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A quantitative version of Picard's theorem

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Abstract. Let f be an entire function of order at least $\frac{1}{2}$, $M(r)=\max_{|z|=r}|f(z)|$, and $n(r,a)$ the number of zeros of $f(z)-a$ in $|z|\le r$. It is shown that $\limsup_{r\to\infty} n(r,a)/\log M(r)\ge 1/2\pi$ for all except possibly one $a\in\mathbf{C}$.

1. Introduction and results

Let f be a transcendental entire function. Picard's theorem [6] says that there exists at most one value $a\in\mathbf{C}$ such that $f(z)-a$ has only finitely many zeros. Borel's theorem [2] gives a quantitative version of this result by saying that

$$\limsup_{r\to\infty}\frac{\log n(r,a)}{\log r}=\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r} \tag{1}$$

for all $a\in\mathbf{C}$, with at most one exception. Here $n(r,a)$ denotes the number of zeros of $f(z)-a$ in $|z|\le r$, counted according to multiplicity, and $M(r)=\max_{|z|=r}|f(z)|$ is the maximum modulus of f. The quantity on the right side of (1) is called the order of f and denoted by ϱ.

The "true" quantitative version of Picard's theorem is of course given by Nevanlinna's theory on the distribution of values and by Ahlfors's theory of covering surfaces, see [3], [5]. Here we will use the theories of Nevanlinna and Ahlfors to prove a quantitative version of Picard's theorem whose statement uses only "pre-Nevanlinna" terminology.

Theorem 1. *Let f be an entire function of order at least $\frac{1}{2}$. Then*

$$\limsup_{r\to\infty}\frac{n(r,a)}{\log M(r)}\ge\frac{1}{2\pi} \tag{2}$$

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for all $a \in \mathbf{C}$*, with at most one exception.*

The proof will show that there exists an unbounded sequence (r_j) depending only on f such that $n(r_j, a) \geq (1/2\pi - o(1)) \log M(r_j)$ for all except possibly one value of a and $\log\log M(r_j)/\log r_j \to \varrho$ as $j \to \infty$. Thus Theorem 1 can be considered as a strong form of Borel's theorem for the case that $\varrho \geq \frac{1}{2}$.

We note that the character of the problem is different if $\varrho < \frac{1}{2}$. A classical result of Pólya [7] and Valiron [9], [10] says that if $0 < \varrho < 1$, then

$$\limsup_{r\to\infty} \frac{n(r,a)}{\log M(r)} \geq \frac{\sin \pi\varrho}{\pi} \tag{3}$$

for all $a \in \mathbf{C}$. Simple examples show that if $0 \leq \varrho \leq \frac{1}{2}$, then we may have equality for all $a \in \mathbf{C}$ here. Thus for $0 \leq \varrho \leq \frac{1}{2}$ the constant $1/2\pi$ in (2) has to be replaced by $\sin \pi\varrho/\pi$, and this bound is sharp. (For $\frac{1}{6} < \varrho \leq \frac{1}{2}$ this bound is better than $1/2\pi$).

Theorem 1, however, which deals with the case $\varrho \geq \frac{1}{2}$, is probably not sharp. Note that for $\frac{1}{2} \leq \varrho < \frac{5}{6}$ the estimate (3) is better than (2). It seems likely that the constant $1/2\pi$ on the right side of (2) can be replaced by $1/\pi$. This would be best possible. In fact, we have

$$\limsup_{r\to\infty} \frac{n(r,a)}{\log M(r)} = \frac{1}{\pi} \tag{4}$$

if $f(z) = \exp z$ and $a \neq 0$. If $\frac{1}{2} \leq \varrho < \infty$, $\varrho \neq 1$, and $f(z) = E_{1/\varrho}(z)$, then (4) holds for all $a \in \mathbf{C}$. Here E_α denotes Mittag-Leffler's function. Note that $E_{1/\varrho}$ has order ϱ. Another example (of order 2) is $f(z) = \int_0^z \exp(-t^2)\, dt$, where (4) holds for $a = \pm\sqrt{\pi}/2$. A function of infinite order satisfying (4) for all $a \in \mathbf{C}$ is the function E_0 considered by Hayman [3, p. 81] and Pólya and Szegő [8, Vol. I, p. 115]. Other examples of infinite order can be obtained from the functions constructed in [1, Theorem 3].

Theorem 1 is an immediate consequence of the following result.

Theorem 2. *Let f be an entire function of order at least $\frac{1}{2}$. Then*

$$\limsup_{r\to\infty} \frac{n(r,a)+n(r,b)}{\log M(r)} \geq \frac{1}{\pi} \tag{5}$$

if $a, b \in \mathbf{C}$, $a \neq b$.

In a certain sense Theorem 2 is sharp, because we have equality in (5) if $f(z) = \exp z$ and $a = 0$. But it seems likely that the constant $1/\pi$ on the right side of (5) can be replaced by $2/\pi$ if the order of f is sufficiently large and, in particular, if f has infinite order.

2. A growth lemma for real functions

Lemma. *Let $\Phi(x)$ be increasing and twice differentiable for $x \geq x_0$ and assume that $\Phi(x) \geq cx$ for some positive constant c and arbitrarily large x. Then there exist sequences (x_j), (M_j), and (ε_j) such that $x_j \to \infty$, $M_j \to \infty$, and $\varepsilon_j \to 0$ as $j \to \infty$, $\Phi'(x_j) \geq c/8$, $\Phi''(x_j) \leq 2\Phi'(x_j)^2/\Phi(x_j)$, and $\Phi(x_j+h) \leq \Phi(x_j)+\Phi'(x_j)h+\varepsilon_j$ for $|h| \leq M_j/\Phi'(x_j)$.*

Without the claim about $\Phi''(x_j)$, this was proved in [1, Lemma 1]. The proof given there, however, also yields the above version. To see this, let F and v be as in [1, p. 168], and put $D(x)=F(x)-\Phi(x)$. Then D has a local minimum at v. Hence $D''(v) \geq 0$. Since $D(v)=D'(v)=0$, we deduce that

$$\Phi''(v) \leq F''(v) = \frac{2F'(v)^2}{F(v)} = \frac{2\Phi'(v)^2}{\Phi(v)}$$

and the claim made about $\Phi''(x_j)$ follows.

3. Proof of Theorem 2

Let $\Phi(x)=\log T(e^x)$, where $T(r)$ denotes the Ahlfors–Shimizu characteristic of f. Let $A(r)=rT'(r)$, that is,

$$A(r) = \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\, dx\, dy.$$

Then

$$\Phi'(x) = \frac{A(r)}{T(r)} \tag{6}$$

and

$$\Phi''(x) = \frac{rA'(r)}{T(r)} - \left(\frac{A(r)}{T(r)}\right)^2 \tag{7}$$

where $r=e^x$.

Suppose first that f has infinite order. We choose x_j according to the lemma and define $r_j=\exp x_j$. As shown in [1, p. 171], we have

$$\log M(r_j) \leq (1+o(1))\pi A(r_j). \tag{8}$$

From (6), (7), and the lemma we deduce that

$$r_jA'(r_j)=\Phi''(x_j)T(r_j)+\frac{A(r_j)^2}{T(r_j)}\leq\frac{2\Phi'(x_j)^2}{\Phi(x_j)}T(r_j)+\frac{A(r_j)^2}{T(r_j)}$$
$$=\left(\frac{2}{\log T(r_j)}+1\right)\frac{A(r_j)^2}{T(r_j)}.$$

Thus

$$r_jA'(r_j)\leq(1+o(1))\frac{A(r_j)^2}{T(r_j)} \tag{9}$$

as $j\to\infty$.

Suppose now that f has finite order $\varrho\geq\frac{1}{2}$. We shall show that (8) and (9) hold again for a suitable sequence (r_j). Let $\varrho^*(r)$ be a strong proximate order for $T(r)$, cf. [4, §I.12], and define $\gamma(r)=r^{\varrho^*(r)}$. Let r_j be an unbounded sequence satisfying $T(r_j)=\gamma(r_j)$. Then (r_j) is also a sequence of Pólya peaks (of order ϱ) for $T(r)$ and the arguments of [1, p. 164] show that (8) holds again. Define $x_j=\log r_j$ and $\Psi(x)=\log\gamma(e^x)$. Then $\Phi(x)\leq\Psi(x)$ with equality for $x=x_j$ and thus $\Phi'(x_j)=\Psi'(x_j)$ and $\Phi''(x_j)\leq\Psi''(x_j)$. It follows from the properties of strong proximate orders that $\Psi'(x)\to\varrho$ and $\Psi''(x)\to 0$ as $x\to\infty$. Thus $\Phi'(x_j)\to\varrho$ and $\Phi''(x_j)\leq o(1)$ as $j\to\infty$. Combining this with (6) and (7) we see that (9) also holds again.

Define (cf. [3, p. 144], [8, p. 348])

$$L(r)=\int_{|z|=r}\frac{|f'(z)|}{1+|f(z)|^2}\,|dz|.$$

Then

$$L(r)^2\leq 2\pi^2rA'(r)$$

by Schwarz's inequality (cf. [3, p. 144], [8, p. 349]). Together with (9) it follows that

$$L(r_j)\leq(1+o(1))\sqrt{2}\,\pi\frac{A(r_j)}{\sqrt{T(r_j)}}=o(A(r_j)).$$

Hence

$$n(r_j,a)+n(r_j,b)\geq(1-o(1))A(r_j) \tag{10}$$

by the main result of Ahlfors's theory of covering surfaces (cf. [3, p. 148], [8, p. 349, inequality (II′)]). Combining (8) and (10) we obtain (5).

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