alpha(K(\omega))$:

$$\begin{aligned} H^\alpha(K(\omega)) &\leq \sup_{\delta>0} \sum_{\sigma\in D_{m(\delta)}} \operatorname{diam}(X_\sigma(\omega)+\varrho^{(0)} \dots \varrho^{(m)} M)^\alpha \\ &= \sup_{\delta>0} \sum_{\sigma\in D_{m(\delta)}} (\varrho^{(0)} \dots \varrho^{(m)})^\alpha \operatorname{diam}(M)^\alpha \\ &= \operatorname{diam}(M)^\alpha \sup_{\delta>0} \prod_{k=1}^{m(\delta)} N^{(k)} (\varrho^{(k)})^\alpha \overset{(3)}{=} \operatorname{diam}(M)^\alpha. \end{aligned}$$

This implies $\dim_H(K(\omega)) \leq \alpha$. □

4. A lower bound for the Fourier dimension

For the construction of K we assumed the variables $X_j^{(k)}$ ($k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$) to be independent and uniformly bounded. To establish α as an almost surely lower bound for the Fourier dimension of the random set K we need some more conditions on the variables $X_j^{(k)}$. These conditions seem to be very technical, but we will later see that they are rather canonical.

Definition 4.1. The sequence of random variables $X_j^{(k)}$ will be called admissible if for their characteristic functions

$$\varphi_{j,k}(x) = \int e^{i\langle x, X_j^{(k)}(\omega)\rangle} P(d\omega) \quad (x\in\mathbf{R}^d,\ k\in\mathbf{N},\ j=1,\dots,N^{(k)})$$

there exist constants $c_k>0$ (not depending on j) and an $\varepsilon>0$ with

(i) $|\varphi_{j,k}(x)| \leq c_k |x|^{-\varepsilon} \quad \forall x\in\mathbf{R}^d,\ k\in\mathbf{N},\ j=1,\dots,N^{(k)}$;

(ii) $\log\Big(\max_{1\leq k\leq p+1} c_k\Big) = o\Big(\sum_{k=1}^{p} \log(N^{(k)})\Big) \quad (p\to\infty)$.

Example 4.2. Let $X_j^{(k)}$ ($k\in\mathbf{N}, j=1,\dots,N^{(k)}$) be independent random variables in $\mathbf{R}^d$ with independent coordinate variables uniformly distributed in the unit interval $[0,1]$. Then by elementary calculations we see that the sequence $X_j^{(k)}$ ($k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$) is admissible with $\varepsilon=1$ and $c_k=2d^{1/2}$ ($k\in\mathbf{N}$).

In the sequel we always assume the sequence $X_j^{(k)}$ ($j=1,\dots,N^{(k)}$, $k\in\mathbf{N}$) to be admissible. Additionally we make the following assumption on the sequence $(N^{(k)})_{k\in\mathbf{N}}$:

$$\log(N^{(p+1)}) = o\Big(\sum_{k=1}^{p} \log(N^{(k)})\Big) \quad (p\to\infty). \tag{5}$$

For example such a sequence may be defined by $N^{(k)}:=(k+1)^d$.

Now we define a random measure $\mu(\omega)$ with support in $K(\omega)$ through distribution of mass in equal portions in every step of a dissection. For this purpose we put

$$\mu_m(\,\cdot\,,\omega):=\mathop{\ast}_{k=1}^{m}\left(\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}}\delta_{\varrho^{(0)}\cdots\varrho^{(k-1)}X_j^{(k)}(\omega)}(\cdot)\right)\quad(m\in\mathbf{N},\ \omega\in\Omega),$$

where δ_y denotes the Dirac measure in the point $y\in\mathbf{R}^d$, and $*$ means convolution.

Theorem 4.3. *For every $\omega\in\Omega$ the sequence of measures $\mu_m(\omega)$ $(m\in\mathbf{N})$ converges weakly to a measure $\mu(\omega)$ whose support is carried by $K(\omega)$.*

Proof. It is easy to check that we can write $\mu_m(\omega)$ as

$$\mu_m(\,\cdot\,,\omega)=\frac{1}{N^{(1)}}\cdots\frac{1}{N^{(m)}}\sum_{\sigma\in D_m}\delta_{X_\sigma(\omega)}(\cdot). \tag{6}$$

This implies

$$\hat{\mu}_m(x,\omega):=\frac{1}{N^{(1)}}\cdots\frac{1}{N^{(m)}}\sum_{\sigma\in D_m}e^{i\langle x,X_\sigma(\omega)\rangle}.$$

Using (3), $|e^{i\langle x,y\rangle}-e^{i\langle x,z\rangle}|\le|x||y-z|$, and estimating $|\hat{\mu}_m(x,\omega)-\hat{\mu}_n(x,\omega)|$ $(m,n$ large) we see that the Fourier transforms of the measures $\mu_m(\omega)$ converge uniformly on compact sets. This implies weak convergence.

The representation (6) shows that the support of $\mu(\omega)$ must be contained in $K(\omega)$. □

In the sequel we use the following form of the Fourier transforms of the measures $\mu_m(\omega)$:

$$\hat{\mu}_m(x,\omega)=\prod_{k=1}^{m}\left(\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}}e^{i\varrho^{(0)}\cdots\varrho^{(k-1)}\langle x,X_j^{(k)}(\omega)\rangle}\right). \tag{7}$$

The usual method to get an upper estimate for $\hat{\mu}(\omega)$ is to estimate $E(|\hat{\mu}(x)|^{2q})$ $(q\in\mathbf{N})$. This technique was developed by Kahane [8] and Salem [13]. The next two lemmas prepare such a mean value estimation. By Π_q $(q\in\mathbf{N})$ we denote the set of all permutations of $\{1,\dots,q\}$.

Lemma 4.4. *Let $q,k\in\mathbf{N}$, $x\in\mathbf{R}^d$ and $i_1,\dots,i_q,j_1,\dots,j_q\in\{1,\dots,N^{(k)}\}$. Under the assumption*

$$\forall\pi\in\Pi_q:\pi(i_1,\dots,i_q)\neq(j_1,\dots,j_q) \tag{8}$$

the following estimation holds:

$$\left|E\left(e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,\sum_{n=1}^{q}X_{i_n}^{(k)}-X_{j_n}^{(k)}\rangle}\right)\right|\leq c_k|\varrho^{(0)}\dots\varrho^{(k-1)}x|^{-\varepsilon}.$$

Proof. There exist numbers $h_1,\dots,h_{N^{(k)}}\in\mathbf{Z}$ (independent of $\omega\in\Omega$) with $|h_j|\leq q$ and

$$\sum_{n=1}^{q}X_{i_n}^{(k)}(\omega)-X_{j_n}^{(k)}(\omega)=\sum_{j=1}^{N^{(k)}}h_jX_j^{(k)}(\omega)\quad(\omega\in\Omega).$$

From assumption (8) we know the existence of at least one j_0 with $|h_{j_0}|\geq1$. The variables $X_j^{(k)}$ are admissible and independent. This implies the following estimation, where $\varphi_{j,k}$ denotes the characteristic function of $X_j^{(k)}$:

$$\begin{aligned}\left|E\left(e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,\sum_{n=1}^{q}X_{i_n}^{(k)}-X_{j_n}^{(k)}\rangle}\right)\right|&=\prod_{j=1}^{N^{(k)}}|\varphi_{j,k}(h_j\varrho^{(0)}\dots\varrho^{(k-1)}x)|\\&\leq|\varphi_{j_0,k}(h_{j_0}\varrho^{(0)}\dots\varrho^{(k-1)}x)|\\&\leq c_k|\varrho^{(0)}\dots\varrho^{(k-1)}x|^{-\varepsilon}.\end{aligned}$$

Therefore the assertion of the Lemma is proved. □

Lemma 4.5. *Let $q,k\in\mathbf{N}$. If $x\in\mathbf{R}^d$ fulfills the condition*

$$c_kq^{-q}(N^{(k)})^q\leq(\varrho^{(0)}\dots\varrho^{(k-1)})^{\varepsilon}|x|^{\varepsilon}, \tag{9}$$

then we obtain the following estimation:

$$E\left(\left|\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}}e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,X_j^{(k)}\rangle}\right|^{2q}\right)\leq\frac{2q^q}{[N^{(k)}]^q}.$$

Proof. We have the following equality:

$$\begin{aligned}&E\left(\left|\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}}e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,X_j^{(k)}\rangle}\right|^{2q}\right)\\&\qquad=\sum_{\substack{i_1,\dots,i_q,\\j_1,\dots,j_q=1}}^{N^{(k)}}\frac{1}{[N^{(k)}]^{2q}}E\left(e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,\sum_{n=1}^{q}X_{i_n}^{(k)}-X_{j_n}^{(k)}\rangle}\right).\end{aligned}$$

Splitting the sum and using Lemma 4.4 we conclude

$$E\left(\left|\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}} e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,X_j^{(k)}\rangle}\right|^{2q}\right)\le\frac{q!}{[N^{(k)}]^q}+c_k|\varrho^{(0)}\dots\varrho^{(k-1)}x|^{-\varepsilon},$$

which implies the assertion of the Lemma. □

The following lemma will be the key to establish α as an almost surely lower bound for the Fourier dimension of the random set K.

Lemma 4.6. *Let* $q\in\mathbf{N}$ *and* $0<\theta<1$. *Then there exists a bound* $\Theta(\theta,q)>0$ *with*

$$E(|\hat{\mu}(x)|^{2q})\le|x|^{-\theta\alpha q+1}\quad\forall x\in\mathbf{R}^d,\ |x|\ge\Theta(\theta,q).$$

Proof. Let q and θ be given. If for $x\in\mathbf{R}^d$ the condition

$$\left(\max_{1\le k\le p+1}c_k\right)q^{-q}(N^{(p+1)})^q\le(\varrho^{(0)}\dots\varrho^{(p)})^{\varepsilon}|x|^{\varepsilon}\tag{10}$$

is fulfilled then we can apply Lemma 4.5 for $k=1,\dots,p+1$. Because the variables $X_j^{(k)}$ are independent, we get

$$\begin{aligned}E(|\hat{\mu}(x)|^{2q})&\le\prod_{k=2}^{p+1}E\left(\left|\frac{1}{N^{(k)}}\sum_{j=1}^{N^{(k)}}e^{i\varrho^{(0)}\dots\varrho^{(k-1)}\langle x,X_j^{(k)}\rangle}\right|^{2q}\right)\\&\le\frac{2^pq^{pq}}{[N^{(2)}\dots N^{(p+1)}]^q}\le\frac{2^pq^{pq}}{[N^{(1)}\dots N^{(p)}]^q}.\end{aligned}$$

Now we use a technique developed by Salem [13]. Condition (10) is equivalent to

$$\log\left(\max_{1\le k\le p+1}c_k\right)-q\log(q)+q\log(N^{(p+1)})\le\varepsilon\log(|x|)-\frac{\varepsilon}{\alpha}\sum_{k=1}^{p}\log(N^{(k)}).\tag{11}$$

We choose an $x\in\mathbf{R}^d$ and a $p\in\mathbf{N}$ such that condition (11) is fulfilled. Then we assume $p=p(x)$ to be chosen maximal to $x\in\mathbf{R}^d$ such that (11) is true. This guarantees that for $p(x)+1$ the opposite inequality of (11) holds. Of course we have the implication

$$|x|\to\infty\quad\Longrightarrow\quad p(x)\to\infty.\tag{12}$$

The sequence $X_j^{(k)}$ $(k\in\mathbf{N},\ j=1,\dots,N^{(k)})$ is admissible, therefore we have the asymptotic relation

$$\log\left(\max_{1\le k\le p+1}c_k\right)=o\left(\sum_{k=1}^{p}\log(N^{(k)})\right)\quad(p\to\infty)$$

(cf. Definition 4.1, (ii)). For the sequence $(N^{(k)})_{k\in\mathbf{N}}$ we made the asymptotic assumption (5). Dividing (11) by $(\varepsilon/\alpha)\sum_{k=1}^{p}\log(N^{(k)})$ and using the mentioned asymptotic relations, we get the existence of a number $0<\theta_{p(x)}<1$ with

$$\sum_{k=1}^{p(x)}\log(N^{(k)})=\theta_{p(x)}\alpha\log(|x|) \tag{13}$$

for x large enough and $p=p(x)$. Using the maximality of $p(x)$ and the implication (12) we see that for x large enough $\theta_{p(x)}$ can be chosen arbitrarily close to one. Especially it is possible to have $\theta_{p(x)}\geq\theta$ for all $x\in\mathbf{R}^d$ with $|x|\geq\Theta'(\theta,q)$ with a certain bound $\Theta'(\theta,q)$. We conclude

$$E(|\hat{\mu}(x)|^{2q})\leq\frac{2^{p(x)}q^{p(x)q}}{[N^{(1)}\dots N^{(p(x))}]^q}\leq\frac{2^{p(x)}q^{p(x)q}}{|x|^{\theta\alpha q}}\quad\forall x\in\mathbf{R}^d,\ |x|\geq\Theta'(\theta,q).$$

Moreover we have the asymptotic relation

$$p=o\biggl(\sum_{k=1}^{p}\log(N^{(k)})\biggr)\quad(p\to\infty).$$

Because of (12) and (13) we are able to find a bound $\Theta''(q)$ with

$$p(x)\log(2q^q)\leq\log(|x|)\quad\forall x\in\mathbf{R}^d,\ |x|\geq\Theta''(q).$$

Now set $\Theta(\theta,q):=\max\{\Theta'(\theta,q),\Theta''(q)\}$. □

Theorem 4.7. *The Fourier dimension of K is almost surely minorized by α.*

Proof. With Lemma 4.6 in mind the conclusion is standard (cf. Kahane [8, Chapters 17–18], resp. Salem [13, p. 360–361]). As usual the estimation of $E(|\hat{\mu}(x)|^{2q})$ leads to the almost sure absolute convergence of an appropriate random series. □

Corollary 4.8. *The random set K is a random Salem set of dimension α in $\mathbf{R}^d$.*

Now we will give an example which shows how easily one can construct random Salem sets in $\mathbf{R}^d$ with the help of the random recursive construction method.

Example 4.9. Fix $0<\alpha<d$, set $N^{(k)}:=(k+1)^d$, $\varrho^{(k)}:=(N^{(k)})^{-1/\alpha}$ and choose random variables $X_j^{(k)}$ uniformly distributed in $[0,1]^d$ and independent (as in Example 4.2). Then the random compact set

$$K:\omega\mapsto K(\omega)=\{X_\sigma(\omega)\mid\sigma\in D_\infty\}$$

is a random Salem set with dimension α in $\mathbf{R}^d$.

5. Salem sets with topological dimension zero

In this paragraph we show that under certain separation conditions on the variables $X_j^{(k)}$ we get totally disconnected sets $K(\omega)$ $(\omega\in\Omega)$. From dimension theory it follows that these sets have topological dimension zero (cf. [6]).

Instead of a complicated general technical setting we give a concrete random recursive construction which leads to totally disconnected Salem sets. It is not difficult to adapt the construction to general situations.

Fix now $\alpha\in]0,d[$ and set $N^{(k)}:=(k+1)^d$ and $\varrho^{(k)}:=(N^{(k)})^{-1/\alpha}$ $(k\in\mathbf{N})$. The idea consists of constructing an admissible sequence $X_j^{(k)}$ in such a way that the resulting sets in every step of the recursive construction (4) have a positive distance to each other. We choose independent random variables $Z_j^{(k)}$ in $\mathbf{R}^d$ with independent coordinate variables uniformly distributed in $\left[0,\frac{1}{3}((k+1)^{-1}-\varrho^{(k)})\right]$, $k\in\mathbf{N}$, $j=1,\dots,N^{(k)}$. For every $k\in\mathbf{N}$ we define numbers $\beta_1^{(k)},\dots,\beta_{k+1}^{(k)}\in[0,1]$ inductively by

$$\beta_1^{(k)}:=0,\quad \beta_i^{(k)}:=\beta_{i-1}^{(k)}+\varrho^{(k)}+\frac{2}{3}\left(\frac{1}{k+1}-\varrho^{(k)}\right)\quad (i=2,\dots,k+1).$$

Now for every $k\in\mathbf{N}$ we have a family of vectors $b_j^{(k)}\in[0,1]^d$ $(j=1,\dots,N^{(k)})$ defined by

$$\{b_1^{(k)},\dots,b_{N^{(k)}}^{(k)}\}=\{\beta_1^{(k)},\dots,\beta_{k+1}^{(k)}\}^d.$$

Then we set

$$X_j^{(k)}:=b_j^{(k)}+Z_j^{(k)}\quad (k\in\mathbf{N},\ j=1,\dots,N^{(k)}).$$

By construction the variables $X_j^{(k)}$ are independent and uniformly bounded.

Theorem 5.1. *The random variables* $X_j^{(k)}$ $(k\in\mathbf{N},\ j=1,\dots,N^{(k)})$ *are admissible.*

Proof. By elementary calculations one finds that conditions (i) and (ii) in Definition 4.1 are fulfilled. For details we refer to the author's forthcoming doctoral thesis [2]. □

Theorem 5.2. *The random set* $K:\omega\mapsto K(\omega)=\{X_\sigma(\omega)\,|\,\sigma\in D_\infty\}$ *has totally disconnected realizations.*

Proof. The assertion of the theorem is an immediate consequence of the special construction of the sequence $X_j^{(k)}$ $(k\in\mathbf{N},\ j=1,\dots,N^{(k)})$. □

Corollary 5.3. *The random set K is a random Salem set of dimension α whose realizations have topological dimension zero.*

Proof. Theorems 5.1 and 5.2. □

Remark 5.4. In the introduction we mentioned that under a certain condition the images of compact sets under fractional Brownian motion are random Salem sets. In a paper of Kahane (cf. [7, p. 153]) we find that for classical Brownian motion these random sets can be chosen to have topological dimension zero.

6. An existence theorem

Professor Kölzow asked whether it is possible to find sets with arbitrary given topological, Fourier and Hausdorff dimension. From a result in [6] it follows that the topological dimension is an integer which is always majorized by the Hausdorff dimension. For the Fourier dimension the same relation holds. But are there other dependences between the three dimensions? The next theorem answers the question for $\mathbf{R}^d$ with $d \geq 2$ negatively. For the proof it is essential that we know the existence of Salem sets with topological dimension zero (cf. Corollary 5.3.)

In the sequel we write $\dim_T(K)$ for the topological dimension of $K \subseteq \mathbf{R}^d$.

Theorem 6.1. *Let $d \geq 2$, $\alpha, \beta \in]0, d[$ and $m \in \{0, \dots, d-1\}$ with $m, \alpha \leq \beta$. Then there exists a compact set $K \subset \mathbf{R}^d$ with topological dimension m, Fourier dimension α and Hausdorff dimension β.*

Proof. To prove the theorem it is enough to establish the existence of compact sets K_α, K_β, $K_m \subset \mathbf{R}^d$ with the following dimensions:

$$\begin{aligned} &\text{(i)} && \dim_T(K_m) = m, && \dim_F(K_m) = 0, && \dim_H(K_m) = m; \\ &\text{(ii)} && \dim_T(K_\alpha) = 0, && \dim_F(K_\alpha) = \alpha, && \dim_H(K_\alpha) = \alpha; \\ &\text{(iii)} && \dim_T(K_\beta) = 0, && \dim_F(K_\beta) = 0, && \dim_H(K_\beta) = \beta. \end{aligned}$$

Then the set $K := K_\alpha \cup K_\beta \cup K_m$ is a solution to our problem.

The existence of sets K_m with (i) is clear. From the last paragraph we know that there exist totally disconnected Salem sets K_α with dimension α. If $\beta \leq d-1$ choose a compact set $K'_\beta \subset \mathbf{R}^{d-1}$ with Hausdorff dimension β and topological dimension zero. Then $K_\beta := K'_\beta \times \{x_0\}$ with a fixed $x_0 \in \mathbf{R}$ solves (iii), because the Fourier dimension disappears by imbedding. If $\beta > d-1$, we cannot work with imbeddings. Therefore we need another method.

Let C_ξ denote a Cantor set in $[0,1]$ constructed like the ternary Cantor set but with ratio $0 < \xi < \frac{1}{2}$. If ξ^{-1} is a P.V. number, then it is well known that the Fourier dimension of C_ξ is zero (cf. Kechris & Louveau [9]). The Hausdorff dimension of C_ξ is equal to $\log(2)/(-\log(\xi))$. From a result of Salem (cf. [12, Theorem IV]) there are P.V. numbers $\xi^{-1}>2$ with $\dim_H(C_\xi)$ as close to 1 as we please. Therefore it is possible to choose a P.V. number $\xi_1^{-1}>2$ with $\dim_H(C_{\xi_1})>\beta-(d-1)$. Further choose numbers $\xi_2, \ldots, \xi_d$ with

$$\dim_H(C_{\xi_1})+\ldots+\dim_H(C_{\xi_d})=\beta.$$

Then from a result in [3, p. 95], it follows that

$$K_\beta := C_{\xi_1}\times\ldots\times C_{\xi_d}$$

fulfills (iii). This concludes the proof. □

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