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Counting eigenvalues using coherent states with an application to Dirac and Schrödinger operators in the semi-classical limit

William Desmond Evans, Roger T. Lewis, Heinz Siedentop and Jan Philip Solovej

1. Introduction

Coherent states have been successfully used for obtaining leading order asymptotics for spectral properties of Schrödinger operators (e.g., Berezin [2], Lieb [11], and Thirring [16]) and other systems. They have been used so far mainly to *sum* the negative eigenvalues, i.e., computing the first Riesz moment, or to evaluate traces of other convex or concave functions of the operator using the Berezin–Lieb inequalities. Another quantity of interest is the dimension of the discrete spectral subspace. At this point we would like to mention the paper of Li and Yau [10] which actually does not mention coherent states at all but which may be reinterpreted in terms of coherent states. Li and Yau treat the counting problem for Schrödinger operators H in a compact domain and evaluate the trace of the semi-group which requires H to have discrete spectrum only. In some sense our approach is related to theirs.

The purpose of this note is to show that the counting problem is accessible to a coherent states analysis: based on some rudimentary functional calculus we roll the problem back to the first Riesz moment. However, instead of developing a general theory we demonstrate the usefulness of the technique with a nontrivial example, the Dirac operator in the semi-classical limit, but our general result also includes the Schrödinger operator. We shall recover the leading order asymptotics of the number of eigenvalues in the spectral gap of the Dirac operator, namely—as predicted by Planck—the phase space volume of the corresponding energy shell divided by the Planck constant h raised to the power d where d is the underlying dimension, which in this case is three. In the following we shall choose the units such that the spectrum of the free Dirac operator is $(-\infty,-1]\cup[1,\infty)$, namely, in physical terms, such that the velocity of light and the rest energy of the electron are both one.

In the case of a proper subinterval (a,b), with $-1<a<b<1$ (equality is not allowed), of the spectral gap $(-1,1)$ this result with a strong error term has been previously obtained by Helffer and Robert [6]. For any closed subinterval, including the whole interval $[-1,1]$, a first order result was obtained by Levendorskiĭ [9]. All of these consider smooth potentials and use pseudo-differential and/or Fourier integral operator methods. However, the approximation method we use also gives from their result the first order result for the counting functions for more general potentials.

We consider potentials $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$. With this minimal regularity one cannot expect any error estimates. However, we establish the asymptotics for the number of *all* the eigenvalues in the spectral gap.

One might question the usefulness of developing another rough first order counting method for eigenvalues besides the Dirichlet–Neumann bracketing technique (Weyl [19], [20], see also Courant and Hilbert [5]). There are, however, important examples that do not easily yield to Dirichlet–Neumann bracketing; the Dirac operator is one of them. The ability to deal with the Dirac operator, in particular, demonstrates the power of the technique. We stress, however, that the method used here is by no means restricted to this example. In fact our main result gives the semi-classical spectral asymptotics for the counting function of Dirac *and* Schrödinger operators.

We wish to mention some related work by Klaus [8] and Birman and Laptev [3]. These papers give the first order asymptotics for the number of eigenvalues entering and leaving the interval $(-1,1)$ in the strong coupling limit. The number of eigenvalues within $(-1,1)$, being the difference of these two numbers that are equal in first order, is not determined. Note, however, that—in contrast to the Schrödinger operator—the strong coupling limit for the Dirac operator is not directly related to the semi-classical limit. Finally Tamura [15] and Klaus [8] gave, for long range potentials, the asymptotics of the number of eigenvalues in an interval (a,b) of $(-1,1)$ with $-1<a$, as $b\to1$.

2. Basic facts

The Dirac operator to be considered—for convenience we pick units such that the electron mass and the velocity of light are one—is a self-adjoint realization in $[L^2(\mathbf{R}^3)]^4$ of

$$D:=\alpha\cdot\frac{\hbar}{i}\nabla+\beta+V,$$

where $(\hbar/i)\nabla$ is the momentum operator and $\alpha:=(\alpha_1,\alpha_2,\alpha_3)$, and β are the Dirac matrices acting on $[\mathfrak{D}(\mathbf{R}^3)]^4:=[C_0^\infty(\mathbf{R}^3)]^4$ in the obvious way. We shall take the standard representation

$$\alpha_\nu=\begin{pmatrix} 0 & \sigma_\nu \\ \sigma_\nu & 0 \end{pmatrix}, \quad \nu=1,2,3,$$

where the σ_ν are the Pauli matrices

$$\sigma_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2=\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\beta=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

For later purposes we note that any pair of different Dirac matrices anti-commutes and each Dirac matrix yields the identity upon squaring.

Under our general assumption, namely $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$, D is self-adjoint with domain $[H^1(\mathbf{R}^3)]^4$ and $[C_0^\infty(\mathbf{R}^3)]^4$ is a core. Under this hypothesis the spectrum of D is discrete in $(-1,1)$ (Weidmann [17, Theorem 10.37]).

Given a self-adjoint operator H we shall set $N(H):=\operatorname{tr}[\chi_{(-\infty,0)}(H)]$. Here and in the following we use the convention that the trace of a nonnegative self-adjoint operator, which is not trace class, is equal to infinity. The standard inner product and norm on $[L^2(R^3)]^4$ will be denoted by $(\,\cdot\,,\cdot\,)$ and $\|\cdot\|$ respectively.

Lemma 1. *If $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$, the square D^2 of the Dirac operator, as defined by the spectral theorem for D, is the self-adjoint operator associated with the quadratic form $(D\psi,D\psi)$ defined on $\mathfrak{D}:=[H^1(\mathbf{R}^3)]^4$. Moreover*

$$\operatorname{tr}[\chi_{(-1,1)}(D)]=N(D^2-1). \tag{1}$$

Proof. Since D is self-adjoint, and hence closed, with domain $\mathfrak{D}$, the nonnegative symmetric form $(D\psi,D\psi)$ is closed on $\mathfrak{D}$ and the lemma follows from the first representation theorem for sesquilinear forms. □

For the reader's convenience we collect some facts from the literature which are indeed easy to verify:

• Given a function $h\in L^1_{\mathrm{loc}}$ and $g\in L^2(B_1(0))$ with $\|g\|_2=1$, we denote the dilation of g by $g_\varrho(x)=g(x/\varrho)/\varrho^{3/2}$ for any positive ϱ, and set

$$\langle h\rangle_\varrho(\mathfrak{q}):=\int_{\mathbf{R}^3} h(x)(g_\varrho(x-\mathfrak{q}))^2\,dx$$

which we can interpret as the average value of h in the ball of radius ϱ centered at $\mathfrak{q}$ with respect to the measure $(g_\varrho(x-\mathfrak{q}))^2\,dx$. We have:

1. For any distribution $h\in\mathfrak{D}'(\mathbf{R}^3)$, the function $\langle h\rangle_\varrho$ is smooth, if g is smooth.
2. If $h\in L^p(\mathbf{R}^3)$ with $p\in[1,\infty)$, then $\|\langle h\rangle_\varrho\|_p\le\|h\|_p$ by Jensen's inequality.
3. For any $h\in L^p(\mathbf{R}^3)$ with $p\in[1,\infty)$ we have $\langle h\rangle_\varrho\to h$ in $L^p(\mathbf{R}^3)$ as $\varrho\to0$.

• Let A be a self-adjoint operator which is bounded from below. Denote by $A_-=-\chi_{(-\infty,0)}(A)A$ its negative part. Then the minimax principle implies (see, e.g., Weinstein and Stenger [18])

$$-\operatorname{tr}(A_-)=\inf\{\operatorname{tr}(Ad)\mid Ad\in\mathfrak{S}_1, d \text{ self-adjoint}, 0\le d\le 1\} \tag{2}$$

where $\mathfrak{S}_1$ denotes the trace class operators. Note that both sides of (2) can be $-\infty$ simultaneously: a "minimizer" is $\chi_{(-\infty,0)}(A)$ which is uniquely determined in the case of finite infimum up to the projection to zero.

• We write $\gamma:=(z,\mu):=(\mathfrak{p},\mathfrak{q},\mu)$ for a point in the classical phase space, here $\Gamma:=\mathbf{R}^3\times\mathbf{R}^3\times\{1,2,3,4\}$. Furthermore assume g to be a spherically symmetric smooth function with support in the unit ball and L^2-norm equal to one. Again we denote its dilation by $g_\varrho(x)=g(x/\varrho)/\varrho^{3/2}$ for any positive ϱ. We assume V and φ to be real-valued functions with $\varphi, V^2\in L^1_{\text{loc}}(\mathbf{R}^3)$. Given these, we define the two 4×4 matrix-valued functions of $z=(\mathfrak{p},\mathfrak{q})\in\mathbf{R}^3\times\mathbf{R}^3$

$$\mathcal{O}(z):=\mathfrak{p}^2+2\langle V\rangle_\varrho(\mathfrak{q})(\mathfrak{p}\cdot\alpha+\beta)+\langle V^2-\varphi\rangle_\varrho(\mathfrak{q}) \tag{3}$$

which will be of relevance for an upper bound on the sum of eigenvalues and

$$\mathcal{U}(z):=(1-\delta)\mathfrak{p}^2-\varphi(\mathfrak{q})+V(\mathfrak{q})^2+2V(\mathfrak{q})(\mathfrak{p}\cdot\alpha+\beta) \tag{4}$$

which will be of relevance for the lower bound on the sum of eigenvalues. Note that these matrices depend on the three functions g, V, and φ which we shall pick later, as well as the parameters ϱ and δ also to be specified later. For each z each of the two matrices have four eigenvalues defining the following phase space functions, namely

$$o(\gamma):=o(z,\mu):=\mathfrak{p}^2+\langle V^2-\varphi\rangle_\varrho(\mathfrak{q})+\mathrm{s}(\mu)2|\langle V\rangle_\varrho(\mathfrak{q})|\sqrt{\mathfrak{p}^2+1}, \tag{5}$$

where $\mathrm{s}(1):=\mathrm{s}(2):=-1$ and $\mathrm{s}(3):=\mathrm{s}(4):=1$, are the eigenvalues of $\mathcal{O}(z)$ and

$$u(\gamma)=(1-\delta)\mathfrak{p}^2-\varphi+V(\mathfrak{q})^2+\mathrm{s}(\mu)2|V(\mathfrak{q})|\sqrt{\mathfrak{p}^2+1} \tag{6}$$

are the eigenvalues of $\mathcal{U}(z)$.(1) We also pick four orthonormal eigenvectors for each of the matrices $\mathcal{O}(z)$ and $\mathcal{U}(z)$ corresponding to the above eigenvalues and denote them by $\mathfrak{o}_\mu(z)$ and $\mathfrak{u}_\mu(z)$.

We are now in a position to define the normalized coherent states used here:

$$F_\gamma^O(x) := e^{i\mathfrak{p}x/\hbar} g_\varrho(x-\mathfrak{q})\mathfrak{o}_\mu(z) \tag{7}$$

are the states to be used for the upper bound (construction of a trial density matrix) and

$$F_\gamma^U(x) := e^{i\mathfrak{p}x/\hbar} g_\varrho(x-\mathfrak{q})\mathfrak{u}_\mu(z) \tag{8}$$

will be the states that we shall use to construct an approximate operator for the lower bound. Note that these functions depend also on $\hbar$ and ϱ which we suppress. The parameter $\hbar$ is positive, and for later use we set $h:=2\pi\hbar$. Note also that g is often chosen not to be a function of compact support, but a Gaußian.

• By $d\!\!\!{}^-\Omega$ we denote the following measure on the phase space Γ

$$d\!\!\!{}^-\Omega(\gamma) := \frac{1}{h^3} d\Omega(\gamma) := \frac{d\mathfrak{p}d\mathfrak{q}}{h^3}\sum_\mu,$$

i.e., it is the "natural" product measure, namely—up to a factor—the Lebesgue measure in the first six factors of Γ (variables $z=(\mathfrak{p},\mathfrak{q})$) and the counting measure in the last factor, namely in the variable μ (see also the definition of Γ above).

• We have for any $\psi,\varphi\in[L^2(\mathbf{R}^3)]^4$ and any set of coherent states, in particular the ones defined in (7) and (8),

$$(\psi,\varphi) = \int_\Gamma d\!\!\!{}^-\Omega(\gamma)(\psi,F_\gamma)(F_\gamma,\varphi), \tag{9}$$

i.e., coherent states are complete.

• Coherent states preserve positivity, i.e., given any function M on the phase space with values in an interval $[a,b]$ we have that the corresponding quasi-classical operator d satisfies

$$a \le d := \int_\Gamma d\!\!\!{}^-\Omega(\gamma)M(\gamma)|F_\gamma\rangle\langle F_\gamma| \le b \tag{10}$$

(1) That $o(\gamma)$ and $u(\gamma)$ are indeed the eigenvalues of $\mathcal{O}(z)$ and $\mathcal{U}(z)$ may be either seen directly or by using the anti-commutation relations of the Dirac matrices: set $a:=\omega\cdot\alpha+\omega_4\beta$ with $\omega^2+\omega_4^2=1$. Because of the anti-commutation relations of the Dirac matrices, we have $a^2=1$ implying that the eigenvalues of a are ±1. Since a is traceless both have multiplicity two. Now note that both $\mathcal{O}(z)$ and $\mathcal{U}(z)$ are just of this form except for a multiplication by a scalar and a shift by a multiple of the unit matrix which proves the stated formulae.

as an operator. (Note that we use Dirac notation to denote projections; in particular we pick the scalar product such that it is linear in its second variable but conjugate linear in the first one.)

Finally we give a nonasymptotic bound for the number of eigenvalues which is a simple consequence of the Cwickel–Lieb–Rosenblum inequality, abbreviated henceforth by CLR, (Cancelier et al. [4]).

Lemma 2. *If* $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$ *and* $W\in L^{3/2}(\mathbf{R}^3)$, *then*

$$\operatorname{tr}\chi_{(-\infty,0]}(D^2-1+W)\le 4(2^{1/2}/\hbar)^3 d_{CLR}\int_{\mathbf{R}^3}(2|V|+V^2-W)_+^{3/2}, \tag{11}$$

where d_{CLR} *is the optimal constant in CLR.*

Note that for $W=0$ the left-hand side of (11) equals the number of eigenvalues of the Dirac operator D in $[-1,1]$.

Proof. For all $\phi\in[C_0^\infty(\mathbf{R}^3)]^4$

$$\begin{aligned}(\phi,(D^2-1)\phi)&=\left\|\alpha\cdot\frac{\hbar}{i}\nabla\phi\right\|^2+2\operatorname{Re}\left(\alpha\cdot\frac{\hbar}{i}\nabla\phi,V\phi\right)+(\phi,(2V\beta+V^2)\phi)\\&\ge\hbar^2(1-\varepsilon)\|\nabla\phi\|^2+\left(\phi,\left[-2|V|+\left(1-\frac{1}{\varepsilon}\right)V^2\right]\phi\right).\end{aligned}\tag{12}$$

For definiteness we pick $\varepsilon=\frac12$. The result follows on adding the expectation of W and using CLR. □

3. The sum of eigenvalues

We begin with an upper bound on the phase space volume $\int_{o(\gamma)<0}d\Omega(\gamma)$ appearing in Lemma 4.

Lemma 3. *Let* $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$ *and* $\varphi\in L^{3/2}(\mathbf{R}^3)$. *Then*

$$\int_{o(\gamma)<0}d\Omega(\gamma)\le\frac{2^{11/2}\pi}{3h^3}\int_{\mathbf{R}^3}(\varphi(\mathfrak{q})_++V(\mathfrak{q})^2+2|V(\mathfrak{q})|)^{3/2}\,d\mathfrak{q}.$$

Proof. We have

$$\begin{aligned}o(\gamma)&=\mathfrak{p}^2+\langle V^2-\varphi\rangle_\varrho(\mathfrak{q})+2s(\mu)|\langle V\rangle_\varrho(\mathfrak{q})|\sqrt{\mathfrak{p}^2+1}\\&\ge\mathfrak{p}^2+\langle V^2-\varphi\rangle_\varrho(\mathfrak{q})-2|\langle V\rangle_\varrho(\mathfrak{q})|-\tfrac12\mathfrak{p}^2-2\langle V\rangle_\varrho(\mathfrak{q})^2\\&\ge\tfrac12\mathfrak{p}^2-\langle\varphi\rangle_\varrho(\mathfrak{q})_+-2|\langle V\rangle_\varrho(\mathfrak{q})|-\langle V\rangle_\varrho(\mathfrak{q})^2.\end{aligned}$$

Thus, if $0>o(\gamma)$,

$$\mathfrak{p}^2 \le 2(\langle\varphi\rangle_\varrho(\mathfrak{q})_+ + |\langle V\rangle_\varrho(\mathfrak{q})|^2 + 2|\langle V\rangle_\varrho(\mathfrak{q})|).$$

Thus, integration over $\mathfrak{p}$ and summation over μ yields

$$\int_{o(\gamma)<0} d\Omega(\gamma) \le \frac{2^{11/2}\pi}{3h^3}\int_{\mathbf{R}^3}(\langle\varphi\rangle_\varrho(\mathfrak{q})_+ + \langle V\rangle_\varrho(\mathfrak{q})^2 + 2|\langle V\rangle_\varrho(\mathfrak{q})|)^{3/2}\,d\mathfrak{q}.$$

Applying Jensen's inequality yields the right-hand side of the claimed inequality. □

We now give an upper bound on the sum of the negative eigenvalues.

Lemma 4. *Assume* $V\in L^{3/2}(\mathbf{R}^3)\cap L^5(\mathbf{R}^3)$ *and* $\varphi\in L^{3/2}(\mathbf{R}^3)\cap L^{5/2}(\mathbf{R}^3)$. *Set*

$$d := \int_\Gamma d\Omega(\gamma) M(\gamma) |F_\gamma^O\rangle\langle F_\gamma^O|$$

with $M := \chi_{\{\gamma | o(\gamma)<0\}}$. *Then*

$$-\operatorname{tr}(D^2-1-\varphi)_- \le \operatorname{tr}[(D^2-1-\varphi)d] = \int_{o(\gamma)<0} (o(\gamma)+\hbar^2\|\nabla g_\varrho\|^2)\,d\Omega(\gamma)$$

with $o(\gamma)$ *defined by* (5).

Proof. By (2) and (10)

$$-\operatorname{tr}(D^2-\varphi-1)_- \le \operatorname{tr}[(D^2-\varphi-1)d]$$

and by (9)

$$\operatorname{tr}[(D^2-\varphi-1)d] = \int_{o(\gamma)<0} d\Omega(\gamma)(F_\gamma^O, (D^2-\varphi-1)F_\gamma^O)$$

with F_γ^O as defined in (7). We calculate using (12) and get

$$\begin{aligned}(F_\gamma^O,&(D^2-\varphi-1)F_\gamma^O)\\ &= \mathfrak{o}_\mu(z)^*\{\mathfrak{p}^2+\hbar^2\|\nabla g_\varrho\|^2+2\langle V\rangle_\varrho(\mathfrak{q})(\mathfrak{p}\cdot\alpha+\beta)+\langle V^2-\varphi\rangle_\varrho(\mathfrak{q})\}\mathfrak{o}_\mu(z)\\ &= \mathfrak{o}_\mu(z)^*\{\mathcal{O}(z)+\hbar^2\|\nabla g_\varrho\|^2\}\mathfrak{o}_\mu(z) = o(\gamma)+\hbar^2\|\nabla g_\varrho\|^2. \quad \square\end{aligned}$$

We return to this upper bound below, but we now seek a similar lower bound for $-\operatorname{tr}(D^2-1-\varphi)_-$. As usual we try to approximate the original operator by an operator A having a coherent state symbol. An analysis of A and the remainder $R := D^2-1-\varphi-A$ then produces our estimate.

For convenience we introduce the notation $\mathfrak{d}_\varrho f := f - \langle f\rangle_\varrho$ for the difference of a function f and $f * g_\varrho^2$.

Lemma 5. *Define* $A: [H^2(\mathbf{R}^3)]^4 \to [L^2(\mathbf{R}^3)]^4$ *by*

$$A := \int_\Gamma d\Omega(\gamma) u(\gamma) |F_\gamma^U\rangle\langle F_\gamma^U|$$

and the self-adjoint operator R *by the form*

$$(\psi, R\psi) := \delta\hbar^2(\nabla\psi, \nabla\psi) - (1-\delta)\hbar^2 \|\nabla g_\varrho\|^2 (\psi, \psi) \\ + 2\,\mathrm{Re}\left(\psi, \mathfrak{d}_\varrho(V)\alpha\cdot\frac{\hbar}{i}\nabla\psi\right) + (\psi, [2\mathfrak{d}_\varrho(V)\beta + \mathfrak{d}_\varrho(V^2-\varphi)]\psi)$$

with form domain $[H^1(\mathbf{R}^3)]^4$. *Then,*

$$A = (1-\delta)\hbar^2(-\Delta + \|\nabla g_\varrho\|^2) + \alpha\cdot\frac{\hbar}{i}\nabla\langle V\rangle_\varrho + 2\langle V\rangle_\varrho\left(\alpha\cdot\frac{\hbar}{i}\nabla + \beta\right) + \langle V^2-\varphi\rangle_\varrho$$

and $D^2-\varphi-1$ *is the sum*

$$D^2 - \varphi - 1 = A + R$$

in the form sense.

Proof. First we remark that R is easily seen to be equal to the Laplacian (multiplied by δh^2) plus an operator with relative—to the Laplacian—form bound zero, so that R has form domain $[H^1(\mathbf{R}^3)]^4$.

It is easy to verify the claimed identities for the corresponding sesquilinear forms on $C_0^\infty(\mathbf{R}^3)^4 \times C_0^\infty(\mathbf{R}^3)^4$. This, however, is enough, since $\langle V\rangle_\varrho$ and $\langle V^2-\varphi\rangle_\varrho$ and all their derivatives are obviously smooth functions decaying at infinity, implying that A is just the sum of the Laplacian (multiplied by $(1-\delta)h^2$) plus a perturbation of relative bound 0. □

Lemma 6. *Let* $V \in L^{3/2}(\mathbf{R}^3) \cap L^5(\mathbf{R}^3)$, $\varphi \in L^{3/2}(\mathbf{R}^3) \cap L^{5/2}(\mathbf{R}^3)$. *Then*

$$-\,\mathrm{tr}(D^2-1-\varphi)_- \geq -\int_\Gamma u(\gamma)_- d\Omega(\gamma) + \mathrm{tr}[R d_{\min}]$$

where R *is defined in Lemma* 5 *and* $d_{\min}$ *is the projection to the negative subspace of* $D^2-1-\varphi$, *i.e.,* $d_{\min} := \chi_{(-\infty,0)}(D^2-1-\varphi)$.

Proof. From (12) we have

$$D^2 - 1 - \varphi \geq -(\hbar^2/4)\Delta - (2|V| + V^2) - (\hbar^2/4)\Delta - \varphi$$

and hence

$$\mathrm{tr}(D^2-1-\varphi)_- \leq \mathrm{tr}(-(\hbar^2/4)\Delta - (2|V|+V^2))_- + \mathrm{tr}(-(\hbar^2/4)\Delta - \varphi)_-.$$

Thus $(D^2-1-\varphi)_-\in\mathfrak{S}_1$ by the Lieb–Thirring inequality. It follows from (2) and the fact that A is a relatively bounded perturbation of the Laplacian with relative bound zero that

(13)
$$\begin{aligned}-\operatorname{tr}(D^2-1-\varphi)_- &= \inf\{\operatorname{tr}[(D^2-1-\varphi)d]\mid d \text{ self-adjoint, } 0\le d\le 1,\ d,\Delta d\in\mathfrak{S}_1\}\\ &= \operatorname{tr}[(A+R)d_{\min}]\\ &\ge \inf\{\operatorname{tr}(Ad)\mid d \text{ self-adjoint, } 0\le d\le 1,\ d,\Delta d\in\mathfrak{S}_1\}+\operatorname{tr}(Rd_{\min}).\end{aligned}$$

(Note that $A+R$ occurring above is to be understood in the form sense.) Furthermore let $\phi_1,\phi_2,\dots,$ be any orthonormal basis of the negative spectral subspace of A. Then

$$\begin{aligned}-\operatorname{tr}(A)_- &= \sum_\nu(\phi_\nu,A\phi_\nu)=\sum_\nu\int_\Gamma d\Omega(\gamma)u(\gamma)(\phi_\nu,F_\gamma^U)(F_\gamma^U,\phi_\nu)\\ &=\int_\Gamma d\Omega(\gamma)u(\gamma)\sum_\nu|(\phi_\nu,F_\gamma^U)|^2\ge-\int_\Gamma d\Omega(\gamma)u(\gamma)_-\end{aligned}$$

by the Bessel inequality. The stated inequality now follows by inserting this expression in (13). □

The following estimate of the remainder term in Lemma 6 will be useful later.

Lemma 7. *Let $V\in L^{3/2}(\mathbf{R}^3)\cap L^5(\mathbf{R}^3)$, $\varphi\in L^{3/2}(\mathbf{R}^3)\cap L^{5/2}(\mathbf{R}^3)$, and $d_{\min}$ as in Lemma 6. Then*

$$\operatorname{tr}(Rd_{\min})\ge-\operatorname{const}\left[\frac{1}{\delta^{3/2}\hbar^3}\left(\frac{1}{\delta^{5/2}}\|\mathfrak{d}_\varrho V\|_5^5+\|\mathfrak{d}_\varrho V\|_{5/2}^{5/2}+\|\mathfrak{d}_\varrho(V^2-\varphi)\|_{5/2}^{5/2}\right)+\frac{1}{\hbar\varrho^2}\right].$$

Proof. By completing squares we have the following chain of inequalities in the sense of quadratic forms—in particular with regard to the gradient of $\mathfrak{d}_\varrho V$—

$$\begin{aligned}R &= -\delta\hbar^2\Delta+\mathfrak{d}_\varrho V\left(\alpha\cdot\frac{\hbar}{i}\nabla\right)+\left(\alpha\cdot\frac{\hbar}{i}\nabla\right)(\mathfrak{d}_\varrho V)\\ &\quad+2(\mathfrak{d}_\varrho V)\beta+\mathfrak{d}_\varrho(V^2-\varphi)-(1-\delta)\hbar^2\|\nabla g_\varrho\|^2\\ &\ge-\delta\hbar^2\Delta+\left(\sqrt{\delta}\alpha\cdot\frac{\hbar}{\sqrt{2}i}\nabla+\frac{\sqrt{2}}{\sqrt{\delta}}\mathfrak{d}_\varrho V\right)^2+\frac{\delta}{2}\hbar^2\Delta-\frac{2}{\delta}(\mathfrak{d}_\varrho V)^2\\ &\quad+2(\mathfrak{d}_\varrho V)\beta+\mathfrak{d}_\varrho(V^2-\varphi)-(1-\delta)\hbar^2\|\nabla g_\varrho\|^2\\ &\ge-\frac{\hbar^2}{2}\delta\Delta-\frac{2}{\delta}(\mathfrak{d}_\varrho V)^2-2|\mathfrak{d}_\varrho V|+\mathfrak{d}_\varrho(V^2-\varphi)-(1-\delta)\hbar^2\|\nabla g_\varrho\|^2.\end{aligned}$$

If R' denotes the first four terms in the last expression we have by (2)

$$\begin{aligned}\operatorname{tr}[R'd_{\min}] &\geq \inf\{\operatorname{tr}[R'd] \mid 0\leq d\leq 1,\ d \text{ self-adjoint},\ R'd\in\mathfrak{S}_1\}\\ &\geq -\frac{\text{const}}{\delta^{3/2}h^3}\left(\frac{1}{\delta^{5/2}}\|\partial_\varrho V\|_5^5+\|\partial_\varrho V\|_{5/2}^{5/2}+\|\partial_\varrho(V^2-\varphi)\|_{5/2}^{5/2}\right)\end{aligned}$$

using the Lieb–Thirring inequality [13]. Also, by Lemma 2, $\operatorname{tr}(d_{\min})=O(h^{-3})$; thus the lemma follows. □

4. Convergence of the density and the dimension of the discrete spectral subspace

The above results which bound the sum of eigenvalues suffice already to prove a convergence theorem for the density, namely

Theorem 1. *Suppose that* $V\in L^{3/2}(\mathbf{R}^3)\cap L^5(\mathbf{R}^3)$, $U\in L^{3/2}(\mathbf{R}^3)\cap L^{5/2}(\mathbf{R}^3)$, *and* $W\in L^{3/2}(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$. *Denote by D the Dirac operator with potential V and by* ψ_ν, $\nu=1,2,3,\ldots$, *the eigenfunctions of* D^2-1+W *with negative eigenvalues. We write for the sum of the density of these states* $n(x):=\sum_\nu\sum_{\sigma=1}^4|\psi_\nu(x,\sigma)|^2$. *Then, as* $h\to 0$

$$\begin{aligned}(14)\quad &\int_{\mathbf{R}^3} U(x)n(x)\,dx\\ &=\frac{1}{h^3}\int_{\mathbf{R}^6} U(\mathfrak{q})\operatorname{tr}[\chi_{(-\infty,0)}(\mathfrak{p}^2+V(\mathfrak{q})^2+W(\mathfrak{q})+2V(\mathfrak{q})(\mathfrak{p}\cdot\alpha+\beta))]\,d\mathfrak{p}d\mathfrak{q}+o(h^{-3}).\end{aligned}$$

Proof. First we note that we can without loss of generality assume that U is positive: we can always decompose U into its positive and negative parts. Proving the theorem separately for these and finally subtracting the resulting equations from each other gives the claimed inequality for general U.

Let $d:=\chi_{(-\infty,0)}(D^2-1+W)$. By the minimax principle

$$-\operatorname{tr}(D^2-1+W-\varepsilon U)_-\leq \operatorname{tr}[(D^2-1+W)d]-\varepsilon\int U(x)n(x)\,dx$$

and therefore

$$\varepsilon\int U(x)n(x)\,dx\leq -\operatorname{tr}(D^2-1+W)_-+\operatorname{tr}(D^2-1+W-\varepsilon U)_-$$

which implies for positive ε the upper bound

$$(15)\qquad \int U(x)n(x)\,dx\leq[-\operatorname{tr}(D^2-1+W)_-+\operatorname{tr}(D^2-1+W-\varepsilon U)_-]/\varepsilon$$

and for negative ε the lower bound

$$\int U(x)n(x)\,dx \geq -[\operatorname{tr}(D^2-1+W)_- - \operatorname{tr}(D^2-1+W-\varepsilon U)_-]/\varepsilon. \tag{16}$$

To estimate these traces, we select different functions φ in (3) and (4) and therefore in Lemma 4 and Lemma 6. We choose $\varphi:=-W$ in (3) and $\varphi:=-W+\varepsilon U$ in (4).

1. The upper bound ($\varepsilon>0$): From Lemma 4 and Lemma 6 we have

$$\begin{aligned}\int_{\mathbf{R}^3} U(x)n(x)\,dx \leq \frac{1}{\varepsilon}\int_\Gamma (u(\gamma)_- - o(\gamma)_-)\,d\Omega(\gamma)\\ -\frac{1}{\varepsilon}\left\{\operatorname{tr}[Rd_{\min}] - \frac{\hbar^2}{\varrho^2}\|\nabla g\|^2 \int_{o(\gamma)<0} d\Omega(\gamma)\right\}\end{aligned} \tag{17}$$

with $o(\gamma)$ and $u(\gamma)$ as defined in (5) and (6) with φ replaced by $-W$ and $-W+\varepsilon U$ respectively. Multiplication by h^3 together with Lemma 7 and Lemma 3 yields

$$\begin{aligned} h^3\int_{\mathbf{R}^3} U(x)n(x)\,dx \leq \frac{1}{\varepsilon}\int_\Gamma (u(\gamma)_- - o(\gamma)_-)\,d\Omega(\gamma)\\ +\frac{\text{const}}{\varepsilon}\left[\frac{1}{\delta^{3/2}}\left(\frac{1}{\delta^{5/2}}\|\mathfrak{d}_\varrho V\|_5^5 + \|\mathfrak{d}_\varrho V\|_{5/2}^{5/2} + \|\mathfrak{d}_\varrho(V^2+W-\varepsilon U)\|_{5/2}^{5/2}\right)+\frac{\hbar^2}{\varrho^2}\right]\end{aligned} \tag{18}$$

and thus

$$\begin{aligned} \limsup_{h\to 0}\left(h^3\int_{\mathbf{R}^3} U(x)n(x)\,dx\right) \leq \frac{1}{\varepsilon}\int_\Gamma (u_\varepsilon(\gamma)_- - o_\varrho(\gamma)_-)\,d\Omega(\gamma)\\ +\frac{\text{const}}{\varepsilon\delta^{3/2}}\left(\frac{1}{\delta^{5/2}}\|\mathfrak{d}_\varrho V\|_5^5 + \|\mathfrak{d}_\varrho V\|_{5/2}^{5/2} + \|\mathfrak{d}_\varrho(V^2+W-\varepsilon U)\|_{5/2}^{5/2}\right).\end{aligned} \tag{19}$$

(Note that we have written o_ϱ and u_ε instead of o and u to indicate with these subscripts the explicit parameter dependence of these two quantities.)

Next we take the limit $\varrho\to 0$ which makes the last term in (19) vanish. Also, on choosing a particular subsequence ϱ_ν tending to zero as ν tends to infinity, o_ϱ can be replaced—using dominated convergence—by o_0, its pointwise limit, in the resulting inequality. To see this, we first substitute

$$\int_\Gamma o_\varrho(\gamma)_-\,d\Omega(\gamma) \geq \int_{\{\zeta|\zeta\in\operatorname{supp}(o_0)_-\}} o_\varrho(\gamma)_-\,d\Omega(\gamma)$$

into (19), and proceed to prove that

$$\lim_{\varrho\to 0}\int_{\{\zeta|\zeta\in\operatorname{supp}(o_0)_-\}} o_\varrho(\gamma)_-\,d\Omega(\gamma) = \int_\Gamma o_0(\gamma)_-\,d\Omega(\gamma)$$

at least for a subsequence ϱ_ν which we are free to choose. Observe that the left-hand side has its support contained in a ϱ independent set of finite measure, namely the support of $(o_0)_-$. Furthermore,

$$\begin{aligned} o_\varrho(\gamma)_- &= (\langle V^2\rangle_\varrho(\mathfrak{q}) - \langle V\rangle_\varrho^2(\mathfrak{q}) + \big(\sqrt{\mathfrak{p}^2+1} + s(\mu)|\langle V\rangle_\varrho(\mathfrak{q})|\big)^2 + \langle W\rangle_\varrho(\mathfrak{q}) - 1)_- \\ &\leq 1 + \|W\|_\infty \end{aligned}$$

because $\langle V^2\rangle_\varrho(\mathfrak{q}) \geq \langle V\rangle_\varrho^2(\mathfrak{q})$ by Jensen's inequality. Thus

$$\chi_{\{\zeta|\zeta\in\mathrm{supp}(o_0)_-\}}(o_\varrho)_- \leq \chi_{\{\zeta|\zeta\in\mathrm{supp}(o_0)_-\}}(1+\|W\|_\infty)$$

yielding the integrable dominator. Furthermore, since $\langle V\rangle_\varrho$ converges to V in $L^{3/2}(\mathbf{R}^3)$, $\langle V\rangle_\varrho$ converges in measure to V. Since $\mathbf{R}^3$ with the Lebesgue measure is σ-finite, there exists a subsequence $\langle V\rangle_{\varrho_\nu}$, $\nu=1,2,3,\dots$, such that $\langle V_{\varrho_\nu}\rangle$ converges pointwise almost everywhere to V (see Bauer [1, Theorems 19.3 and 19.6]). Since $(o_{\varrho_\nu})_-$ depends continuously on $\langle V\rangle_{\varrho_\nu}$ and $\langle V^2\rangle_{\varrho_\nu}$, the function $(o_{\varrho_\nu})_-$ converges pointwise almost everywhere to $(o_0)_-$ which yields the convergence hypothesis of the dominated convergence theorem.

Next take the limit $\delta\to 0$ which—again by dominated convergence—simply replaces the δ in u_ε by 0. (Note that we chose to suppress the δ-dependence in our notation.) Finally, to perform the $\varepsilon\to 0$ limit we want to apply dominated convergence once more. To this end note that $u_\varepsilon = u_0 - \varepsilon U$ and $u_0 = o_0$. Thus

$$\begin{aligned} &\frac{1}{\varepsilon}\int_\Gamma ((u_\varepsilon)_- - (o_0)_-)\, d\Omega(\gamma) \\ (20)\qquad &\leq \int_{u_0(\gamma)\leq 0} U(\mathfrak{q})\, d\Omega(\gamma) + \int_\Gamma d\Omega(\gamma) \frac{-u_0(\gamma)+\varepsilon U}{\varepsilon} \chi_{\{\zeta\in\Gamma|u_0(\zeta)>0>u_0(\zeta)-\varepsilon U\}}(\gamma). \end{aligned}$$

The first integral on the right-hand side of (20) is independent of ε. For $\varepsilon\leq 1$ the modulus of the integrand of the last integral is bounded by $U\chi_{\{\zeta\in\Gamma|0>u_0(\zeta)-U\}}$ uniformly in ε. Now, note that

$$\int_{0>u_0-\varepsilon U} U(\mathfrak{q})\, d\Omega(\gamma) \leq \mathrm{const}\int (|V|^{3/2}+|V|^3+U^{3/2})U\, d\mathfrak{q}$$

which is integrable under our hypothesis on U. Thus dominated convergence is indeed applicable and the limit $\varepsilon\to 0$ gives the desired result.

2. To obtain the lower bound pick ε negative, take the $\liminf_{h\to 0}$ and repeat the argument. □

Theorem 2. *Assume that* $V\in L^{3/2}(\mathbf{R}^3)\cap L^3(\mathbf{R}^3)$ *and* $W\in L^{3/2}(\mathbf{R}^3)$. *Let* D *be the Dirac operator with potential* V. *Then*

$$\begin{aligned}&\operatorname{tr}\chi_{(-\infty,0)}(D^2-1+W)\\&\qquad=\frac{1}{\hbar^3}\int_{\mathbf{R}^6}\operatorname{tr}\chi_{(-\infty,0)}[\mathfrak{p}^2+W+V(\mathfrak{q})^2+2V(\mathfrak{q})(\mathfrak{p}\cdot\alpha+\beta)]\,d\mathfrak{p}\,d\mathfrak{q}+o(\hbar^{-3})\end{aligned}$$

as $\hbar\to 0$.

We note—as for the weak convergence of the density—that Theorem 2 can be easily generalized to any subinterval of the interval $(-1,1)$. In fact the proof is easier for any subinterval (a,b), $-1<a<b<1$. But we restrict again to the full interval for the sake of clarity of the presentation.

Proof. We shall first prove the theorem only when $V,W\in C_0^\infty(\mathbf{R}^3)$.

1. Upper bound: We pick two functions $I\in C_0^\infty(\mathbf{R}^3)$, $A\in C^\infty(\mathbf{R}^3)$ such that

- for all $x\in\mathbf{R}^3$ the identity $I(x)^2+A(x)^2=1$ holds,
- $\operatorname{supp}V\cup\operatorname{supp}W\subset\operatorname{supp}I$,
- $\operatorname{distance}(\operatorname{supp}V\cup\operatorname{supp}W,\operatorname{supp}A)\geq 1$.

We set $P:=D^2-1+W-|\hbar\nabla I|^2-|\hbar\nabla A|^2$, $P_I:=IPI$ and $P_A:=APA$.

Given any nonnegative smooth function θ not exceeding one the minimax principle implies for the nth negative eigenvalue $e_n(\theta)$ of $P_\theta:=\theta P\theta$ that it is not less than the nth eigenvalue e_n of P: let $\mathfrak{H}_{n-1}$ denote the linear subspace spanned by the eigenvectors of P with eigenvalues $e_1,\dots,e_{n-1}$. Then

(21)
$$\begin{aligned}e_n(\theta)&=\sup\{\inf\{(\psi,P_\theta\psi)\mid\psi\in\mathfrak{D},\ \psi\perp\mathfrak{H},\ \|\psi\|\leq 1\}\mid\mathfrak{H}\subset[L^2(\mathbf{R}^3)]^4,\ \dim(\mathfrak{H})\leq n-1\}\\&\geq\inf\{(\theta\psi,P(\theta\psi))\mid\psi\in\mathfrak{D},\ \psi\perp\theta\mathfrak{H}_{n-1},\ \|\psi\|\leq 1\}\\&\geq\inf\{(\psi,P\psi)\mid\psi\in\mathfrak{D},\ \psi\perp\mathfrak{H}_{n-1},\ \|\psi\|\leq 1\}=e_n.\end{aligned}$$

Thus we have

$$N(P_A)\leq N[\hbar^2(-\Delta-|\nabla I|^2-|\nabla A|^2)]=N(-\Delta-|\nabla I|^2-|\nabla A|^2)=:C,$$

which is finite because of CLR and independent of $\hbar$.

Applying the above argument and a similar one for the case of Dirichlet boundary conditions and using the fact that $D^2-1+W=P_I+P_A$ yields

$$N(D^2+W-1)\leq N(P_I)+N(P_A)\leq N(P|_{H_0^1(\operatorname{supp}I)})+C$$

where, given any measurable $\Omega\subset\mathbf{R}^3$ and any semi-bounded self-adjoint operator B with form core $[C_0^\infty(\mathbf{R}^3)]^4$, the symbol $B|_{H_0^1(\Omega)}$ denotes the operator defined by the closure of the quadratic form $(\psi, B\psi)$ on $[C_0^\infty(\Omega)]^4$, i.e., the operator B acting on functions on Ω with Dirichlet boundary conditions on $\partial\Omega$. Again, by the minimax principle, the nth negative eigenvalue of $P|_{H_0^1(\mathrm{supp}\, I)}$ is not bigger than the nth negative eigenvalue of P_I and the nth negative eigenvalue of $(P-\varepsilon)|_{H_0^1(\mathrm{supp}\, I)}$ is not less than the nth negative eigenvalue of $P-\varepsilon\chi_{\mathrm{supp}\, I}$. Thus we have for positive ε

$$\begin{aligned} N(P_I) \leq N(P|_{H_1^0(\mathrm{supp}\, I)}) &\leq \frac{1}{\varepsilon}\{-\operatorname{tr}[P|_{H_0^1(\mathrm{supp}\, I)}]_- + \operatorname{tr}[P|_{H_0^1(\mathrm{supp}\, I)} - \varepsilon]_-\} \\ &\leq \frac{1}{\varepsilon}\{-\operatorname{tr}(P_I)_- + \operatorname{tr}[(P-\varepsilon\chi_{\mathrm{supp}\, I})|_{H_0^1(\mathrm{supp}\, I)}]_-\} \\ &\leq \frac{1}{\varepsilon}\{\operatorname{tr}(Pd) + \operatorname{tr}(P-\varepsilon\chi_{\mathrm{supp}\, I})_-\} \end{aligned}$$

where d is as in the statement of Lemma 4 with $\varphi=-W$. In the last step we have used that $IdI=d$. This follows, since, when $\varrho<\frac{1}{2}$, $g_\varrho(x-\mathfrak{q})$ vanishes for all $(\mathfrak{p}, \mathfrak{q}, \mu) \in \mathrm{supp}\, M$ and $x \in \mathrm{supp}\, A$.

To evaluate these two traces we use again Lemma 4 and Lemma 6. In Lemma 4 we pick $\varphi=-W$; in Lemma 6 we pick $\varphi=\hbar^2|\nabla I|^2+\hbar^2|\nabla A|^2-W\ -\varepsilon\chi_{\mathrm{supp}\, I}$.

We can now follow the corresponding argument in the proof of Theorem 1 step by step and obtain, on noticing that the term $\hbar^2(|\nabla I|^2+|\nabla A|^2)$ and the constant C can be absorbed at no cost into the error term,

$$N(D^2-1+W) \leq \int_{u_0(\gamma)<0} d\Omega(\gamma) + o(h^{-3}).$$

2. Lower bound: by Theorem 1 we have

$$N(D^2-1+W) \geq \int \chi_{\mathrm{supp}\, V \cup \mathrm{supp}\, W}(x) n(x)\, dx = \int_{u_0(\gamma)<0} d\Omega(\gamma) + o(h^{-3})$$

where, as previously, $u_0(\gamma)=\mathfrak{p}^2+W(\mathfrak{q})+V(\mathfrak{q})^2+s(\mu)|V(\mathfrak{q})|(\mathfrak{p}^2+1)^{1/2}$ and we use that $u_0(\mathfrak{p}, \mathfrak{q}, \mu) \geq 0$ for $\mathfrak{q} \notin \mathrm{supp}\, V \cap \mathrm{supp}\, W$.

The general case follows now by approximation and Lemma 2 (see Reed and Simon [14, Theorem XIII.80], for the analogous argument for Schrödinger operators): Pick $\delta>0$ arbitrary and suppose now $V=U+X$ and $W=T+Y$ with $U, T \in C_0^\infty(\mathbf{R}^3)$ and $\|X\|_3+\|X\|_{3/2}+\|Y\|_{3/2} \leq \delta$. Write

$$D^2-1+W = \widetilde{D}_U^2 - 1 + T - \frac{2\varepsilon}{1-\varepsilon} U^2 + R$$

where

$$\widetilde{D}_U = \frac{\tilde{h}}{i}\alpha\cdot\nabla + \beta + U$$

with $\tilde{h}:=(1-\varepsilon)\hbar$ and

$$R:=-(2-\varepsilon)\varepsilon\hbar^2\Delta+2\operatorname{Re}\left[(\hbar V-\tilde{h}U)\frac{1}{i}\alpha\cdot\nabla\right]+2(V-U)\beta+V^2-U^2+W-T+\frac{2\varepsilon}{1-\varepsilon}U^2$$
$$=-(2-\varepsilon)\varepsilon\hbar^2\Delta+2\operatorname{Re}\left[(\varepsilon V+\tilde{h}X)\frac{1}{i}\alpha\cdot\nabla\right]+2X\beta+X(V+U)+Y+\frac{2\varepsilon}{1-\varepsilon}U^2.$$

Now,

$$N(D^2-1+W)\le N\left(\widetilde{D}_U^2-1+T-\frac{2\varepsilon}{1-\varepsilon}U^2\right)+N(R). \tag{22}$$

By the estimate that we just proved for smooth potentials we have

$$N\left(\widetilde{D}_U^2-1+T-\frac{2\varepsilon}{1-\varepsilon}U^2\right)$$
$$=\frac{1}{(2\pi\tilde{h})^3}\int_{\mathfrak{p}^2+U(\mathfrak{q})^2+T(\mathfrak{q})-(2\varepsilon U(\mathfrak{q})^2/(1-\varepsilon))+2s(\mu)|U(\mathfrak{q})|\sqrt{\mathfrak{p}^2+1}<0}d\Omega(\gamma)+o(\tilde{h}^{-3}).$$

We bound R from below. To this end note that for any $\gamma,\widetilde{\gamma}>0$ the first order derivative term in R can be estimated as follows

$$2\operatorname{Re}\left[(\hbar V-\tilde{h}U)\frac{1}{i}\alpha\cdot\nabla\right]=2\operatorname{Re}\left[(X-\varepsilon U)\frac{\hbar}{i}\alpha\cdot\nabla\right]\ge-\frac{X^2}{\widetilde{\gamma}}+\widetilde{\gamma}\hbar^2\Delta-\frac{\varepsilon^2\hbar^2}{\gamma}U^2+\gamma\Delta.$$

We pick $\gamma:=(1-\varepsilon)\varepsilon\hbar^2/2$ and $\widetilde{\gamma}:=\varepsilon$ and obtain

$$R\ge-[(1-\varepsilon)\varepsilon\hbar^2/2]\Delta-\frac{1}{\varepsilon}X^2-2|X|-|X|\,|V+U|-|Y|.$$

Thus, R is bounded from below for ε small enough and

$$N(R)\le\frac{\text{const}}{((1-\varepsilon)\varepsilon\hbar^2)^{3/2}}\int\varepsilon^{-3/2}|X|^3+|X|^{3/2}+|X|^{3/2}|V+U|^{3/2}+|Y|^{3/2}\,d\mathfrak{q}.$$

Taking first the limit $\delta\to0$ and then $\varepsilon\to0$ gives, using dominated convergence as in Theorem 1,

$$\limsup_{h\to0}[h^3N(D^2-1+W)]\le\int_{\mathfrak{p}^2+V(\mathfrak{q})^2+W(\mathfrak{q})+2s(\mu)|V(\mathfrak{q})|\sqrt{\mathfrak{p}^2+1}<0}d\Omega(\gamma)$$

which is the desired upper bound.

To obtain the reverse bound we reverse the roles and write

$$D_U^2 - 1 + T + \frac{2\varepsilon}{1-\varepsilon} U^2 = \left(\frac{\tilde{\hbar}}{i}\alpha\cdot\nabla + \beta + V\right)^2 - 1 + W + F$$

with

$$D_U := \frac{\hbar}{i}\alpha\cdot\nabla + \beta + U, \quad \widetilde{D}_V := \frac{\tilde{\hbar}}{i}\alpha\cdot\nabla + \beta + V,$$
$$F := -(2-\varepsilon)\varepsilon\hbar\Delta + 2\,\mathrm{Re}\left[(U-(1-\varepsilon)V)\frac{\hbar}{i}\alpha\cdot\nabla\right] + 2(U-V)\beta + U^2 - V^2$$

and again $\tilde{\hbar} = (1-\varepsilon)\hbar$, and repeat the previous argument. □

Note that:

1. The proofs of Theorems 1 and 2 both yield a weak convergence for the density. Similar arguments have been previously used, e.g., by Lieb and Simon [12] or Iantchenko et al. [7].

2. The proof of Theorem 2 generalizes easily to cover any subinterval $(a,b) \subset (-\infty, 0)$

$$\mathrm{tr}\,\chi_{(a,b)}(D^2 - 1 + W) = \int_{\mathbf{R}^6} \mathrm{tr}\{\chi_{(a,b)}((\alpha\cdot\mathfrak{p} + \beta + V(\mathfrak{q}))^2 - 1 + W(\mathfrak{q}))\}\, d\mathfrak{p}\, d\mathfrak{q} + o(h^{-3}).$$

Corollary 1. *Assume D to be a Dirac operator with potential $V \in L^{3/2}(\mathbf{R}^3) \cap L^3(\mathbf{R}^3)$. Then*

$$\mathrm{tr}\,\chi_{(-1,1)}(D) = \frac{1}{h^3}\int_{\mathbf{R}^6} \mathrm{tr}[\chi_{(-1,1)}(\alpha\cdot\mathfrak{p} + \beta + V(\mathfrak{q}))]\, d\mathfrak{p}\, d\mathfrak{q} + o(\hbar^{-3}).$$

Corollary 2. *Assume $S := -\hbar^2\Delta + W$ to be a Schrödinger operator on $H^2(\mathbf{R}^3)$ with potential $W \in L^{3/2}(\mathbf{R}^3)$. Then*

$$N(S) = \frac{1}{h^3}\int_{\mathfrak{p}^2 + W(\mathfrak{q}) < 0} d\mathfrak{p}\, d\mathfrak{q} + o(\hbar^{-3}).$$

Proof. The claims follow from Theorem 2 by setting $W=0$ and application of Lemma 1 in the Dirac case and by setting $V=0$ and dividing by four in the Schrödinger case. □

We would like to repeat our introductory statement that these results can be obtained also directly from known results by the approximation argument used at the end of the proof of Theorem 2 using the phase-space bounds of Cwickel, Lieb, and Rozenblum. However, we repeat also that the main purpose of the article was to show that coherent states can be used in this context.

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William Desmond Evans
School of Mathematics
University of Wales, Cardiff
Senghennydd Road
Cardiff CF2 4YH
Great Britain
email: smawde@cardiff.ac.uk

Roger T. Lewis
Department of Mathematics
University of Alabama at Birmingham
Birmingham, AL 35294-1170
U.S.A.
email: lewis@math.uab.edu

Heinz Siedentop
Matematisk institutt
Universitetet i Oslo
Postboks 1053
N-0316 Oslo
Norway
email: heinz@math.uio.no

Jan Philip Solovej
Institut for Matematiske Fag
Aarhus Universitet
Ny Munkegade
DK-8000 Århus C
Denmark
email: solovej@mi.aau.dk