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Relative vanishing theorems in characteristic p

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Introduction

In [I], L. Illusie proved a decomposition theorem of the relative de Rham complex for a morphism $f: X \to Y$ of smooth schemes defined over a perfect field k of characteristic $p>0$, generalizing an earlier result of [DI] which treats the absolute case. He deduced from the theorem several vanishing results for the direct image of line bundles.

Let $f: X \to Y$ be a k-morphism and E a vector bundle on X. In this paper we introduce the *p-cohomological dimension relative to* f, which will be denoted by $\mathrm{pcd}(E, f)$, of E. By means of this notion, we extend some of the vanishing theorems obtained in [I] to the case of higher rank bundles. As in [I], we need the assumption that f is semistable along a normal crossing divisor $D_Y \subset Y$ and f is liftable to $W_2(k)$, the ring of length two Witt vectors.

In Section 1, we prove a vanishing of direct image sheaves of vector bundles for a semistable morphism. The cohomology vanishing of the Gauss–Manin systems will be considered in Section 2. In Section 3, we treat the case of open varieties and generalize a theorem in [BK2] to the relative situation.

1. Relative vanishing for semistable morphisms

Let k be a perfect field of characteristic $p>0$. Let X be a smooth scheme defined over k. We denote by $F_X: X \to X$ the absolute Frobenius of X. A vector bundle E on X yields a bundle E' on X'. Let $F^n E := (F_X^*)^n E$ denote the bundle on X obtained by the nth iterated pull-back of E by F_X.

Let Y be a smooth scheme over k and $f: X \to Y$ a k-morphism. We define the *p-cohomological dimension relative to* f of E, which we denote by $\mathrm{pcd}(E, f)$, as the smallest integer $\alpha \geq 0$ such that for every coherent sheaf $\mathcal{F}$, there exists $n_0 = n_0(\mathcal{F})$ satisfying $R^k f_*(\mathcal{F} \otimes F^n E) = 0$ for all $n \geq n_0$ and all $k > \alpha$.

Remark. If $Y = \operatorname{Spec} k$, we simply write $\operatorname{pcd}(E)$ instead of $\operatorname{pcd}(E, f)$. We note that $\operatorname{pcd}(\mathcal{O}_X)$ is identical with $\operatorname{cd}(X)$, the cohomological dimension of X. Furthermore, if $f: X \to Y = \operatorname{Spec} k$ is a projective morphism, then $\operatorname{pcd}(E)=0$ if and only if E is cohomologically p-ample (cf. [H], [K]).

In this section we prove a vanishing theorem for the direct image of vector bundles in terms of $\operatorname{pcd}(E, f)$. For this purpose, we recall the decomposition theorem of Illusie. We follow the notation of [I]. Let $f: X \to Y$ be a k-morphism of smooth schemes. Let $D_Y \subset Y$ be a normal crossing divisor and let $D_X = f^{-1}(D_Y)$. We denote by X' the scheme deduced from X by the Frobenius of Y and let $\omega^i_{X'/Y} = \omega^i_{X/Y} \otimes \mathcal{O}_{X'}$. We say that $f: (X, D_X) \to (Y, D_Y)$ is *semistable* if, locally for the etale topology on X, f is the product of the following k-morphisms:

(1) $pr_1: \mathbf{A}^n_k \to \mathbf{A}^1_k$, $D_Y = \emptyset$;

(2) $h: \mathbf{A}^n_k \to \mathbf{A}^1_k$, $h^* y = x_1 \dots x_n$,

where $\mathbf{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, $\mathbf{A}^1_k = \operatorname{Spec} k[y]$ and $D_Y = (y)$. If $f: (X, D_X) \to (Y, D_Y)$ is semistable, we define the sheaf $\Omega^1{}_{X/Y}(\log D_X/D_Y)$ to be the cokernel of the natural injection $f^*\Omega^1_Y(\log D_Y) \subset \Omega^1_X(\log D_X)$. Setting $\omega^i{}_{X/Y} = \bigwedge^i \Omega^1{}_{X/Y}(\log D_X/D_Y)$, we obtain the complex $\omega^{\cdot}{}_{X/Y}$, which is called *the relative de Rham complex with logarithmic poles along* D_X.

We assume that f, together with D_X, D_Y, has a semistable lifting over $W_2(k)$, the ring of Witt vectors of length two over k. We denote it by $\tilde{f}: (\widetilde{X}, \widetilde{D}_X) \to (\widetilde{Y}, \widetilde{D}_Y)$. We also assume the existence of a lifting $\widetilde{F}_Y: \widetilde{Y} \to \widetilde{Y}$ of the absolute Frobenius of Y such that $\widetilde{F}_Y^{-1}(\widetilde{D}_Y) = p\widetilde{D}_Y$. The following result is due to L. Illusie.

Proposition 1.1. ([I, 2.8]) *Under the assumptions above, there exists an isomorphism in* $D(X')$

$$\phi(\tilde{f}, \widetilde{F}_Y): \bigoplus_{i<p} \omega^i_{X'/Y}[-i] \xrightarrow{\sim} \tau_{<p} F_* \omega^{\cdot}{}_{X/Y}$$

where $D(X')$ *is the derived category of the category of* $\mathcal{O}_{X'}$*-modules.*

Lemma 1.2. *In addition to the assumptions above, suppose that* f *is purely of relative dimension* $d<p$. *Let* M *be a vector bundle on* X. *If*

$$(*) \qquad R^j f_*(F^*_X M \otimes \omega^i{}_{X/Y}) = 0 \quad \text{for } i+j=n,$$

then we have

$$R^j f_*(M \otimes \omega^i{}_{X/Y}) = 0 \quad \text{for } i+j=n.$$

Proof. We follow the argument in [DI, 2.9]. Let M' be the bundle induced by the Frobenius base change from M. We have $F^* M' = F^*_X M$. By the projection formula,

$$R^j f_*\Big(F_X^* M \underset{\mathcal{O}_X}{\otimes} \omega^i{}_{X/Y}\Big) = R^j f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^i{}_{X/Y}\Big).$$

We have the spectral sequence

$$E_1^{ij} = R^j f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^i{}_{X/Y}\Big) \quad\Longrightarrow\quad \mathbf{R}^{i+j} f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^{\cdot}{}_{X/Y}\Big)$$

where $\mathbf{R}^n f_*$ denotes the nth hyperdirect image and

$$\mathbf{R}^n f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^{\cdot}{}_{X/Y}\Big) = \mathbf{R}^n f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} \tau_{<p} F_*\omega^{\cdot}{}_{X/Y}\Big).$$

By assumption $(*)$, $R^j f'_*(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^i{}_{X/Y}) = 0$ for $i+j=n$. Hence Proposition 1.1 yields

$$\begin{aligned}\mathbf{R}^n f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} F_*\omega^{\cdot}{}_{X/Y}\Big) &= \bigoplus_i R^{n-i} f'_*\Big(M' \underset{\mathcal{O}_{X'}}{\otimes} \omega^i{}_{X'/Y}\Big) \\ &= \bigoplus_i F_Y^* R^{n-i} f_*\Big(M \underset{\mathcal{O}_X}{\otimes} \omega^i{}_{X/Y}\Big) = 0.\end{aligned}$$

Hence we obtain

$$R^{n-i} f_*\Big(M \underset{\mathcal{O}_X}{\otimes} \omega^i{}_{X/Y}\Big) = 0$$

for all i. This proves the lemma. □

Theorem 1.3. *Let f be as above and assume further that f is proper. If E is a vector bundle on X with* $\mathrm{pcd}(E,f) \le \alpha$, *then*

(1.3.1) $$R^j f_*(E \otimes \omega^i{}_{X/Y}) = 0 \quad \textit{for } i+j > d+\alpha,$$

(1.3.2) $$R^j f_*(E^\vee \otimes \omega^i{}_{X/Y}) = 0 \quad \textit{for } i+j < d-\alpha.$$

Proof. Since $\mathrm{pcd}(E,f) \le \alpha$, we have

$$R^j f_*(F^n E \otimes \omega^i{}_{X/Y}) = 0$$

for $i+j > d+\alpha$ and for sufficiently large n. If we apply Lemma 1.2 to $M = F^n E$, we obtain (1.3.1) by descending induction on n. Since Grothendieck duality yields

$$\mathbf{R} f_*(E^\vee \otimes \omega^{d-i}{}_{X/Y}) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om(\mathbf{R} f_*(E \otimes \omega^i{}_{X/Y}), \mathcal{O}_Y)[-d],$$

(1.3.2) easily follows from (1.3.1). □

Let $f: X \to Y$ be a proper k-morphism and $l \geq 0$ a nonnegative integer. A line bundle L on X is said to be *l-ample relative to f* if there exists $n \in \mathbf{N}$ such that

(1) the canonical map $f^* f_* L^{\otimes n} \to L^{\otimes n}$ is surjective;

(2) the fibers of the canonical Y-morphism

$$X \to \mathbf{P}(f_* L^{\otimes n})$$

are at most l-dimensional.

In the case when $Y = \operatorname{Spec} k$, the above notion coincides with the l-ampleness in the sense of Sommese. By [BK1, 5.2], we have $\operatorname{pcd}(L, f) \leq l$ if L is l-ample relative to f. Hence Theorem 1.3 yields the following result.

Corollary 1.4. *Let f be as in Theorem 1.3 and $l \geq 0$. If L is a line bundle on X which is l-ample relative to f, then we have*

$$R^j f_*(L \otimes \omega^i{}_{X/Y}) = 0 \quad \text{for } i+j > d+l, \tag{1.4.1}$$

$$R^j f_*(L^\vee \otimes \omega^i{}_{X/Y}) = 0 \quad \text{for } i+j < d-l. \tag{1.4.2}$$

2. Vanishing for Gauss–Manin systems

Let $f: (X, D_X) \to (Y, D_Y)$ be as in the previous section. We define the graded $\mathcal{O}_Y$-module

$$H = \bigoplus_i \mathbf{R}^i f_* \omega^{\cdot}{}_{X/Y}.$$

Then there exists the Gauss–Manin connection

$$d: H \to \omega^1{}_Y \otimes H$$

which leads to the following complex

$$\omega^{\cdot}{}_Y(H) = (H \xrightarrow{d} \omega^1{}_Y \otimes H \longrightarrow \dots \xrightarrow{d} \omega^i{}_Y \otimes H \longrightarrow \dots).$$

The Hodge filtration on H can be extended to the following filtration of $\omega^{\cdot}{}_Y(H)$:

$$\operatorname{Fil}^i \omega^{\cdot}{}_Y(H) = (\operatorname{Fil}^i H \longrightarrow \omega^1{}_Y \otimes \operatorname{Fil}^{i-1} H \longrightarrow \dots \longrightarrow \omega^j{}_Y \otimes \operatorname{Fil}^{i-j} H \longrightarrow \dots).$$

We denote the associated graded complex by $\operatorname{gr}^{\cdot} \omega^{\cdot}{}_Y(H)$. In this section we prove a vanishing for the cohomology of bundles of the form $E \otimes \operatorname{gr}^i \omega^{\cdot}{}_Y(H)$. The following is a consequence of [I, 4.7].

Proposition 2.1. *Assume that* $f\colon(X,D_X)\to(Y,D_Y)$ *is proper and semistable and admits a lifting* $\tilde{f}\colon(\widetilde{X},\widetilde{D}_X)\to(\widetilde{Y},\widetilde{D}_Y)$ *over* $W_2(k)$ *which is also semistable. If* f *is purely of relative dimension* $d<p$*, then there exists an isomorphism*

$$\phi\colon\bigoplus_i \mathrm{gr}^i\,\omega^{\cdot}{}_{Y'}(H')\xrightarrow{\sim}F_*\omega^{\cdot}{}_Y(H).$$

Lemma 2.2. *Let* f *be as above. If* M *is a vector bundle on* Y*, then we have for all* n

$$\sum_{i+j=n}\dim H^{i+j}(Y,M\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))\le\sum_{i+j=n}\dim H^{i+j}(Y,F^*M\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H)).$$

Proof. By the spectral sequence

$$E_1^{i,j}=H^{i+j}(Y',M'\otimes F_*\,\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))\quad\Longrightarrow\quad\mathbf{H}^n(Y',M'\otimes F_*\omega^{\cdot}{}_Y(H)),$$

we obtain

$$\dim\mathbf{H}^n(Y',M'\otimes F_*\omega^{\cdot}{}_Y(H))\le\sum_{i+j=n}\dim H^{i+j}(Y,F^*M\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H)).$$

On the other hand, Proposition 2.1 yields

$$M'\otimes F_*\omega^{\cdot}{}_Y(H)\cong\bigoplus_i M'\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_{Y'}(H).$$

Hence we have

$$\dim\mathbf{H}^n(Y,M'\otimes F_*\omega^{\cdot}{}_Y(H))=\sum_{i+j=n}H^{i+j}(Y,M\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H)).$$

Thus the desired inequality follows. □

Theorem 2.3. *Let* f *be as above and assume further that* Y *is proper over* k*. Let* $e=\dim Y$*. If* E *is a vector bundle on* Y *with* $\mathrm{pcd}(E)\le\alpha$*, then we have*

(2.3.1) $$H^j(Y,E\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))=0\quad\textit{for } i+j>e+\alpha,$$

(2.3.2) $$H^j(Y,E^\vee(-D_Y)\otimes\mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))=0\quad\textit{for } i+j<e-\alpha.$$

Proof. By assumption and Lemma 2.2, we obtain (2.3.1) as in the proof of (1.3.1). Let d be the relative dimension of f. Then

$$\mathrm{gr}^i\,\omega^{\cdot}{}_Y(\mathbf{R}^j f_*\omega^{\cdot}{}_{X/Y})$$

is Serre dual (with coefficients in ω_Y^e) to

$$\mathrm{gr}^{d+e-i}\,\omega^{\cdot}{}_Y(\mathbf{R}^{2d-j}f_*\omega^{\cdot}{}_{X/Y})[e].$$

Hence

$$H^n(Y, E\otimes \mathrm{gr}^i\,\omega^{\cdot}{}_Y(\mathbf{R}^j f_*\omega^{\cdot}{}_{X/Y}))$$

is dual to

$$H^{2e-n}(Y, E^\vee(-D_Y)\otimes \mathrm{gr}^{d+e-i}\,\omega^{\cdot}{}_Y(\mathbf{R}^{2d-j}f_*\omega^{\cdot}{}_{X/Y})).$$

Therefore (2.3.2) follows. □

Corollary 2.4. *Let f be as in Theorem 2.3. If L is an l-ample line bundle on Y, then we have*

(2.4.1) $$H^j(Y, L\otimes \mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))=0 \quad \textit{for } i+j>e+l,$$

(2.4.2) $$H^j(Y, L^\vee(-D_Y)\otimes \mathrm{gr}^i\,\omega^{\cdot}{}_Y(H))=0 \quad \textit{for } i+j<e-l.$$

3. The case of open varieties

In [BK2], a vanishing theorem has been proved for line bundles on an open variety whose complement is a divisor with globally generated normal bundle. The purpose of this section is to generalize the result to the relative case.

Lemma 3.1. *Let X be a scheme proper over k and L a line bundle on X which is globally generated. Let $f\colon X\to Y$ be a proper k-morphism to a scheme Y. Let E be a vector bundle on X with $\mathrm{pcd}(E,f)\le\alpha$. Then for every coherent sheaf $\mathcal{F}$ on X, there exists $n_0=n_0(\mathcal{F})$ such that for all $n\ge n_0$ we have*

$$R^j f_*(F^nE\otimes\mathcal{F}\otimes L^{\otimes m})=0 \quad \textit{for } j>\alpha \textit{ and } m\ge 0.$$

Proof. We proceed by induction on $(m, \dim X)$. Clearly the claim holds if $m=0$ or $\dim X=0$. Since L is globally generated, we can choose a section $\mathcal{O}_X\to L$ which is injective. Let Z be its zero scheme. We have the exact sequence

$$0\longrightarrow F^nE\otimes\mathcal{F}\otimes L^{\otimes(m-1)}\longrightarrow F^nE\otimes\mathcal{F}\otimes L^{\otimes m}\longrightarrow F^nE\otimes\mathcal{F}\otimes L^{\otimes m}{}_{|Z}\longrightarrow 0.$$

By inductive hypothesis, we can find n_0 such that for all $n\ge n_0$ and $j>\alpha$, we have $R^jf_*(F^nE\otimes\mathcal{F}\otimes L^{\otimes(m-1)})=R^jg_*(F^nE\otimes\mathcal{F}\otimes L^{\otimes m}{}_{|Z})=0$, where $g:=f_{|Z}$. Hence $R^jf_*(F^nE\otimes\mathcal{F}\otimes L^{\otimes m})=0$. This proves the lemma. □

Lemma 3.2. *Let X be a scheme which is smooth and proper over k and let $f: X \to Y$ be a smooth and proper k-morphism to a scheme Y. Let $D \subset X$ be a divisor such that the normal bundle $N_{D/X}$ is globally generated and let $U := X \setminus D$. Let E be a vector bundle on X with $\operatorname{pcd}(E, f) \leq \alpha$. Then for every vector bundle $\mathcal{F}$ on U, there exists $n_0 = n_0(\mathcal{F})$ such that for all $n \geq n_0$ we have*

$$R^j(f|U)_*(F^n E \otimes \mathcal{F}) = 0 \quad \textit{for } j > \alpha.$$

Proof. Let $\mathcal{I}$ be the ideal sheaf of D and let $g := f_{|D}$. By [BK1, 5.4], for every vector bundle $\mathcal{F}$ on X, we have an isomorphism

$$\mathcal{E}xt^j_f(\mathcal{I}^m/\mathcal{I}^{m+1}, \mathcal{F}) \cong R^{j-1} g_*(\mathcal{F}_{|D} \otimes (\mathcal{I}^{m+1}/\mathcal{I}^{m+2})^\vee)$$

for all $j > 0$ and $m \geq 0$. Therefore, by Lemma 3.1, we can find n_1 such that

$$\mathcal{E}xt^j_f(\mathcal{I}^m/\mathcal{I}^{m+1}, F^n E \otimes \mathcal{F}) = 0$$

for all $n \geq n_1$, $j > \alpha + 1$, $m > 0$ since we have $(\mathcal{I}^m/\mathcal{I}^{m+1})^\vee \cong {N_{D/X}}^{\otimes m}$.

This implies, by Grothendieck duality,

$$R^j g_*((\mathcal{I}^m/\mathcal{I}^{m+1}) \otimes \mathcal{F}^\vee \otimes (F^n E)^\vee \otimes {\omega^d}_{X/Y}) = 0$$

for all $n \geq n_1$, $j < d - \alpha - 1$, $m > 0$. Putting $\mathcal{G}_n := \mathcal{F}^\vee \otimes (F^n E)^\vee \otimes {\omega^d}_{X/Y}$, we see that the natural map

$$R^j g_*(\mathcal{I}^{m+1} \mathcal{G}_n) \longrightarrow R^j g_*(\mathcal{I}^m \mathcal{G}_n)$$

is bijective for $n \geq n_1$, $j < d - \alpha - 1$ and $m > 0$. By [BK1, 3.1], for some m_0, we obtain a bijection

$$R^j(f|U)_*(\mathcal{F} \otimes F^n E) \cong R^{d-j} g_*(\mathcal{I}^{m_0} \mathcal{G}_n)^\vee$$

for $j > d + \alpha + 1$ and a surjection

$$R^{d-j} g_*(\mathcal{I}^{m_0} \mathcal{G}_n)^\vee \longrightarrow R^j(f|U)_*(\mathcal{F} \otimes F^n E)$$

for $j = d + \alpha + 1$. Then by duality and the assumption on $\operatorname{pcd}(E, f)$, we can find n_2 such that

$$R^j f_*(\mathcal{F}(m_0 D) \otimes F^n E) = 0$$

for $n \geq n_2$, $j > \alpha$. If we set $n_0 = \max(n_1, n_2)$, we have for all $n \geq n_0$,

$$R^j(f|U)_*(F^n E \otimes \mathcal{F}) = 0 \quad \text{for } j > \alpha. \quad \square$$

Applying Lemma 1.2 and Lemma 3.2 to $\mathcal{F} = {\Omega^i}_{U/Y}$, descending induction on n yields the following result.

Theorem 3.3. *Let $f: X \to Y$ be a k-morphism which is smooth and proper. Let $D \subset X$ be a divisor such that the normal bundle $N_{D/X}$ is globally generated. Let $U := X \setminus D$. Assume that $f|U: U \to Y$ is smooth, purely of relative dimension $d < p$ and liftable to $W_2(k)$. If E is a vector bundle on X with* $\mathrm{pcd}(E, f) \leq \alpha$, *we have*

$$R^j (f|U)_* (E \otimes \Omega^i{}_{U/Y}) = 0 \quad \textit{for } i+j > d+\alpha.$$

As a consequence, we obtain the following generalization of a theorem of Bauer–Kosarew [BK2, 6.3].

Corollary 3.4. *Let $f: X \to Y$ be as in Theorem* 3.3. *If L is a line bundle on X which is l-ample relative to f, then we have*

$$R^j (f|U)_* (L \otimes \Omega^i{}_{U/Y}) = 0 \quad \textit{for } i+j > d+l.$$

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