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Generalized Hardy inequalities and pseudocontinuable functions

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Abstract. Given positive integers $n_1<n_2<\dots$, we show that the Hardy-type inequality

$$\sum_{k=1}^{\infty}\frac{|\hat{f}(n_k)|}{k}\le \text{const}\,\|f\|_1$$

holds true for all $f\in H^1$, provided that the n_k's satisfy an appropriate (and indispensable) regularity condition. On the other hand, we exhibit inifinite-dimensional subspaces of H^1 for whose elements the above inequality is always valid, no additional hypotheses being imposed. In conclusion, we extend a result of Douglas, Shapiro and Shields on the cyclicity of lacunary series for the backward shift operator.

1. Introduction

Let $\mathbf{T}$ denote the unit circle $\{z\in\mathbf{C}:|z|=1\}$ and m the normalized arclength measure on $\mathbf{T}$. Given a function $f\in L^1(\mathbf{T},m)$, set $\hat{f}(n)\overset{\text{def}}{=}\int_{\mathbf{T}} f\bar{z}^n\,dm$ $(n\in\mathbf{Z})$ and

$$\operatorname{spec} f\overset{\text{def}}{=}\{n\in\mathbf{Z}:\hat{f}(n)\neq 0\}.$$

The Hardy space H^1 is defined by

$$H^1\overset{\text{def}}{=}\{f\in L^1(\mathbf{T},m):\operatorname{spec} f\subset[0,\infty)\}$$

and endowed with the L^1-norm $\|\cdot\|_1$.

A remarkable result of McGehee, Pigno and Smith [MPS] states that, for some absolute constant $C>0$, one has

$$\sum_{k=1}^{\infty}\frac{|\hat{f}(n_k)|}{k}\le C\|f\|_1 \tag{1.1}$$

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whenever $\{n_k\}_{k=1}^{\infty}$ is an increasing subsequence of $\mathbf{N}$ (as usual, $\mathbf{N}$ denotes the positive integers) and f is an H^1 function with

$$\operatorname{spec} f \subset \{n_k\}. \tag{1.2}$$

The "generalized Hardy inequality" (1.1) (reducing to the classical Hardy inequality [H, Chapter 5] when $n_k = k$) has attracted a great deal of attention, because it provided an answer to the famous Littlewood conjecture (see e.g. [GM]) on the L^1-norm of an exponential sum.

In this paper we are trying to find out what happens if one drops the assumption (1.2). Although (1.1) is, in general, non-valid for arbitrary H^1 functions, we show that it does become true under a certain regularity condition on the growth of the n_k's (in particular, this is the case if the ratios n_k/k form a nondecreasing sequence). The arising regularity condition is then shown to be sharp: once slightly relaxed, it is no longer sufficient for (1.1) to hold, nor for any of its natural l^p-analogs.

The described result is stated and proved in Section 3 below. The method involved is different from that in [MPS]; it relies on some multiplier theorems for H^1 that are cited previously in Section 2.

The rest of the paper is related to the so-called star-invariant subspaces and pseudocontinuable functions. Given an inner function θ (see [H, Chapter 5] or [G, Chapter ii]), we define the corresponding star-invariant subspace of H^1 by

$$K_\theta^1 \stackrel{\text{def}}{=} H^1 \cap \theta \bar{z} \overline{H}^1, \tag{1.3}$$

where z stands for the independent variable and the bar denotes complex conjugation. (The term "star-invariant" means "invariant under the backward shift operator"; it is known that all such subspaces in H^1 are given by (1.3).) Further, a function $f \in H^1$ is called pseudocontinuable if it belongs to $\bigcup_\theta K_\theta^1$, where θ ranges over the inner functions. Equivalently, pseudocontinuable functions are precisely the non-cyclic vectors of the backward shift.

In Section 4, we prove that the generalized Hardy inequality (1.1) holds true for $f \in K_\theta^1$, provided that θ is a Blaschke product whose zero sequence is sufficiently sparse. This time we impose no restrictions on the n_k's (instead, we replace H^1 by a smaller set).

Finally, in Section 5 we extend a result of Douglas, Shapiro and Shields [DSS] on the spectrum of a pseudocontinuable function. Their result is that if $f \in H^2$ is pseudocontinuable then $\operatorname{spec} f$ is never lacunary, unless finite (i.e., $\operatorname{spec} f$ cannot be of the form $\{n_k\}_{k=1}^{\infty}$, where $\inf_k n_{k+1}/n_k > 1$). Now we prove that if $f \in K_\theta^1$, where the inner function θ is "sufficiently smooth", then smaller gaps in $\operatorname{spec} f$ are forbidden as well (e.g., for suitable θ's, $\operatorname{spec} f$ cannot be an infinite subset of $\{k^2 : k \in \mathbf{N}\}$). To this end, the Hardy-type inequality (1.1) is exploited.

2. Preliminaries

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of nonnegative numbers. Denote by $b(H^1)$ the unit ball in H^1.

Theorem A. *Necessary and sufficient that*

$$S_1 \stackrel{\text{def}}{=} \sup\left\{\sum_{k=1}^{\infty} a_k|\hat{f}(k)| : f \in b(H^1)\right\} < \infty$$

is the condition

$$S_2 \stackrel{\text{def}}{=} \sup_{N\in\mathbf{N}} \sum_{j=1}^{\infty}\left(\sum_{k=jN}^{(j+1)N-1} a_k\right)^2 < \infty. \tag{2.1}$$

Moreover, there are absolute constants $c_1>0$ and $c_2>0$ such that

$$c_1 S_1 \le \sqrt{S_2} \le c_2 S_1. \tag{2.2}$$

Theorem B. *Necessary and sufficient that*

$$\sup\left\{\sum_{k=1}^{\infty} a_k|\hat{f}(k)|^p : f \in b(H^1)\right\} < \infty \tag{2.3}$$

for some (every) $p\in[2,\infty)$ is the condition

$$\sup_{N\in\mathbf{N}} \sum_{k=N}^{2N-1} a_k < \infty. \tag{2.4}$$

Theorem A is an unpublished result of C. Fefferman, as stated in [AS]; a proof can be found in [SS]. Theorem B is due (in a stronger form) to Stein and Zygmund [SZ] for $p=2$ and to Sledd and Stegenga [SS] for $p>2$. The last mentioned paper also contains similar criteria for the case $1<p<2$.

3. Generalized Hardy inequalities for H^1

Given an increasing sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ and a number $p>0$, set

$$\Lambda(\{n_k\},p) \stackrel{\text{def}}{=} \sup\left\{\sum_{k=1}^{\infty} \frac{|\hat{f}(n_k)|^p}{k} : f \in b(H^1)\right\}.$$

It is not hard to see that

$$\Lambda(\{n_k\},p_1) \ge \Lambda(\{n_k\},p_2), \quad \text{if } p_1 < p_2. \tag{3.1}$$

Theorem 1. (i) *Let* $\{n_k\}_{k=1}^{\infty} \subset \mathbf{N}$ *be an increasing sequence satisfying*

$$\delta \stackrel{\text{def}}{=} \inf_k \frac{k}{n_k}(n_{k+1}-n_k) > 0. \tag{3.2}$$

Then, for every $p \in [1,\infty)$,

$$\Lambda(\{n_k\},p) \le \text{const}\left(1+\frac{1}{\delta}\right), \tag{3.3}$$

where the constant is numerical.

(ii) *Given any decreasing sequence* $\{\delta_k\}_{k=1}^{\infty} \subset \mathbf{R}_+$ *with* $\lim_{k\to\infty} \delta_k = 0$, *there exist positive integers* $n_1 < n_2 < \dots$ *such that*

$$\frac{k}{n_k}(n_{k+1}-n_k) \ge \delta_k, \quad k \in \mathbf{N}, \tag{3.4}$$

and

$$\Lambda(\{n_k\},p) = \infty \tag{3.5}$$

for all $p \in [1,\infty)$.

Proof. (i) In view of (3.1), it suffices to show that (3.2) implies (3.3) with $p=1$. To this end, we apply Theorem A with $a_{n_k} = 1/k$, $a_l = 0$ ($l \ne n_k$) and verify (2.1). More precisely, taking (2.2) into account, we have to check that, for any $N \in \mathbf{N}$, (3.2) yields

$$\sum_{j=1}^{\infty}\Bigg(\sum_{jN \le n_k < (j+1)N} \frac{1}{k}\Bigg)^2 \le \text{const}\left(1+\frac{1}{\delta}\right)^2 \tag{3.6}$$

with an absolute constant on the right.

Let $N \in \mathbf{N}$ be fixed. For $j = 1,2,\dots$ consider the intervals $I_j \stackrel{\text{def}}{=} [jN,(j+1)N)$. Set

$$\begin{aligned}
J_0 &\stackrel{\text{def}}{=} \{j \in \mathbf{N} : I_j \cap \{n_k\} = \emptyset\},\\
J_1 &\stackrel{\text{def}}{=} \{j \in \mathbf{N} : \#(I_j \cap \{n_k\}) = 1\},\\
J_2 &\stackrel{\text{def}}{=} \{j \in \mathbf{N} : \#(I_j \cap \{n_k\}) \ge 2\}.
\end{aligned}$$

We obviously have $\mathbf{N} = J_0 \cup J_1 \cup J_2$, the three sets being disjoint, and so the left-hand side of (3.6) equals

$$\sum_{j\in\mathbf{N}}\Bigg(\sum_{k:n_k\in I_j}\frac{1}{k}\Bigg)^2 = \sum_{j\in J_1}\Bigg(\sum_{k:n_k\in I_j}\frac{1}{k}\Bigg)^2 + \sum_{j\in J_2}\Bigg(\sum_{k:n_k\in I_j}\frac{1}{k}\Bigg)^2 \stackrel{\text{def}}{=} s_1+s_2. \tag{3.7}$$

(Clearly, the contribution of J_0 is zero.) We proceed by estimating s_1 and s_2.

Given $j \in J_1$, let $k(j)$ be the (unique) number $k \in \mathbf{N}$ for which $n_k \in I_j$. Because

$$I_{j'} \cap I_{j''} = \emptyset \quad \text{for } j' \neq j'', \tag{3.8}$$

we see that $k(j') \neq k(j'')$ whenever $j', j'' \in J_1$ and $j' \neq j''$. Therefore,

$$s_1 = \sum_{j \in J_1} \left(\frac{1}{k(j)} \right)^2 \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{3.9}$$

Now fix $j \in J_2$. Denote by $k_1 = k_1(j)$ and $k_2 = k_2(j)$ the minimal and the maximal elements of the set $\{k : n_k \in I_j\}$, so that

$$jN \leq n_{k_1} < n_{k_2} < (j+1)N. \tag{3.10}$$

We have

$$\sum_{k: n_k \in I_j} \frac{1}{k} = \sum_{k=k_1}^{k_2} \frac{1}{k} = \sum_{k=k_1}^{k_2-1} \frac{1}{k} + \frac{1}{k_2}. \tag{3.11}$$

Rewriting (3.2) in the form

$$\frac{1}{k} \leq \frac{n_{k+1} - n_k}{\delta n_k},$$

we get

$$\begin{aligned} \sum_{k=k_1}^{k_2-1} \frac{1}{k} &\leq \frac{1}{\delta} \sum_{k=k_1}^{k_2-1} \frac{n_{k+1} - n_k}{n_k} \leq \frac{1}{\delta n_{k_1}} \sum_{k=k_1}^{k_2-1} (n_{k+1} - n_k) \\ &= \frac{1}{\delta} \frac{n_{k_2} - n_{k_1}}{n_{k_1}} \leq \frac{1}{\delta} \frac{(j+1)N - jN}{jN} = \frac{1}{\delta j}, \end{aligned} \tag{3.12}$$

where the last but one passage relies on (3.10). Substituting the resulting estimate from (3.12) into (3.11), we obtain

$$\sum_{k: n_k \in I_j} \frac{1}{k} \leq \frac{1}{\delta j} + \frac{1}{k_2(j)},$$

whence

$$\left(\sum_{k: n_k \in I_j} \frac{1}{k} \right)^2 \leq 2 \left(\frac{1}{\delta^2 j^2} + \frac{1}{(k_2(j))^2} \right). \tag{3.13}$$

We now observe that, in view of (3.8), $k_2(j') \neq k_2(j'')$ whenever j' and j'' are two distinct elements of J_2. Consequently, (3.13) yields

$$
\begin{aligned}
s_2 &= \sum_{j\in J_2}\Bigl(\sum_{k:n_k\in I_j}\frac{1}{k}\Bigr)^2 \le 2\Bigl(\frac{1}{\delta^2}\sum_{j\in J_2}\frac{1}{j^2}+\sum_{j\in J_2}\frac{1}{(k_2(j))^2}\Bigr) \\
&\le 2\Bigl(\frac{1}{\delta^2}+1\Bigr)\sum_{j=1}^{\infty}\frac{1}{j^2} = \frac{\pi^2}{3}\Bigl(\frac{1}{\delta^2}+1\Bigr) \le \frac{\pi^2}{3}\Bigl(\frac{1}{\delta}+1\Bigr)^2.
\end{aligned}
\tag{3.14}
$$

Finally, substituting (3.9) and (3.14) into (3.7), we arrive at (3.6).

(ii) Let $\{\delta_k\}_{k=1}^{\infty}$ be a decreasing sequence of positive numbers tending to 0. Assume, for the sake of convenience, that $\delta_1 \le \frac{1}{4}$. Set $N_1=1$ and then define the positive integers $N_2, N_3, \dots$ inductively by

$$N_{j+1} \overset{\mathrm{def}}{=} \Bigl[\frac{N_1+\dots+N_j}{2\delta_j}\Bigr]. \quad (^2)$$

It is not hard to see that $N_{j+1} \ge 2N_j$ for all $j\in\mathbf{N}$ and

$$\frac{1}{2\delta_j}-1 < \frac{N_{j+1}}{N_1+\dots+N_j} \le \frac{1}{2\delta_j}. \tag{3.15}$$

Further, put

$$E \overset{\mathrm{def}}{=} \mathbf{N}\cap\bigcup_{j=1}^{\infty}[N_j, 2N_j-1]$$

and let $\{n_k\}_{k=1}^{\infty}$ be the (natural) enumeration of E satisfying $n_1<n_2<\dots$. Note that, for a fixed j, the inclusion $n_k\in[N_j, 2N_j-1]$ is equivalent to

$$k\in[N_1+\dots+N_{j-1}+1, N_1+\dots+N_j].$$

In order to verify (3.5), we apply Theorem B with $a_{n_k}=1/k$, $a_l=0$ $(l\neq n_k)$. We have

$$
\begin{aligned}
\sum_{N_j\le n_k<2N_j}\frac{1}{k} &= \sum_{k=N_1+\dots+N_{j-1}+1}^{N_1+\dots+N_j}\frac{1}{k} \ge \log\frac{N_1+\dots+N_j+1}{N_1+\dots+N_{j-1}+1} \\
&= \log\Bigl(1+\frac{N_j}{N_1+\dots+N_{j-1}+1}\Bigr),
\end{aligned}
$$

where the latter quantity tends to ∞, as is readily seen from (3.15). This means that, for the present choice of the a_k's, condition (2.4) is violated, and so is (2.3)

(^{2}) Here [ ] denotes integral part.

with $p\geq 2$. In other words, (3.5) holds true for $p\geq 2$, and hence also for all $p>0$ (see (3.1) for an explanation).

It remains to check (3.4). To this end, we distinguish two cases.

Case 1. For some j, one has $k=N_1+...+N_j$. In this case $n_k=2N_j-1$ and $n_{k+1}=N_{j+1}$, whence

$$\frac{k}{n_k}(n_{k+1}-n_k)=\frac{N_1+...+N_j}{2N_j-1}(N_{j+1}-2N_j+1)\geq\frac{N_j}{2N_j}\cdot 1=\frac{1}{2}\geq\delta_k.$$

Case 2. For some j, one has

$$k\in[N_1+...+N_j+1, N_1+...+N_{j+1}-1].$$

It then follows that $n_k\in[N_{j+1}, 2N_{j+1}-2]$ and $n_{k+1}=n_k+1$; hence

$$\frac{k}{n_k}(n_{k+1}-n_k)=\frac{k}{n_k}\geq\frac{N_1+...+N_j}{2N_{j+1}}\geq\delta_j\geq\delta_k.$$

Here the last but one inequality relies on (3.15), while the last one holds because $k>j$ and the sequence $\{\delta_k\}$ is decreasing.

Thus, in both cases the desired property is established. The proof is therefore complete. □

Of course, part (i) of the theorem just means that (3.2) implies the "generalized Hardy inequality"

$$\sum_{k=1}^{\infty}\frac{|\hat{f}(n_k)|}{k}\leq\text{const}\left(1+\frac{1}{\delta}\right)\|f\|_1,\quad f\in H^1.$$

In particular, we have the following results supplementing the McGehee–Pigno–Smith inequality (1.1) and the Littlewood conjecture.

Corollary 1. *There is a constant $C>0$ with the following property: Whenever $\{n_k\}_{k=1}^{\infty}$ is an increasing subsequence of* $\mathbf{N}$ *such that $\{n_k/k\}_{k=1}^{\infty}$ is nondecreasing, one has*

$$\sum_{k=1}^{\infty}\frac{|\hat{f}(n_k)|}{k}\leq C\|f\|_1$$

for any $f\in H^1$.

Proof. Under the stated condition on $\{n_k\}$, we have

$$\frac{n_{k+1}}{n_k}\geq\frac{k+1}{k}=1+\frac{1}{k},$$

and so (3.2) is fulfilled with $\delta\geq 1$. □

Corollary 2. *Given positive integers* $n_1<n_2<\ldots<n_N$ *with* $\{n_k/k\}_{k=1}^N$ *nondecreasing, every function* $f\in H^1$ *for which*

$$\min\{|\hat f(n_1)|,\ldots,|\hat f(n_N)|\}\ge 1$$

satisfies $\|f\|_1\ge c\log N$, *where* $c>0$ *is an absolute constant.*

Proof. Apply Corollary 1. $\square$

4. Generalized Hardy inequalities for star-invariant subspaces

Let $\{z_j\}_{j=1}^\infty$ be a sequence of pairwise distinct points of the disk $\mathbf{D}\stackrel{\text{def}}{=}\{|z|<1\}$ satisfying $\sum_j(1-|z_j|)<\infty$. Consider the corresponding Blaschke product

$$B(z)\stackrel{\text{def}}{=}\prod_{j=1}^\infty \frac{\bar z_j}{|z_j|}\frac{z_j-z}{1-\bar z_j z}$$

(if $z_j=0$, set $\bar z_j/|z_j|=-1$) and the star-invariant subspace K_B^1 (see the Introduction), formed from B via the formula

$$K_B^1\stackrel{\text{def}}{=}H^1\cap B\bar z\overline{H}^1.$$

We remark that K_B^1 is known to coincide with the closed linear hull of the rational fractions

$$r_j(z)\stackrel{\text{def}}{=}(1-\bar z_j z)^{-1},\quad j\in\mathbf{N},$$

whereas the whole of H^1 is similarly generated by a larger family $\{(1-\bar\zeta z)^{-1}:\zeta\in\mathbf{D}\}$.

In this section we prove that, under a certain sparseness condition on $\{z_j\}$, the generalized Hardy inequality (1.1) holds true for each $f\in K_B^1$ whenever $n_1<n_2<\ldots$ (no kind of regularity assumption of the form (3.2) is now needed).

First we cite the following result due to the author (cf. Theorem 12(b) in [D3]).

Theorem C. *Suppose that* $\{z_j\}\subset\mathbf{D}$ *is a sequence satisfying*

$$|z_j-z_k|\ge c(1-|z_j|)^s,\quad j\ne k, \tag{4.1}$$

for some fixed $c>0$ *and* $s\in\left(0,\frac12\right)$. *Let* B, K_B^1 *and* r_j *be defined as above. Given a linear operator* T, *defined originally on the (non-closed) linear hull of* $\{r_j\}$ *and taking values in a Banach space* Y, *the existence of a bounded linear extension* $T\colon K_B^1\to Y$ *is equivalent to the condition*

$$\|Tr_j\|_Y=O\left(\log\frac{1}{1-|z_j|}\right),\quad \text{as } j\to\infty. \tag{4.2}$$

Now we state the main result of this section.

Theorem 2. *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing subsequence of $\mathbf{N}$, and let $\{z_j\}\subset\mathbf{D}$ satisfy* (4.1) *with some $c>0$ and $0<s<\frac{1}{2}$. We have then*

$$\sum_{k=1}^{\infty}\frac{|\hat{f}(n_k)|}{k}\le C\|f\|_1,\quad f\in K_B^1,$$

where $C=C(c,s)$ is a positive constant independent of f.

Proof. We apply Theorem C to the case where $Y=l^1$ and T is the map given by

$$Tf\stackrel{\text{def}}{=}\{\hat{f}(n_k)/k\}_{k=1}^{\infty},\tag{4.3}$$

f being a finite linear combination of the r_j's. For every such f, its coefficients $\hat{f}(\cdot)$ must decrease exponentially, so that Tf is indeed in l^1. Further, since $\hat{r}_j(n)=\bar{z}_j^n$ and $n_k\ge k$, we have

$$\|Tr_j\|_{l^1}=\sum_{k=1}^{\infty}\frac{|z_j|^{n_k}}{k}\le\sum_{k=1}^{\infty}\frac{|z_j|^k}{k}=\log\frac{1}{1-|z_j|}.$$

Thus (4.2) is established. It now follows from Theorem C that T has a bounded linear extension going from K_B^1 to l^1. Of course, this extension is again given by (4.3), so the proof is complete. □

In connection with coefficients of pseudocontinuable functions, we would like to mention the papers [D1] and [D2] containing some more results on that topic.

5. On the spectra of pseudocontinuable functions

Let H^p $(1\le p\le\infty)$ stand for the classical Hardy space of the circle, defined e.g. by $H^p\stackrel{\text{def}}{=}H^1\cap L^p(\mathbf{T},m)$. As usual, elements of H^p are also treated as analytic functions on $\mathbf{D}$. Recall that a function $\theta\in H^\infty$ is called inner if $|\theta|=1$ a.e. on $\mathbf{T}$. With each inner function θ we associate the star-invariant subspace $K_\theta^p\stackrel{\text{def}}{=}H^p\cap\theta\bar{z}\overline{H}^p$.

In this section we use the generalized Hardy inequality (1.1) to ascertain how the smoothness of an inner function θ affects the spectrum $\operatorname{spec} f$ of a function $f\in K_\theta^1$. Our results supplement those of Douglas, Shapiro and Shields [DSS].

We begin with some technical preparations. Given a function $h\in L^1(\mathbf{T},m)$ and a number $n\in\mathbf{Z}_+$ $(=\mathbf{N}\cup\{0\})$, we form the partial sum

$$(S_nh)(\zeta)\stackrel{\text{def}}{=}\sum_{k=-n}^{n}\hat{h}(k)\zeta^k,\quad\zeta\in\mathbf{T},$$

and introduce the quantity

$$r(n,h) \stackrel{\text{def}}{=} \|h - S_n h\|_1.$$

Further, for an inner function θ and any fixed point $z \in \mathbf{D}$, we set

$$k_z(\zeta) = k_{\theta,z}(\zeta) \stackrel{\text{def}}{=} \frac{1 - \overline{\theta(z)}\theta(\zeta)}{1 - \bar{z}\zeta}.$$

It is well known that k_z is the reproducing kernel (for evaluation at z) in the Hilbert space K_θ^2. In other words, k_z is in K_θ^2 and, for every $f \in K_\theta^2$, one has

$$f(z) = \int_{\mathbf{T}} f(\zeta)\overline{k_z(\zeta)}\, dm(\zeta). \tag{5.1}$$

This last relation actually holds for $f \in K_\theta^1$ as well.

We require the following facts.

Lemma 1. *Let θ be an inner function and $n \in \mathbf{Z}_+$. For m-almost all $\zeta \in \mathbf{T}$, we have*

$$\lim_{r \to 1^-} |\hat{k}_{r\zeta}(n)| = |\theta(\zeta) - (S_n\theta)(\zeta)|. \tag{5.2}$$

Proof. For $z \in \mathbf{D}$, a straightforward computation yields

$$\hat{k}_z(n) = \bar{z}^n - \overline{\theta(z)} \sum_{k=0}^{n} \hat{\theta}(k) \bar{z}^{n-k}.$$

Letting $z = r\zeta$, $0 < r < 1$, and making r tend to 1^-, one arrives at (5.2) whenever $\zeta \in \mathbf{T}$ is a point at which $\lim_{r \to 1^-} |\theta(r\zeta)| = 1$. $\square$

Lemma 2. *If θ is inner and $f \in K_\theta^1$, then*

$$\|f\|_1 \le \sum_{n=0}^{\infty} |\hat{f}(n)| r(n,\theta). \tag{5.3}$$

Proof. Rewriting (5.1) in the form

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{k}_z(n)},$$

we get

$$|f(z)| \le \sum_{n=0}^{\infty} |\hat{f}(n)|\, |\hat{k}_z(n)|, \quad z \in \mathbf{D}.$$

Taking radial limits as $z \to \zeta$ and using Lemma 1, we obtain

$$|f(\zeta)| \le \sum_{n=0}^{\infty} |\hat{f}(n)|\, |\theta(\zeta) - (S_n\theta)(\zeta)|$$

for almost all $\zeta \in \mathbf{T}$. Integrating, we arrive at (5.3). $\square$

Now we state

Theorem 3. *There is a constant $\gamma>0$ making the following statement true: Whenever $\{n_k\}_{k=1}^{\infty}\subset\mathbf{N}$ is an increasing sequence and θ is an inner function with*

$$r(n_k,\theta)\le\frac{\gamma}{k},\quad k\in\mathbf{N},\tag{5.4}$$

the inclusions $f\in K_\theta^1$ and $\operatorname{spec} f\subset\{n_k\}$ imply $f\equiv 0$.

Proof. For any function $f\in K_\theta^1$ with $\operatorname{spec} f\subset\{n_k\}$, Lemma 2 gives

$$\|f\|_1\le\sum_{k=1}^{\infty}|\hat f(n_k)|r(n_k,\theta).$$

Combining this result with (5.4), we get

$$\|f\|_1\le\gamma\sum_{k=1}^{\infty}\frac{|\hat f(n_k)|}{k}\le\gamma C\|f\|_1,\tag{5.5}$$

where $C>0$ is the constant appearing in the McGehee–Pigno–Smith inequality (1.1). Finally, letting $\gamma\in(0,1/C)$, we see that (5.5) can only hold if $f\equiv 0$. □

We remark that condition (5.4) above expresses a certain smoothness property of θ. Before proceeding with the next theorem, in which smoothness is involved more explicitly, we introduce some notation.

Given $0<\alpha<1$ and $1\le p<\infty$, let $\mathrm{Lip}(\alpha,p)$ denote the (Lipschitz-type) space of those functions $h\in L^p(\mathbf{T},m)$ for which

$$\left(\int_{\mathbf{T}}|h(e^{i\tau}\zeta)-h(\zeta)|^p\,dm(\zeta)\right)^{1/p}=O(|\tau|^\alpha),\quad \tau\in\mathbf{R}.$$

Further, for $h\in L^p(\mathbf{T},m)$ and $n\in\mathbf{Z}_+$, denote by $E_p(n,h)$ the L^p-distance between h and the subspace of trigonometric polynomials of degree $\le n$:

$$E_p(n,h)\stackrel{\text{def}}{=}\inf\{\|h-Q\|_p:\operatorname{spec} Q\subset[-n,n]\}$$

(here $\|\cdot\|_p$ is the natural norm in $L^p(\mathbf{T},m)$).

The classical Jackson–Bernstein theorems (cf. [N, Chapter 5]) tell as that

$$h\in\mathrm{Lip}(\alpha,p)\iff E_p(n,h)=O(n^{-\alpha}).\tag{5.6}$$

Another auxiliary result, to be used later on, is

Lemma 3. *Let* $1<p<\infty$ *and* $h\in L^p(\mathbf{T},m)$. *There exists a constant* $B=B(p)>0$ *such that*

$$r(n,h)\le BE_p(n,h).$$

Proof. Hölder's inequality gives

$$r(n,h)\le \|h-S_nh\|_p.$$

Recalling that the partial sum operators, S_n, are uniformly bounded in $L^p(\mathbf{T},m)$ (see e.g. [G, Chapter iii]), one easily finds that

$$\|h-S_nh\|_p\le BE_p(n,h),$$

where $B=B(p)$ is a suitable constant. □

Now comes

Theorem 4. *Let* $0<\alpha<1$ *and* $1<p<\infty$. *Suppose* θ *is an inner function of class* $\mathrm{Lip}(\alpha,p)$. *There exists a constant* $M=M(\alpha,p,\theta)>0$ *with the following property*: *Whenever* $\{n_k\}_{k=1}^{\infty}\subset\mathbf{N}$ *is an increasing sequence such that*

$$n_k^{\alpha}\ge Mk,\quad k\in\mathbf{N}, \tag{5.7}$$

the inclusions $f\in K_\theta^1$ *and* $\operatorname{spec} f\subset\{n_k\}$ *imply* $f\equiv 0$.

Proof. Applying Lemma 3 and the equivalence relation (5.6), we get

$$r(n_k,\theta)\le BE_p(n_k,\theta)\le \frac{A}{n_k^{\alpha}},$$

where $A>0$ and $B>0$ are suitable constants. Combining this with (5.7) gives

$$r(n_k,\theta)\le \frac{A}{Mk},\quad k\in\mathbf{N}. \tag{5.8}$$

Now, for M large enough, (5.8) clearly implies (5.4), where γ is the same as in Theorem 3. Thus, the required result follows from the preceding one. □

In connection with the hypothesis of Theorem 4, we remark that there are certain explicit conditions ensuring the inclusion $\theta\in\mathrm{Lip}(\alpha,p)$. For example, the following result, due to I. E. Verbitskii, might be helpful to ascertain whether

$$B_{\{z_k\}}\in\mathrm{Lip}(\alpha,p), \tag{5.9}$$

where $B_{\{z_k\}}$ stands for the Blaschke product with zeros $\{z_k\}$.

Theorem D. (See [V]) (a) *If* $1-|z_k|=O(k^{-1/(1-\alpha p)})$, *with* $1\le p<\infty$ *and* $0<\alpha<1/p$, *then* (5.9) *holds true.*

(b) *For* $p>1$ *and* $\alpha=1/p$, (5.9) *holds if and only if*

$$\sup_j \frac{1}{1-|z_j|}\sum_{k>j}(1-|z_k|)<\infty.$$

Some related criteria for membership of an inner function in various smoothness spaces are contained in [A] and [D2].

We conclude with the following amusing fact.

Theorem 5. *If* θ *is an inner function such that*

$$r(n,\theta)<\frac{1}{\pi(n+1)},\quad n\in\mathbf{Z}_+, \tag{5.10}$$

then $\theta\equiv$const.

Proof. Suppose not; then we can find a nonzero function $f\in K^1_\theta$ (e.g. put $f=1-\overline{\theta(0)}\theta$). Using Lemma 2 and the hypothesis (5.10), we get

$$\|f\|_1\le\sum_{n=0}^{\infty}|\hat f(n)|r(n,\theta)<\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{|\hat f(n)|}{n+1}.$$

By virtue of the classical Hardy inequality (cf. [G, Exercise 8 in Chapter ii]), the latter quantity is $\le\|f\|_1$. This contradiction proves the theorem. □

It might be interesting to compare Theorem 5 with the following result due to Newman and Shapiro [NS]: If θ is an inner function with $\limsup_{n\to\infty} n|\hat\theta(n)|<1/\pi$, then θ is a finite Blaschke product.

Since $r(n,\theta)\ge|\hat\theta(n+1)|$, the conclusion of the Newman–Shapiro theorem holds a fortiori if

$$\sup_{n\in\mathbf{Z}_+}(n+1)r(n,\theta)<\frac{1}{\pi}. \tag{5.11}$$

Now in Theorem 5 we replace (5.11) by the weaker assumption (5.10) and arrive at a stronger conclusion.

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