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# Extension of smooth CR mappings between non-essentially finite hypersurfaces in $\mathbf{C}^3$

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## 0. Introduction

Let $M$ be a real analytic hypersurface in $\mathbf{C}^3$ containing 0 and let $M'$ be the algebraic hypersurface in $\mathbf{C}^3$ defined by

$$\operatorname{Im} w' = |z_1'|^2 + \operatorname{Re} w' |z_2'|^2, \quad (z_1', z_2', w') \in \mathbf{C}^3. \tag{0.1}$$

For any $b'<0$, the function $(z', w') \mapsto 1/(w'-ib')$ is holomorphic in $\mathbf{C}^3 \setminus \{w'=ib'\} \supseteq M'$; therefore its restriction to $M'$ is a CR function which does not extend holomorphically around $(0,0,ib')$. A classical argument using Baire's category theorem (see [HT, p. 125]) guarantees the existence of a CR function on $M'$ which does not extend to a full neighborhood of $0 \in \mathbf{C}^3$. In contrast, for CR mappings we have the following result.

**Theorem 1.** *If $h: M \to M'$ is a smooth CR local diffeomorphism at 0 with $h(0)=0$, then $h$ extends to a holomorphic mapping in a full neighborhood of 0 in $\mathbf{C}^3$.*

As we shall see in Corollary 1.2, if $h$ satisfies the hypothesis of the theorem (more generally if $h$ is of finite multiplicity) then $M$ is of finite type. After Trepreau's theorem we know that any CR function on $M$ extends holomorphically to *one* side of $M$; therefore Theorem 1 is equivalent to a reflection principle (cf. Baouendi and Rothschild [BR3]). Because we do not assume $M$ to be algebraic, and because $M'$ is not essentially finite, Theorem 1 does not follow from the recent results of Baouendi, Huang and Rothschild [BHR] nor from those of Baouendi and Rothschild [BR2]. Notice that $M'$ is holomorphically non-degenerate in the sense of Stanton [S].

Theorem 1 may be generalized as follows.

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**Theorem 2.** *Suppose $h: M \to M'$ is a smooth CR mapping of finite multiplicity at 0 with $h(0)=0$ and the Levi form of $M$ has a nonzero eigenvalue at 0. Then $h$ extends to a holomorphic mapping in a full neighborhood of 0 in $\mathbf{C}^3$.*

*Remark.* We do not know whether the conclusion still holds without the hypothesis of non-degeneracy of the Levi form of $M$.

In order to relax the finite multiplicity condition, let us be precise with some notions. After a local holomorphic change of coordinates, we may assume that $M$ is locally given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w) \tag{0.2}$$

where $\varphi$ is a real analytic function in a neighborhood of $0 \in \mathbf{C}^2 \times \mathbf{R}$ satisfying the identity $\varphi(z, 0, w) \equiv 0$. Such coordinates $(z, w)$ are called *normal coordinates.* Let $h: M \to M'$ be a smooth CR mapping such that $h(0)=0$; we will denote by $(F_1, F_2, G)$ the holomorphic formal Taylor series of $h$ at 0 (cf. [BR1]) and write

$$F_1(z,w) = \sum_{j=0}^{\infty} F_{1j}(z) w^j, \quad F_2(z,w) = \sum_{j=0}^{\infty} F_{2j}(z) w^j.$$

Denote by $l$ the least element of $\mathbf{N} \cup \{\infty\}$ such that $F_{1l}$ or $F_{2l}$ is not constant.

*Definition.* (Cf. [M2]) We say that $h$ is *tangentially finite at* 0 if $l$ is finite and

$$\dim_{\mathbf{C}} \mathbf{C}[[z]]/(F_{1l}(z) - F_{1l}(0), F_{2l}(z) - F_{2l}(0)) < \infty. \tag{0.3}$$

It is called *transversally submersive* [resp. *transversally non-flat*] *at* 0 if

$$\frac{\partial G}{\partial w}(0) \neq 0 \quad [\text{resp. } G \neq 0]. \tag{0.4}$$

Note that because $h(0)=0$, $h$ is tangentially finite with $l=0$ if, and only if, $h$ is of finite multiplicity. The above definition as well as the integer $l$ are independent of the choice of normal coordinates if $h$ is transversally submersive (see [M3]).

We may now state the two other extension results we prove in this paper.

**Theorem 3.** *Let $M$ be a real analytic hypersurface in $\mathbf{C}^3$ non-flat at 0 and let $M'$ be defined by*

$$\operatorname{Im} w' = (\operatorname{Re} w')^{m'} (|z_1'|^2 + \operatorname{Re} w' |z_2'|^2), \quad (z_1', z_2', w') \in \mathbf{C}^3, \ m' \in \mathbf{N}^*. \tag{0.5}$$

*Suppose $h: M \to M'$ is a smooth tangentially finite and transversally non-flat CR mapping with $h(0)=0$. If $h$ extends holomorphically to one side of $M$ near 0, then $h$ extends to a holomorphic mapping in a full neighborhood of 0 in $\mathbf{C}^3$.*

When $m'=0$, the corresponding statement requires more hypotheses.

**Theorem 4.** *Let $M$ be a non-flat real analytic hypersurface in $\mathbf{C}^3$ of infinite type and let $M'$ be defined by* (0.1). *Suppose $h: M \to M'$ is a smooth tangentially finite and transversally submersive CR mapping with $h(0)=0$. If $h$ extends holomorphically to one side of $M$ near 0, then it extends holomorphically to a full neighborhood of* 0 *in $\mathbf{C}^3$.*

*Remark.* Theorems 1–4 have obvious generalizations to the case where $M$ is a real analytic hypersurface in $\mathbf{C}^{p+q+1}$ and $M'$ is the algebraic hypersurface in $\mathbf{C}^{p+q+1}$ defined by

$$\operatorname{Im} w' = (\operatorname{Re} w')^{m'}(|z_1'|^2+\ldots+|z_p'|^2+\operatorname{Re} w'(|z_{p+1}'|^2+\ldots+|z_{p+q}'|^2)). \tag{0.6}$$

Some of the results presented here have been announced in Meylan [M1].

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## 1. The basic relation and its consequences

The following notation will be used throughout:

$$M = \{(z,w) \in \mathbf{C}^3 \mid \operatorname{Im} w = \varphi(z,\bar z, \operatorname{Re} w)\},$$

with $\varphi$ real analytic at 0; $(F_1, F_2, G)$ denotes the holomorphic formal Taylor series of $h$ at 0, with $h(z, s+i\varphi(z,\bar z,s)) = (f_1, f_2, g)(z,\bar z,s)$.

**Lemma 1.1.** *Let $h: M \to M'$ be a smooth CR mapping with $M'$ defined by* (0.1). *Then*

$$\varphi(z,\bar z,0)\left(\frac{\partial G}{\partial w}(0)+O(|z|)\right) = |F_1(z,0)|^2, \quad z \in \mathbf{C}^2. \tag{1.1}$$

*Proof.* On the formal power series level, we have for $(z,w)\in M$:

$$G(z,w)-\overline{G}(\bar z,\bar w) = 2iF_1(z,w)\overline{F}_1(\bar z,\bar w)+i(G(z,w)+\overline{G}(\bar z,\bar w))F_2(z,w)\overline{F}_2(\bar z,\bar w). \tag{1.2}$$

Replacing $w$ by $i\varphi(z,\bar z,0)$ in (1.2), we get an identity for formal power series in $z, \bar z$; putting $\bar z=0$ and using $\varphi(z,0,0)\equiv 0$, we get $G(z,0)\equiv 0$, that is $G(z,w)=w\widetilde{G}(z,w)$ for some formal power series $\widetilde{G}$. Replacing now $w$ by $i\varphi(z,\bar z,0)$ in (1.2), we obtain

$$\begin{aligned}\varphi(z,\bar z,0)(\operatorname{Re}\widetilde{G}(0)+O(|z|)) &= |F_1(z,i\varphi(z,\bar z,0))|^2 = |F_1(z,0)+i\varphi(z,\bar z,0)R(z,\bar z)|^2\\ &= |F_1(z,0)|^2+\varphi(z,\bar z,0)O(|z|)\end{aligned}$$

where $R$ is some formal power series. The lemma will be proved as soon as we know that $\widetilde{G}(0)$ is real. This general fact is quickly verified below.

Because $G$ is the Taylor series of the third component of $h$ at 0, the Taylor series of $g$ at 0 is $G(z, s+i\varphi(z,\bar{z},s))$ and hence

$$\frac{\partial g}{\partial s}(0)=\frac{\partial G}{\partial w}(0). \tag{1.3}$$

When we differentiate the relation

$$g-\bar{g}=2if_1\bar{f}_1+i(g+\bar{g})f_2\bar{f}_2$$

with respect to $s$, we get $(\partial/\partial s)(g-\bar{g})(0)=0$ and hence $(\partial g/\partial s)(0)\in\mathbf{R}$. □

**Corollary 1.2.** *Let $h: M\to M'$ be a smooth CR mapping of finite multiplicity at* 0 *with $M'$ defined by* (0.1). *Then $M$ is of finite type at* 0.

*Proof.* Because $M'$ is of finite type, this result is a consequence of [BR2, p. 486], but we give a direct short proof in our case.

Suppose $M$ is not of finite type at 0. Then $\varphi(z,\bar{z},0)\equiv 0$ and hence, after (1.1), $F_1(z,0)\equiv 0$. Therefore $\dim \mathbf{C}[[z]]/(F_1(z,0),F_2(z,0))=\infty$ which means that $h$ is not of finite multiplicity. □

**Corollary 1.3.** *Suppose $M$, $M'$ and $h$ satisfy the hypotheses of Theorem* 2. *Then $\nabla_z F_1(0)\neq 0$ and the Levi form of $M$ at* 0 *has a unique nonzero eigenvalue whose sign is that of* $(\partial g/\partial s)(0)$.

*Proof.* We may find a linear change of coordinates in $\mathbf{C}^2$ such that

$$\varphi(z,\bar{z},0)=\lambda_1|z_1|^2+\lambda_2|z_2|^2+O(|z|^3), \quad z\to 0\in\mathbf{C}^2$$

with some $\lambda_1,\lambda_2\in\mathbf{R}$. If we also expand $F_1(z,0)$:

$$F_1(z,0)=a_1z_1+a_2z_2+O(|z|^2)$$

we get after (1.1)

$$(\lambda_1|z_1|^2+\lambda_2|z_2|^2)\frac{\partial G}{\partial w}(0)=|a_1|^2|z_1|^2+a_1\bar{a}_2z_1\bar{z}_2+\bar{a}_1a_2\bar{z}_1z_2+|a_2|^2|z_2|^2+O(|z|^3).$$

Therefore

$$\lambda_1\frac{\partial G}{\partial w}(0)=|a_1|^2,\ \lambda_2\frac{\partial G}{\partial w}(0)=|a_2|^2, \quad \text{if } \bar{a}_1a_2=0. \tag{1.4}$$

Because $M'$ is of finite type and $h$ has finite multiplicity, we know that $(\partial G/\partial w)(0)$ is nonzero (see [BR2, Theorem 1]). Hence (1.4) implies $\lambda_1\lambda_2=0$. Because $(\lambda_1,\lambda_2)\neq 0$ we get $(a_1,a_2)\neq 0$ that is $\nabla_z F_1(0)\neq 0$. The last assertion follows from (1.3). □

**Corollary 1.4.** *Suppose $M$, $M'$ and $h$ satisfy the hypothesis of Theorem* 1. *Then the Levi form of $M$ at* 0 *is nonzero. As a consequence, Theorem* 2 *implies Theorem* 1.

*Proof.* The local diffeomorphism assumption implies $\nabla_z F_1(0) \neq 0$ and hence $F_1(z,0)$ has a nonzero linear term. After (1.1), it follows that $\varphi(z,\bar{z},0)$ has a nonzero quadratic term. □

*Remark* 1.5. Because the rank of the Levi form is a CR invariant and since $M$ and $M'$ are CR equivalent, it follows from general theory that the Levi form of $M$ has one zero and one nonzero eigenvalue.

## 2. First properties of the tangential components

**Proposition 2.1.** *Suppose $M$, $M'$ and $h$ satisfy the hypotheses of Theorem* 2. *Then $h$ extends to a neighborhood of $0 \in \mathbf{C}^3$ intersected with the following component of $\mathbf{C}^3 \setminus M$:*

$$\{(\operatorname{Im} w - \varphi(z,\bar{z},\operatorname{Re} w))(\partial g/\partial s)(0) > 0\}. \tag{2.1}$$

*Proof.* From general theory [H, p. 51], if $\lambda$ is a nonzero eigenvalue of the Levi form of $M$ at 0, then every CR-function on $M$ extends to a neighborhood of $0 \in \mathbf{C}^3$ intersected with the following component of $\mathbf{C}^3 \setminus M$: $\{(\operatorname{Im} w - \varphi(z,\bar{z},\operatorname{Re} w))\lambda > 0\}$. Because of Corollary 1.3, this component is identical with (2.1). □

*Remark* 2.2. Because $\varphi$ is analytic, the map

$$(z, s+it) \longmapsto (z, s+it+i\varphi(z,\bar{z},s+it)) \tag{2.2}$$

is well defined in a neighborhood of $0 \in \mathbf{C}^3$. It is a local diffeomorphism at 0 since $(\partial\varphi/\partial s)(0) = 0$ in normal coordinates; the side $\{t > 0\}$ is mapped into the following side $\{\operatorname{Im} w - \varphi(z,\bar{z},\operatorname{Re} w) > 0\}$ of $M$.

*Definition* 2.3. We will say that a germ $u: (\mathbf{C}^2 \times \mathbf{R}, 0) \to (\mathbf{C}, 0)$ of a smooth function *extends up* [resp. *down*] if there exist $\varepsilon > 0$ and a smooth function

$$U: \{(z, s+it) \in \mathbf{C}^3 \mid |z| < \varepsilon,\ |s+it| < \varepsilon\} \longrightarrow \mathbf{C}$$

holomorphic with respect to $s+it$ for $t > 0$ [resp. $t < 0$] which extends $u$.

Recall the following standard notation. Write

$$L_j = \frac{\partial}{\partial \bar{z}_j} - i \frac{\varphi_{\bar{z}_j}}{1 + i\varphi_s} \frac{\partial}{\partial s}, \quad j = 1, 2, \tag{2.3}$$

for the vector fields in $\mathbf{C}^2\times\mathbf{R}$ corresponding to the antiholomorphic tangent vector fields to $M$ and put

$$D=\det(L_j\bar{f}_k)_{1\le j,k\le 2}. \tag{2.4}$$

**Proposition 2.4.** *Let $h\colon M\to M'$ be a smooth CR mapping where $M'$ is defined by* (0.1) *with components $f_1$, $f_2$ and $g$ extending up. Then*

$$D\frac{f_1}{1+if_2\bar{f}_2}\ \textit{and}\ D^5\frac{\bar{g}f_2}{(1+if_2\bar{f}_2)^2}\ \textit{extend down.}$$

*Proof.* Since $h(M)\subseteq M'$, we have

$$\bar{g}=g-\frac{2if_1\bar{f}_1}{1+if_2\bar{f}_2}-g\frac{2if_2\bar{f}_2}{1+if_2\bar{f}_2}. \tag{2.5}$$

Applying $L_1$ and $L_2$ to both sides of (2.5), we obtain

$$L_j\bar{g}=\frac{-2if_1}{1+if_2\bar{f}_2}L_j\bar{f}_1-\left(\frac{2f_1f_2\bar{f}_1}{(1+if_2\bar{f}_2)^2}+g\frac{2if_2}{(1+if_2\bar{f}_2)^2}\right)L_j\bar{f}_2,\quad j=1,2. \tag{2.6}$$

Hence

$$D\frac{f_1}{1+if_2\bar{f}_2}=\frac{i}{2}(L_1\bar{g}L_2\bar{f}_2-L_2\bar{g}L_1\bar{f}_2)=:k_1. \tag{2.7}$$

Because $f_1$, $f_2$ and $g$ extend up, we know that $k_1$ is a smooth function which extends down. This proves the first part.

The same argument also shows that

$$D\left(\frac{-2f_1f_2\bar{f}_1}{(1+if_2\bar{f}_2)^2}+g\frac{-2if_2}{(1+if_2\bar{f}_2)^2}\right) \tag{2.8}$$

extends down. Multiplying (2.7) by $D$ and applying $L_j$ yields

$$-iD^2\frac{f_1f_2}{(1+if_2\bar{f}_2)^2}L_j\bar{f}_2=-2DL_jD\frac{f_1}{1+if_2\bar{f}_2}+L_j(Dk_1);$$

hence $D^3(f_1f_2/(1+if_2\bar{f}_2)^2)$ extends down. This information and (2.8) show that

$$D^3\frac{gf_2}{(1+if_2\bar{f}_2)^2}\quad\text{and}\quad D^3\frac{f_1}{(1+if_2\bar{f}_2)^2}=D^3\frac{f_1(1+if_2\bar{f}_2)}{(1+if_2\bar{f}_2)^2}-D^3\frac{if_1f_2\bar{f}_2}{(1+if_2\bar{f}_2)^2} \tag{2.9}$$

extend down. Applying $L_j$ to (2.9), we get, again by the same type of arguments, that the functions

$$D^5\frac{f_1f_2}{(1+if_2\bar{f}_2)^3}\quad\text{and}\quad D^5\frac{gf_2^2}{(1+if_2\bar{f}_2)^3}$$

extend down. This and (2.5) multiplied by $f_2/(1+if_2\bar{f}_2)^2$ finally proves that the function $D^5\bar{g}f_2/(1+if_2\bar{f}_2)^2$ extends down. □

## 3. Properties of the source hypersurface

**Proposition 3.1.** *Let $h$, $M$ and $M'$ satisfy the hypotheses of Theorem 2. Then $M$ contains a one-dimensional complex submanifold.*

*Proof.* As $h$ is of finite multiplicity and $M'$ is not essentially finite, it follows from [BR2, Theorem 4] that $M$ is not essentially finite. Therefore, by Proposition 4.1 in [BJT], the hypersurface $M$ contains a holomorphic one-dimensional subvariety passing through 0. The curve selection lemma (cf. [L]), guarantees the existence of a non-constant holomorphic curve $c\colon U\to M$, defined in a neighborhood $U$ of $0\in\mathbf{C}$, such that $c(0)=0$. Because we work in normal coordinates we have

$$\frac{\partial^k\varphi}{\partial\bar z_j^k}(0)=\frac{\partial^l\varphi}{\partial z_j^l}(0)=0,\quad \text{for all } k,l\in\mathbf{N},\ j=1,2; \tag{3.1}$$

the chain-rule applied to

$$c_3-\bar c_3=2i\varphi(c_1,\bar c_1,c_2,\bar c_2,(c_3+\bar c_3)/2)$$

shows that $(\partial^k c_3/\partial t^k)(0)=0$, for all $k$, hence $c_3=0$. Because $c$ is holomorphic and $h$ is CR, $h\circ c$ is a holomorphic curve in $M'$ passing through 0. The above argument using (3.1) is also valid for $h\circ c$ and gives $g\circ c=0$, hence $f_1\circ c=0$. After a linear change of coordinates, we may suppose $(\partial F_1/\partial z_1)(0)\neq 0$ (cf. Corollary 1.3) and reparametrizing $c$ we may write

$$c(t)=(c_1(t),t^m,0),\quad \text{with } m\in\mathbf{N}^*. \tag{3.2}$$

On the formal power series level, we have $F_1(c_1(t),t^m,0)\equiv 0$; the formal implicit function theorem applies and gives $c_1(t)=C_1(t^m)$, for some formal power series $C_1$. Because $c_1$ is convergent, $C_1$ is convergent too. The image of the complex curve

$$s\longmapsto (C_1(s),s,0) \tag{3.3}$$

is a complex submanifold contained in $M$. □

**Corollary 3.2.** *Let $h$, $M$ and $M'$ satisfy the hypotheses of Theorem 2. Then there exists a local holomorphic change of coordinates at 0 such that $\varphi$ given by (0.2) satisfies*

$$\varphi(z,\bar z,0)=|z_1|^2\widetilde\varphi(z,\bar z), \tag{3.4}$$

*where $\widetilde{\varphi}$ is real analytic at 0 and $\widetilde{\varphi}(0)\neq 0$ has the sign of $(\partial g/\partial s)(0)$.*

*Proof.* Without loss of generality, we may suppose that $M$ contains the curve (3.3). Let us make the following holomorphic change of coordinates $Z_1=z_1-C_1(z_2)$, $Z_2=z_2$, $W=w$. Then the complex line $\{(0,Z_2,0)\,|\,Z_2\in\mathbf{C}\}$ lies in $M$ and therefore

$$\varphi(0,Z_2,0,\bar{Z}_2,0)\equiv 0.$$

After (1.1), we get $F_1(0,Z_2,0)\equiv 0$ and hence $F_1(Z_1,Z_2,0)=Z_1\widetilde{F}_1(Z_1,Z_2)$ for some power series $\widetilde{F}_1$. Introducing this into (1.1) and using $(\partial G/\partial w)(0)\neq 0$ we obtain (3.4). The question of sign follows from Corollary 1.3 because $\widetilde{\varphi}(0)$ is the nonzero eigenvalue of the Levi form of $M$ at 0. □

## 4. Estimates for the transversal component

In this paragraph we assume that $h, M$ and $M'$ satisfy the hypotheses of Theorem 2. We also assume (without loss of generality) that

$$\frac{\partial g}{\partial s}(0)>0,\quad \varphi(z,\bar{z},0)=|z_1|^2\widetilde{\varphi}(z,\bar{z})\quad\text{and}\quad \widetilde{\varphi}(0)>0 \tag{4.1}$$

(cf. Corollary 3.2). After Remark 2.2 and Proposition 2.1, we know that $h$ has a $C^\infty$ extension $\mathcal{H}$ to a neighborhood $\Omega$ of $0\in\mathbf{C}^3$ which is holomorphic in $\Omega\cap\{\operatorname{Im} w>\varphi(z,\bar{z},\operatorname{Re} w)\}$. Denote by $\mathcal{G}$ the third or transversal component of $\mathcal{H}$ and let

$$g(z,\bar{z},s+it)=\mathcal{G}(z,\bar{z},s+it+i\varphi(z,\bar{z},s+it)).$$

**Proposition 4.1.** *Let $h$, $M$, $M'$, $\mathcal{G}$ and $g$ be as above. Then the following estimates hold*(2):

$$\mathcal{G}(z,w)=o(|z_1|^2)+w\frac{\partial\mathcal{G}}{\partial w}(0)+o(|w|),\quad (z,w)\to 0, \tag{4.2}$$

$$|g(z,\bar{z},s+it)|\geq\frac{1}{8}\frac{\partial g}{\partial s}(0)|s+it|\quad \textit{for } t\geq 0 \textit{ and } |(z,s+it)| \textit{ small.} \tag{4.3}$$

*Proof.* For any $\alpha,\beta\in\mathbf{N}^2$ and any $j,k\in\mathbf{N}$ with $\beta\neq 0$ or $k\neq 0$, we have by continuity $\partial_z^\alpha\partial_{\bar{z}}^\beta\partial_w^j\partial_{\bar{w}}^k\mathcal{G}|_M=0$. Therefore, using $(0,z_2,0)\in M$ for $z_2$ small, we see that the functions

$$z_2\longmapsto\partial_z^\alpha\mathcal{G}(0,z_2,0) \tag{4.4}$$

(2) As usual, $u(z,w)=o(v(z,w))$ means that $u(z,w)/v(z,w)\to 0$ , as $(z,w)\to 0$.

are holomorphic. As already mentioned, at the formal power series level, we may factorize $w$ in $G$ and obtain $\partial_z^\alpha \partial_{\bar z}^\beta \mathcal{G}(0)=0$ for all $\alpha,\beta\in\mathbf{N}^2$. Together with (4.4), this implies

$$\partial_{\bar z}^\alpha \mathcal{G}(0,z_2,0)\equiv 0. \tag{4.5}$$

Taylor's theorem for $z_1\mapsto\mathcal{G}(z_1,z_2,0)$ at 0 shows that for $m\geq 3$, $m\in\mathbf{N}$,

$$\begin{aligned}\mathcal{G}(z_1,z_2,0)&=\mathcal{G}(0,z_2,0)+z_1\frac{\partial\mathcal{G}}{\partial z_1}(0,z_2,0)+\bar z_1\frac{\partial\mathcal{G}}{\partial\bar z_1}(0,z_2,0)+...+O(|z_1|^m)\\&=o(|z_1|^2),\quad z\to 0.\end{aligned}$$

For the same reasons, $(\partial\mathcal{G}/\partial\bar w)(z_1,z_2,0)=o(|z_1|^2)$, $z\to 0$. Finally, Taylor's theorem for $\mathcal{G}$ at $(z_1,z_2,0)$ yields

$$\begin{aligned}\mathcal{G}(z_1,z_2,w)&=o(|z_1|^2)+w\frac{\partial\mathcal{G}}{\partial w}(z_1,z_2,0)+\bar w o(|z_1|^2)+O(|w|^2)\\&=o(|z_1|^2)+w\frac{\partial\mathcal{G}}{\partial w}(0)+o(|w|),\quad (z,w)\to 0.\end{aligned}$$

This proves (4.2). Because $|w(\partial\mathcal{G}/\partial w)(0)+o(|w|)|\geq\frac{1}{2}|w|(\partial\mathcal{G}/\partial w)(0)$, for $|(z,w)|$ small, to prove (4.3), it is enough to verify

$$|s+it+i\varphi(z,\bar z,s+it)|\geq\tfrac{1}{4}(|s+it|+|z_1|^2\widetilde\varphi(0))\quad\text{for } t\geq 0. \tag{4.6}$$

Using (4.1) and $(\partial\varphi/\partial s)(0)=0$, we get after using Taylor's theorem:

$$\varphi(z,\bar z,s+it)=\varphi(z,\bar z,0)+o(|s+it|)=|z_1|^2(\widetilde\varphi(0)+o(1))+o(|s+it|).$$

Hence

$$|s+it+i\varphi(z,\bar z,s+it)|=\big|(s+it)(1+o(1))+i|z_1|^2(\widetilde\varphi(0)+o(1))\big|,$$

as $(z,s+it)\to 0$ and $t\geq 0$. The following elementary inequality

$$|s+it+ir|\geq\frac{1}{\sqrt2}(|s+it|+r)\quad\text{for } r,t\geq 0$$

may be perturbed to give

$$|(s+it)(a+ib)+ir(c+id)|\geq\tfrac{1}{4}(|s+it|+r)\quad\text{for } r,t\geq 0$$

where $a+ib$ and $c+id$ are sufficiently close to 1. The proof of (4.6) follows. □

**Corollary 4.2.** *Under the hypotheses of Theorem 2 with the choice (4.1), the function $D^5f_2/(1+i|f_2|^2)^2$ extends down.*

*Proof.* After Proposition 2.1 and Remark 2.2, we know that $f_1$, $f_2$ and $g$ extend up. Therefore Proposition 2.4 applies and shows that $D^5\bar g f_2/(1+i|f_2|^2)^2$ extends down. Of course, $D^5f_2/(1+i|f_2|^2)^2$ is $C^\infty$ at $0\in\mathbf{C}^2\times\mathbf{R}$. Because $\bar g$ extends down and is bounded below by (4.3), we may apply Lemma 4.5 of [BR3] to conclude that $D^5f_2/(1+i|f_2|^2)^2=(1/\bar g)D^5\bar g f_2/(1+i|f_2|^2)^2$ extends down. □

## 5. Proof of Theorem 2

Let us assume that $h: M \to M'$ satisfy hypotheses of Theorem 2. The finite multiplicity assumption guarantees (cf. [BR1, (3.18)]) the existence of a multi-index $\gamma \in \mathbf{N}^2$ such that

$$L^\gamma D(0) \neq 0 \tag{5.1}$$

where $L$ and $D$ are defined by (2.3) and (2.4). If we take care to choose $\gamma$ minimal with respect to the following lexicographic order on $\mathbf{N}^2$:

$$\alpha \prec \beta \quad \Longleftrightarrow \quad \alpha_1 + \alpha_2 < \beta_1 + \beta_2 \text{ or } \alpha_1 + \alpha_2 = \beta_1 + \beta_2 \text{ and } \alpha_1 < \beta_1 \tag{5.2}$$

then we also have

$$L^{5\gamma} D^5(0) \neq 0. \tag{5.3}$$

Indeed, the order (5.2) satisfies

$$\alpha, \beta \in \mathbf{N}^2, \ \alpha + \beta = 2\gamma \quad \Longrightarrow \quad \alpha \preceq \gamma \text{ or } \beta \preceq \gamma;$$

and Leibniz' formula gives

$$L^{2\gamma} D^2(0) = \sum_{\alpha+\beta=2\gamma} \binom{2\gamma}{\alpha} L^\alpha D(0) L^\beta D(0) = \binom{2\gamma}{\gamma} (L^\gamma D)^2(0) \neq 0.$$

Using now

$$\alpha, \beta \in \mathbf{N}^2, \ \alpha + \beta \prec 2\gamma \quad \Longrightarrow \quad \alpha \prec \gamma \text{ or } \beta \prec \gamma,$$

we see that

$$L^\delta D^2(0) = 0 \quad \text{for } \delta \prec 2\gamma.$$

By iteration, we get (5.3).

Go back to Proposition 2.4 and rewrite (2.7) as

$$D f_1 = (1 + i f_2 \bar{f}_2) k_1. \tag{5.4}$$

Applying $L^\alpha$, for $\alpha \in \mathbf{N}^2$, to both sides gives

$$(L^\alpha D) f_1 = \sum_{\alpha' + \alpha'' = \alpha} \binom{\alpha}{\alpha'} (L^{\alpha'}(1 + i f_2 \bar{f}_2))(L^{\alpha''} k_1)$$

and hence, at 0,

$$0=\sum_{\alpha'+\alpha''=\alpha}\binom{\alpha}{\alpha'}L^{\alpha'}(1+if_2\bar{f}_2)(0)L^{\alpha''}k_1(0).$$

By induction on $\alpha$ and using $(1+if_2\bar{f}_2)(0)\neq 0$ we get $L^{\alpha}k_1(0)=0$ for all $\alpha\in\mathbf{N}^2$. This implies

$$L^{\alpha}(\bar{f}_2k_1)(0)=0 \quad \text{for all } \alpha\in\mathbf{N}^2. \tag{5.5}$$

Application of $L^{\gamma}$ to both sides of (5.4) may also be written as

$$(L^{\gamma}D)f_1=L^{\gamma}k_1+if_2L^{\gamma}(\bar{f}_2k_1)$$

That is, after (5.1)

$$f_1+f_2u_1+v_1=0 \tag{5.6}$$

with $u_1=-iL^{\gamma}(\bar{f}_2k_1)/L^{\gamma}D$ , $v_1=-L^{\gamma}k_1/L^{\gamma}D$ extending down and from (5.5) we have $u_1(0)=0$.

Using $L^{5\gamma}$, Corollary 4.2 and (5.3), we prove in the same manner that $f_2$ satisfies an algebraic relation

$$f_2+f_2u_2+f_2^2u_3+v_2=0 \tag{5.7}$$

with $u_2$, $u_3$, $v_2$ extending down and $u_2(0)=0$. The implicit function theorem enables us to solve the system (5.6)–(5.7) and show that $f_1$ and $f_2$ extend down. From (2.5), we deduce that $g$ also extends down. From Lemma 2.2 of [BJT] it follows that $h$ has a holomorphic extension near 0 in $\mathbf{C}^3$. The proof of Theorem 2 is complete.

## 6. Proofs of Theorems 3 and 4

**Lemma 6.1.** *Let $\omega\in\mathbf{C}\{x_1,x_2,y\}$ be defined by*

$$\frac{y-\omega}{2i}=\left(\frac{y+\omega}{2}\right)^{m'}x_1+\left(\frac{y+\omega}{2}\right)^{m'+1}x_2,\quad m'\geq 1. \tag{6.1}$$

*Then*

$$\omega(x_1,x_2,y)=y+y^{m'}\sum_{\alpha\in\mathbf{N}^2\setminus\{0\}}A_{\alpha}(y)x_1^{\alpha_1}x_2^{\alpha_2},\quad A_{\alpha}\in\mathbf{C}\{y\},$$

*where* $A_{1,0}=-2i$ *and* $A_{0,1}=-2iy$.

*Proof.* After the implicit function theorem, we may write

$$\omega=\sum_{\alpha\in\mathbf{N}^2,\, j\in\mathbf{N}} B_{\alpha,j}x_1^{\alpha_1}x_2^{\alpha_2}y^j.$$

For $x=0$ in (6.1), we have $\sum_j B_{0,j}y^j=y$, and hence

$$\omega=y+\sum_{|\alpha|>0} B_\alpha(y)x_1^{\alpha_1}x_2^{\alpha_2}, \quad \text{with } B_\alpha(y)=\sum_j B_{\alpha,j}y^j.$$

Let us insert this expression into (6.1); we get

(6.2)

$$-\frac{1}{2i}\sum_{|\alpha|>0} B_\alpha(y)x^\alpha=\Big(y+\frac{1}{2}\sum_{|\alpha|>0} B_\alpha(y)x^\alpha\Big)^{m'}x_1+\Big(y+\frac{1}{2}\sum_{|\alpha|>0} B_\alpha(y)x^\alpha\Big)^{m'+1}x_2.$$

When $y=0$, this implies that $B_\alpha(0)=0$ for all $\alpha$. Because the right hand side of (6.2) has a factor $y^{m'}$, each $B_\alpha$ is divisible by $y^{m'}$. The expressions for $A_{1,0}$ and $A_{0,1}$ follow directly. □

*Proof of Theorem* 3. Without loss of generality, we may suppose that $f_1$, $f_2$ and $g$ extend up. Because $h(M)\subseteq M'$, Lemma 6.1 yields

$$\bar{g}=g+g^{m'}\sum_{\alpha\in\mathbf{N}^2\setminus\{0\}} A_\alpha(g)|f_1|^{2\alpha_1}|f_2|^{2\alpha_2}. \tag{6.3}$$

Applying $L_j$, for $j=1,2$, gives

$$\begin{aligned}L_j\bar{g}&=g^{m'}\sum_{|\alpha|>0} A_\alpha(g)(\alpha_1 f_1^{\alpha_1}f_2^{\alpha_2}\bar{f}_1^{\alpha_1-1}\bar{f}_2^{\alpha_2}L_j\bar{f}_1+\alpha_2 f_1^{\alpha_1}f_2^{\alpha_2}\bar{f}_1^{\alpha_1}\bar{f}_2^{\alpha_2-1}L_j\bar{f}_2)\\ &=g^{m'}(-2if_1+S_1(f_1,f_2,g,\bar{f}_1,\bar{f}_2))L_j\bar{f}_1+g^{m'}(-2igf_2+S_2(f_1,f_2,g,\bar{f}_1,\bar{f}_2))L_j\bar{f}_2\end{aligned}$$

with $S_1$, $S_2$ being convergent power series such that

$$S_j(0,0,g,\bar{f}_1,\bar{f}_2)=\frac{\partial S_j}{\partial f_1}(0,0,g,\bar{f}_1,\bar{f}_2)=\frac{\partial S_j}{\partial f_2}(0,0,g,\bar{f}_1,\bar{f}_2)=0.$$

Hence, with $D=L_1\bar{f}_1L_2\bar{f}_2-L_2\bar{f}_1L_1\bar{f}_2$:

$$\begin{aligned}Dg^{m'}(-2if_1+S_1(f_1,f_2,g,\bar{f}_1,\bar{f}_2))&=k_1,\\ Dg^{m'}(-2igf_2+S_2(f_1,f_2,g,\bar{f}_1,\bar{f}_2))&=k_2,\end{aligned} \tag{6.4}$$

where $k_1$ and $k_2$ are functions which extend down.

Under the non-flatness condition (0.4), we may use Lemma 3.12 of [M2], a detailed proof of which is given in [M3]. Therefore $g$ is divisible by $s^k$ for some $k$:

$$g(z,\bar{z},s)=s^k g_1(z,\bar{z},s), \quad \text{with } g_1(0)\neq 0.$$

Division by $\bar{g}$ in the class of functions which extend down is then possible because Corollary 4.8 of [BR3] applies. Conjugation of (6.3) gives

$$g=\bar{g}\Big(1+\bar{g}^{m'-1}\sum_{|\alpha|>0}\bar{A}_\alpha(\bar{g})|f_1|^{\alpha_1}|f_2|^{\alpha_2}\Big) \tag{6.5}$$

and hence $g^{m'}=\bar{g}^{m'}(1+...)$ where the dots do not contain constant terms. From (6.4) we get

$$\begin{aligned} D(f_1+T_1(f_1,f_2,g,\bar{f}_1,\bar{f}_2))&=v_1,\\ D(gf_2+T_2(f_1,f_2,g,\bar{f}_1,\bar{f}_2))&=v_2, \end{aligned} \tag{6.6}$$

where $T_1$, $T_2$ have no linear terms in $f$ and $v_1$, $v_2$ extend down.

From the tangential finiteness assumption, Proposition 3.1 of [M2] asserts that

$$L^\beta D(z,\bar{z},s)=s^{2l}D_\beta(z,\bar{z},s) \quad \text{for all } \beta\in\mathbf{N}^2$$

and there exists $\alpha$ such that $D_\alpha(0)\neq 0$.

Applying $L^\alpha$ to both sides of (6.6), we get

$$\begin{aligned} s^{2l}D_\alpha f_1+s^{2l}R_1(f_1,f_2,u)&=L^\alpha v_1,\\ s^{2l}D_\alpha gf_2+s^{2l}R_2(f_1,f_2,u)&=L^\alpha v_2, \end{aligned}$$

where $u$ is a finite set of functions extending down and $R_1$, $R_2$ do not contain linear terms in $f$. After dividing by $s^{2l}$ and use of the implicit function theorem we obtain that $f_1$ and $gf_2$ extend down. Hence with the same trick as above, $f_1$ and $f_2$ extend down. Finally (6.5) shows that $g$ extends down. □

*Proof of Theorem* 4. Because $M$ is assumed to be of infinite type and $h$ is transversally submersive, it follows from [M3] that

$$g(z,\bar{z},s)=sg_1(z,\bar{u},s) \quad \text{with } g_1(0)\neq 0.$$

Therefore we may divide by $\bar{g}$ in the class of functions which extend down.

Proposition 2.4 applies and with the above remark asserts that

$$\frac{Df_1}{1+if_2\bar{f}_2} \quad \text{and} \quad \frac{D^5 f_2}{1+if_2\bar{f}_2}$$

extend down.

The proof of Theorem 2 in Paragraph 5 may be reproduced with adjunction of a factor $s^{2l}$, where $l$ comes from (0.3), in each formula from (5.1) to (5.5). Indeed, the following observations are used:

(1) If $M$ is of infinite type, $v$ is smooth and $p\in\mathbf{N}$ then $L^\alpha(s^p v)=s^p v_\alpha$, with $v_\alpha$ smooth.

(2) $D(z,\bar{z},s)=s^{2l}D_0(z,\bar{z},s)$ (see [M3]). □

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