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Average decay of Fourier transforms and integer points in polyhedra

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1. Introduction

Let χ_B be the characteristic function of a compact connected set B in $\mathbf{R}^n$. Precise estimates of the decay of the Fourier transform

$$\widehat{\chi}_B(\xi)=\int_B e^{-2\pi i\xi\cdot x}\,dx$$

are crucial for several applications in Fourier analysis, geometry of convex sets and geometry of numbers. See e.g. [10], [9], [7], [8]. In the literature B has often been assumed to be convex, with a smooth boundary of strictly positive curvature. Under these assumptions the decay of $\widehat{\chi}_B$ along a fixed direction represents the global behavior. This is not always true if the boundary ∂B is not smooth or if it is smooth but with curvature vanishing at some points; in both cases the decay of the Fourier transform may depend on the direction. For example, if P is an n-dimensional polyhedron, then $\widehat{\chi}_P(\xi)=\sum_j Q_j(\xi)e^{2\pi i\alpha_j\cdot\xi}$ with Q_j homogeneous of degree $-n$ and it can be seen that $\widehat{\chi}_P(\xi)$ decays as fast as $|\xi|^{-n}$ along almost all directions, but only as $|\xi|^{-1}$ along directions perpendicular to the $(n-1)$-dimensional faces. Therefore, when studying the behavior of the Fourier transform, one may be led to introduce an average decay. This point of view has been exploited e.g. in [14], [13], [20], [11], [2].

In this paper we study an average decay of $\widehat{\chi}_P$ when P is a polyhedron and then we apply our results to obtain estimates for the number of integer points in a dilated copy of P, randomly positioned in the space. We also compare polyhedra with more general domains.

Our first result is the following. Let $\Sigma_{n-1}=\{\sigma\in\mathbf{R}^n:|\sigma|=1\}$ be the unit sphere equipped with the Lebesgue surface measure and let $\varrho\geq 0$. When $n=1$ the polyhedron reduces to a segment, say $\left[-\frac{1}{2},\frac{1}{2}\right]$, and the sphere consists of the two points

$\pm\varrho$. In this case one has the pointwise estimate

$$|\widehat{\chi}_{[-1/2,1/2]}(\pm\varrho)| = \left|\frac{\sin(\pi\varrho)}{\pi\varrho}\right| \le c\,(2+\varrho)^{-1}.$$

When $n\geq 2$ the average decay of $\widehat{\chi}_P$ is different when measured by different norms. We prove that

(i) $\sup_{\lambda>0}\lambda|\{\sigma\in\Sigma_{n-1}:|\widehat{\chi}_P(\varrho\sigma)|>\lambda\}|\leq c(2+\varrho)^{-n}\log^{n-2}(2+\varrho)$,

(ii) $\int_{\Sigma_{n-1}}|\widehat{\chi}_P(\varrho\sigma)|\,d\sigma\leq c(2+\varrho)^{-n}\log^{n-1}(2+\varrho)$,

(iii) $\left(\int_{\Sigma_{n-1}}|\widehat{\chi}_P(\varrho\sigma)|^p\,d\sigma\right)^{1/p}\leq c(2+\varrho)^{-1-(n-1)/p}$, $1<p\leq+\infty$,

and we also show that these estimates are essentially sharp.

Here and in the sequel the letter c denotes a positive constant which may vary from step to step but does not depend on ϱ. Moreover, when A is a measurable set in some measure space, $|A|$ denotes its measure.

Observe that the above estimates may be easily checked in the case of the unit square in the plane,

$$\widehat{\chi}_{[-1/2,1/2]\times[-1/2,1/2]}(\varrho\cos(\theta),\varrho\sin(\theta)) = \frac{\sin(\pi\varrho\cos(\theta))}{\pi\varrho\cos(\theta)}\,\frac{\sin(\pi\varrho\sin(\theta))}{\pi\varrho\sin(\theta)}.$$

The case of a polygon is similar since we still have a quite explicit expression for the Fourier transform. Such a formula gets more complicated for an n-dimensional polyhedron and this general case will be handled through an induction argument on the dimension. Indeed, by the divergence theorem, the n-dimensional Fourier transform of a polyhedron is essentially a sum of the $(n-1)$-dimensional Fourier transform of its faces.

Some of the above estimates of the decay of Fourier transforms do not hold only for polyhedra but also for a large class of "regular" domains. Indeed, we shall see that when the boundary of a domain B has finite Minkowski measure, that is $|\{x\in\mathbf{R}^n:d(x,\partial B)<\varepsilon\}|\leq c\varepsilon$, then

$$\left(\frac{1}{|\{\varrho\le|\xi|\le 2\varrho\}|}\int_{\{\varrho\le|\xi|\le2\varrho\}}|\widehat{\chi}_B(\xi)|^p\,d\xi\right)^{1/p} \le \begin{cases} c(2+\varrho)^{-(n+1)/2}, & 1\le p\le 2,\\ c(2+\varrho)^{-1-(n-1)/p}, & 2\le p\le+\infty.\end{cases}$$

The main result here is the L^2 estimate, which is a quite immediate consequence of the direct and inverse approximation theorems of Jackson and Bernstein. See e.g. [12]. Indeed $\left(\int_{\{|\xi|\geq\varrho\}}|\widehat{\chi}_B(\xi)|^2\,d\xi\right)^{1/2}$ is the best approximation in $L^2(\mathbf{R}^n)$ of the function χ_B by means of entire functions of exponential type ϱ. This best approximation is related to the L^2 modulus of continuity of χ_B and hence to the Minkowski dimension of the boundary ∂B. Observe that for $p\geq 2$ the above estimates match with the corresponding ones for polyhedra. The estimates for $p<2$ are a trivial consequence of the case $p=2$, nevertheless they are sharp as the explicit example of a ball shows.

The second part of this paper is devoted to the classical problem of estimating the number of lattice points in large domains. Let P be our polyhedron and let $\varrho\geq 0$, $\theta\in\mathrm{SO}(n)$, $t\in\mathbf{R}^n$. Define the discrepancy $\mathcal{D}(\varrho,\theta,t)$ as the difference between the number of integer points in the set $\varrho\theta^{-1}P-t$, a dilated, rotated and translated copy of P, and the expected number $|\varrho\theta^{-1}P-t|=\varrho^n|P|$, i.e.

$$\mathcal{D}(\varrho,\theta,t)=\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}P-t}(m)-\varrho^n|P|.$$

Since this function is periodic with respect to translations, we may restrict the variable t to the torus $\mathbf{T}^n=\mathbf{R}^n/\mathbf{Z}^n$.

It is easy to check that $\mathcal{D}(\varrho,\theta,t)$ may be of the order of ϱ^{n-1} as $\varrho\to\infty$. On the other hand, Hardy and Littlewood have shown that, in dimension two and for particular choices of θ which give suitable irrational slopes of the sides of the polygon, the error can be logarithmically small: $|\mathcal{D}(\varrho,\theta,t)|\leq c_{\theta,t}\log(2+\varrho)$. An extension of this result to several variables has recently been proposed by Skriganov. See [5], [6], [15], [16]. See also [14], [19], [20], [1], [11] for related results. Our purpose is to extend, in a probabilistic framework, the result of Hardy and Littlewood to several variables. Our methods are different from the ones developed by the above authors, but we acknowledge the influence of the paper of Kendall [10]. Our result is the following,

(i) $\sup_{\lambda>0}\lambda|\{\theta\in\mathrm{SO}(n),\ t\in\mathbf{T}^n:|\mathcal{D}(\varrho,\theta,t)|>\lambda\}|\leq c\log^{n-1}(2+\varrho)$,

(ii) $\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varrho,\theta,t)|\,dt\,d\theta\leq c\log^n(2+\varrho)$,

(iii) $\left(\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varrho,\theta,t)|^p\,dt\,d\theta\right)^{1/p}\leq c(2+\varrho)^{(n-1)(1-1/p)}$, $1<p\leq+\infty$.

To prove these estimates, the idea is to use the Fourier expansion of the discrepancy, as a function of $t\in\mathbf{T}^n$,

$$\mathcal{D}(\varrho,\theta,t)=\sum_{m\in\mathbf{Z}^n\setminus\{0\}}\varrho^n\widehat{\chi}_P(\varrho\theta m)e^{2\pi i m\cdot t}.$$

The mean square estimate of the discrepancy follows from Parseval's formula and the previous estimates for the L^2 decay of the Fourier transform. However, in our opinion, the main result is (i), since the case $p=+\infty$ is quite immediate and the remaining cases, although they need a direct proof, may be considered as an interpolation between these extreme cases. We shall see that the estimates in (iii) are sharp and we shall also give estimates from below for (i) and (ii). These estimates all together give an idea of the size of the discrepancy, which can be very large, but only around some singular points. In particular (i) shows that

$$|\{\theta\in\mathrm{SO}(n),\ t\in\mathbf{T}^n:|\mathcal{D}(\varrho,\theta,t)|>\varepsilon^{-1}\log^{n-1}(2+\varrho)\}|\leq c\varepsilon.$$

We can give to the above result the following probabilistic interpretation. Throwing at random a dilated polyhedron ϱP in the space, the difference between the number of integer points in it and its volume can be as large as the surface measure of the boundary, $c\varrho^{n-1}$, however, the probability for this difference to be much greater than $\log^{n-1}(2+\varrho)$ is very small.

We also consider the discrepancy associated to domains more general than polyhedra. When the domain has a boundary with finite Minkowski measure, then the estimates on the Fourier transform when applied to the study of the discrepancy give

$$\left(\frac{1}{\varrho}\int_{\varrho}^{2\varrho}\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\tau,\theta,t)|^p\,dt\,d\theta\,d\tau\right)^{1/p}\le\begin{cases}c(2+\varrho)^{(n-1)/2}, & 1\le p\le 2,\\ c(2+\varrho)^{(n-1)(1-1/p)}, & 2\le p\le+\infty.\end{cases}$$

Again these estimates are sharp and it may be interesting to compare them with the corresponding estimates for polyhedra.

Finally, revisiting [1] and [11], we briefly consider the problem of the discrepancy associated to an arbitrary distribution of a finite set $\{z_j\}_{j=1}^M$ of points in $\mathbf{T}^n$. Generalizing the previous definition without changing the notation, for a given domain B contained in $\mathbf{T}^n$ we define the discrepancy as

$$\mathcal{D}(\varepsilon,\theta,t)=\sum_{j=1}^{M}\chi_{\varepsilon\theta^{-1}B-t}(z_j)-M\varepsilon^n|B|.$$

Assuming the set B satisfies $a|h|\le|((B-h)\setminus B)\cup(B\setminus(B-h))|\le b|h|$ for sufficiently small $|h|\le 1$, we prove that

$$\left(\int_q^1\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon\right)^{1/2}\ge cM^{(n-1)/2n}$$

for suitable constants $0<q<1$ and $c>0$ independent of the distribution of points $\{z_j\}_{j=1}^M$.

The 2-dimensional case of this result has been proved by Montgomery in [11], while the n-dimensional case has been proved in [1] by Beck assuming the domain convex.

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2. Average decay of Fourier transforms

Let X be a measure space and let $0<p\leq\infty$. The Lebesgue space $L^p(X)$ consists of all measurable functions with quasi-norm

$$\|g\|_{L^p(X)}=\left(\int_X |g(x)|^p\,dx\right)^{1/p}<+\infty.$$

The space *weak-*$L^p(X)$, or $L^{p,\infty}(X)$, is defined by the quasi-norm

$$\|g\|_{L^{p,\infty}(X)}=\sup_{\lambda>0}\lambda|\{x\in X:|g(x)|>\lambda\}|^{1/p}.$$

See e.g. [18].

The main result of this section is the following.

Theorem 2.1. *Let P be a compact polyhedron in $\mathbf{R}^n$. Then, for $n\geq 2$,*
(i) $\|\widehat{\chi}_P(\varrho\cdot)\|_{L^{1,\infty}(\Sigma_{n-1})}\leq c\,(2+\varrho)^{-n}\log^{n-2}(2+\varrho)$,
(ii) $\|\widehat{\chi}_P(\varrho\cdot)\|_{L^1(\Sigma_{n-1})}\leq c\,(2+\varrho)^{-n}\log^{n-1}(2+\varrho)$,
(iii) $\|\widehat{\chi}_P(\varrho\cdot)\|_{L^p(\Sigma_{n-1})}\leq c(2+\varrho)^{-1-(n-1)/p}$, $1<p\leq+\infty$.

Before starting the proof of the theorem we recall the explicit expression of the Fourier transform of the characteristic function of a polygon in the plane.

Lemma 2.2. *Let P be a polygon in the plane with counterclockwise oriented vertices $\{a_j\}_{j=1}^m$. Denote by σ_j the unit vector parallel to the side $[a_j,a_{j+1}]$ and by v_j the outward unit normal to this side. Then, defining $a_{m+1}=a_1$, we have*

$$\widehat{\chi}_P(\xi)=\int_P e^{-2\pi i\xi\cdot x}\,dx=-(2\pi|\xi|)^{-2}\sum_{j=1}^m(e^{-2\pi i\xi\cdot a_{j+1}}-e^{-2\pi i\xi\cdot a_j})\frac{\xi\cdot v_j}{\xi\cdot\sigma_j}.$$

Proof. By the divergence theorem,

$$\begin{aligned}\int_P e^{-2\pi i\xi\cdot x}\,dx&=-(2\pi|\xi|)^{-2}\int_P \Delta[e^{-2\pi i\xi\cdot x}]\,dx\\&=-(2\pi|\xi|)^{-2}\sum_{j=1}^m\int_{[a_j,a_{j+1}]}\frac{\partial}{\partial v_j}[e^{-2\pi i\xi\cdot x}]\,dx\\&=-(2\pi|\xi|)^{-2}\sum_{j=1}^m(e^{-2\pi i\xi\cdot a_{j+1}}-e^{-2\pi i\xi\cdot a_j})\frac{\xi\cdot v_j}{\xi\cdot\sigma_j}.\quad\square\end{aligned}$$

Proof of Theorem 2.1. From now on we assume ϱ large, since the estimates for ϱ small are immediate. The proof of (i) is by induction on the dimension n, starting with $n=2$. In this case the explicit expression of the Fourier transform of the characteristic function of a polygon shows that $\widehat{\chi}_P(\varrho\cos(\phi),\varrho\sin(\phi))$ is dominated by a finite sum of terms of the form $\varrho^{-2}|\cos(\phi-\phi_j)|^{-1}$. Since the functions $|\cos(\phi-\phi_j)|^{-1}$ are in $L^{1,\infty}(\Sigma_1)$, the desired estimate for $n=2$ follows.

We now consider the case $n>2$. By the divergence theorem

$$\int_P e^{-2\pi i\xi\cdot x}\,dx=\sum_{j=1}^{m}\frac{i\xi\cdot v_j}{2\pi|\xi|^2}\int_{F_j}e^{-2\pi i\xi\cdot x}\,dx,$$

where the F_j's are the faces of P and the v_j's are the outward unit normals to these faces.

Write $x=(t,y)$ with $t\in\mathbf{R}$ and $y\in\mathbf{R}^{n-1}$, $\xi=\varrho\sigma$ with $\varrho\geq 0$ and $\sigma\in\Sigma_{n-1}$. Also write $\sigma=(\cos(\phi),\sin(\phi)\eta)$ with $0\leq\phi\leq\pi$ and $\eta\in\Sigma_{n-2}$. Let us choose a face F, with unit normal v. We can assume that this face lies in the hyperplane $\{t=0\}$ with outward normal $(1,0,\dots,0)$. Then

$$\frac{iv\cdot\xi}{2\pi|\xi|^2}\int_F e^{-2\pi i\xi\cdot x}\,dx=\frac{i\cos(\phi)}{2\pi\varrho}\int_F e^{-2\pi i\varrho\sin(\phi)\eta\cdot y}\,dy=\frac{i\cos(\phi)}{2\pi\varrho}\widehat{\chi}_F(\varrho\sin(\phi)\eta),$$

where $\widehat{\chi}_F$ is an $(n-1)$-dimensional Fourier transform. Hence, roughly speaking, the n-dimensional Fourier transform of the characteristic function of a polyhedron is a sum of the $(n-1)$-dimensional Fourier transforms of its faces, multiplied by a factor ϱ^{-1}. Now we estimate the weak norm of $\widehat{\chi}_P$ using the induction assumption on $\widehat{\chi}_F$. Integrating in polar coordinates we have

$$\begin{aligned}\lambda\left|\left\{(\cos(\phi),\sin(\phi)\eta)\in\Sigma_{n-1}:\left|\frac{i\cos(\phi)}{2\pi\varrho}\widehat{\chi}_F(\varrho\sin(\phi)\eta)\right|>\lambda\right\}\right|\\ =\lambda\int_0^{\pi}\left|\left\{\eta\in\Sigma_{n-2}:|\widehat{\chi}_F(\varrho\sin(\phi)\eta)|>\frac{2\pi\varrho\lambda}{|\cos(\phi)|}\right\}\right|\sin^{n-2}(\phi)\,d\phi.\end{aligned}$$

By induction, the above term is bounded by

$$\begin{aligned}c\varrho^{-1}\int_0^{\pi}\frac{\log^{n-3}(2+\varrho\sin(\phi))}{(2+\varrho\sin(\phi))^{n-1}}\sin^{n-2}(\phi)\,d\phi\leq c\varrho^{-n}\int_0^{\varrho\pi/2}\frac{\log^{n-3}(2+s)}{(2+s)^{n-1}}s^{n-2}\,ds\\ \leq c\varrho^{-n}\log^{n-2}(\varrho).\end{aligned}$$

The proof of (i) is thus complete. In order to prove (ii) and (iii) we start the induction with the trivial case $n=1$. When $n>1$ and $1\leq p<+\infty$, arguing as before, we bound $\|\widehat{\chi}_P(\varrho\cdot)\|^p_{L^p(\Sigma_{n-1})}$ by

$$\begin{aligned}\int_0^\pi\int_{\Sigma_{n-2}}&\left|\frac{i\cos(\phi)}{2\pi\varrho}\widehat{\chi}_F(\varrho\sin(\phi)\eta)\right|^p\sin^{n-2}(\phi)\,d\eta\,d\phi\\&\le\begin{cases}\dfrac{c}{\varrho}\displaystyle\int_0^\pi\frac{\log^{n-2}(2+\varrho\sin(\phi))}{(2+\varrho\sin(\phi))^{n-1}}\sin^{n-2}(\phi)\,d\phi\le c\frac{\log^{n-1}(\varrho)}{\varrho^n} & \text{if } p=1,\\ \dfrac{c}{\varrho^p}\displaystyle\int_0^\pi(2+\varrho\sin(\phi))^{-p-(n-2)}\sin^{n-2}(\phi)\,d\phi\le c\varrho^{-p-(n-1)} & \text{if } p>1.\end{cases}\end{aligned}$$

When $p=+\infty$ it suffices to control the decay of $\widehat{\chi}_P$ with the L^1 modulus of continuity of χ_P as follows.

$$\begin{aligned}\int_{\mathbf{R}^n}e^{-2\pi i\xi\cdot x}\chi_P(x)\,dx&=-\int_{\mathbf{R}^n}e^{-2\pi i\xi\cdot(x+(\xi/2|\xi|^2))}\chi_P(x)\,dx\\&=\frac{1}{2}\int_{\mathbf{R}^n}e^{-2\pi i\xi\cdot x}\left(\chi_P(x)-\chi_P\left(x-\frac{\xi}{2|\xi|^2}\right)\right)dx.\end{aligned}$$

Hence

$$|\widehat{\chi}_P(\xi)|\le\frac{1}{2}\int_{\mathbf{R}^n}\left|\chi_P(x)-\chi_P\left(x-\frac{\xi}{2|\xi|^2}\right)\right|dx\le c|\xi|^{-1}.\quad\square$$

For a generic polyhedron the previous estimates cannot be improved, as the following theorem shows.

Theorem 2.3. *Let S be a simplex in $\mathbf{R}^n$, $n\ge2$. Then*
(i) $\|\widehat{\chi}_S(\varrho\cdot)\|_{L^{1,\infty}(\Sigma_{n-1})}\ge c\,(2+\varrho)^{-n}\log^{n-2}(2+\varrho)$,
(ii) $\|\widehat{\chi}_S(\varrho\cdot)\|_{L^1(\Sigma_{n-1})}\ge c\,(2+\varrho)^{-n}\log^{n-1}(2+\varrho)$,
(iii) $\|\widehat{\chi}_S(\varrho\cdot)\|_{L^p(\Sigma_{n-1})}\ge c(2+\varrho)^{-1-(n-1)/p}$, $1<p\le+\infty$.

Proof. We first prove (ii) and (iii). Arguing as in the proof of Theorem 2.1 we have

$$\int_S e^{-2\pi i\xi\cdot x}\,dx=\sum_{j=0}^n\frac{i\xi\cdot v_j}{2\pi|\xi|^2}\int_{S_j}e^{-2\pi i\xi\cdot x}\,dx,$$

where the S_j's are the faces of S and the v_j's are the corresponding outward unit normal vectors. Since S is a simplex, all the v_j's are different. Write $\xi=\varrho\sigma$ with $\varrho\ge0$ and $\sigma\in\Sigma_{n-1}$. Let $\delta>0$ and let $U=\{\sigma\in\Sigma_{n-1}:\sigma\cdot v_0\ge\cos(\delta)\}$. Then, for large ϱ,

$$\begin{aligned}&\|\widehat{\chi}_F(\varrho\cdot)\|_{L^p(U)}\\&\quad\ge\frac{c}{\varrho}\left(\int_U\left|\int_{S_0}e^{-2\pi i\varrho\sigma\cdot x}\,dx\right|^p d\sigma\right)^{1/p}-\sum_{j=1}^n\frac{c}{\varrho}\left(\int_U\left|\int_{S_j}e^{-2\pi i\varrho\sigma\cdot x}\,dx\right|^p d\sigma\right)^{1/p}.\end{aligned}$$

By induction, as in the proof of Theorem 2.1 we get

$$\frac{1}{\varrho}\left(\int_U\left|\int_{S_0} e^{-2\pi i\varrho\sigma\cdot x}\,dx\right|^p d\sigma\right)^{1/p} \geq \begin{cases} c\dfrac{\log^{n-1}(\varrho)}{\varrho^n} & \text{when } p=1, \\ c\varrho^{-1-(n-1)/p} & \text{when } p>1. \end{cases}$$

Let us now consider the contribution of one of the faces S_j when $j\geq 1$. We may assume that this face lies in the hyperplane orthogonal to $v_j=(1,0,\dots,0)$. Since the normal v_0 to the face S_0 is not parallel to this v_j, if δ is suitably small we may also assume that U is contained in $\{(\cos(\phi),\sin(\phi)\eta):\delta\leq\phi\leq\pi-\delta,\ \eta\in\Sigma_{n-2}\}$. We apply Theorem 2.1 to the $(n-1)$-dimensional Fourier transform of S_j to get

$$\begin{aligned}\frac{1}{\varrho^p}\int_U\left|\int_{S_j} e^{-2\pi i\varrho\sigma\cdot x}\,dx\right|^p d\sigma &\leq \frac{c}{\varrho^p}\int_\delta^{\pi-\delta}\int_{\Sigma_{n-2}}|\widehat{\chi}_{S_j}(\varrho\sin(\phi)\eta)|^p\sin^{n-2}(\phi)\,d\phi\,d\eta \\ &\leq \begin{cases} \dfrac{c}{\varrho}\displaystyle\int_\delta^{\pi/2}\frac{\log^{n-2}(\varrho\phi)}{\varrho^{n-1}\phi}\,d\phi \leq c\frac{\log^{n-2}(\varrho)}{\varrho^n} & \text{if } p=1, \\ \dfrac{c}{\varrho^p}\displaystyle\int_\delta^{\pi/2}\varrho^{2-n-p}\phi^{-p}\,d\phi = c\varrho^{2-n-2p} & \text{if } p>1. \end{cases}\end{aligned}$$

Hence the contribution of the faces S_j, $j\geq 1$, is negligible when compared with the contribution of S_0. Therefore the proof of (ii) and (iii) is complete.

We now prove (i). The idea is to show that if the estimate (i) fails, then (ii) fails as well.

Let g be the non increasing rearrangement of $\widehat{\chi}_S(\varrho\cdot)$, that is g is a non negative non increasing function, defined on the interval $(0,|\Sigma_{n-1}|)$, with the same distribution function as $\widehat{\chi}_S(\varrho\cdot)$: for every $\lambda>0$

$$|\{u\in(0,|\Sigma_{n-1}|):g(u)>\lambda\}| = |\{\sigma\in\Sigma_{n-1}:|\widehat{\chi}_S(\varrho\sigma)|>\lambda\}|.$$

See e.g. [18]. Then we have $g(u)\leq\|\widehat{\chi}_S(\varrho\cdot)\|_{L^\infty(\Sigma_{n-1})}\leq|S|$, and if we assume that $\|\widehat{\chi}_S(\varrho\cdot)\|_{L^{1,\infty}(\Sigma_{n-1})}\leq\varepsilon\varrho^{-n}\log^{n-2}(\varrho)$, we also have $g(u)\leq\varepsilon\varrho^{-n}\log^{n-2}(\varrho)u^{-1}$. Hence

$$\begin{aligned}\|\widehat{\chi}_S(\varrho\cdot)\|_{L^1(\Sigma_{n-1})} &= \int_0^{|\Sigma_{n-1}|} g(u)\,du \\ &\leq |S|\int_0^{\varrho^{-n}} du + \varepsilon\frac{\log^{n-2}(\varrho)}{\varrho^n}\int_{\varrho^{-n}}^{|\Sigma_{n-1}|}\frac{du}{u} \\ &= \varepsilon n\frac{\log^{n-1}(\varrho)}{\varrho^n} + \varepsilon\log(|\Sigma_{n-1}|)\frac{\log^{n-2}(\varrho)}{\varrho^n} + |S|\varrho^{-n}.\end{aligned}$$

Since we know, by (i), that $\|\widehat{\chi}_S(\varrho\cdot)\|_{L^1(\Sigma_{n-1})}\geq c\varrho^{-n}\log^{n-1}(\varrho)$, we deduce that ε cannot be too small. $\square$

The proof shows that the above theorem holds true not only for a simplex but also for every polyhedron with a face not parallel to the others. However we cannot go much further since the example of a cube shows a different behavior. This fact seems to be related to the location of the zeros of the Fourier transform.

Theorem 2.4. *Let* $Q=\left[-\frac{1}{2},\frac{1}{2}\right]^n$ *be the unit cube in* $\mathbf{R}^n$, $n\geq 2$.
(i) *There exist two positive constants* a *and* b *such that for any positive* ϱ,

$$a\frac{\log^{n-1}(2+\varrho)}{(2+\varrho)^n}\leq\|\widehat{\chi}_Q(\varrho\cdot)\|_{L^1(\Sigma_{n-1})}\leq b\frac{\log^{n-1}(2+\varrho)}{(2+\varrho)^n}.$$

(ii) *If* $1<p\leq+\infty$ *then there exist two positive constants* a *and* b *such that*

$$a(2+\varrho)^{-3/2-(2n-3)/2p}\leq\|\widehat{\chi}_Q(\varrho\cdot)\|_{L^p(\Sigma_{n-1})}\leq b(2+\varrho)^{-1-(n-1)/p}.$$

Moreover, if $1+(n-1)/p\leq\gamma\leq\frac{3}{2}+(2n-3)/2p$, *then there exists a constant* c *and a sequence* $\varrho_k\to+\infty$ *such that* $\|\widehat{\chi}_Q(\varrho_k\cdot)\|_{L^p(\Sigma_{n-1})}=c\varrho_k^{-\gamma}$.

Observe that when $n=1$, the Fourier transform $\widehat{\chi}_{[-1/2,1/2]}(\xi)=\sin(\pi\xi)/\pi\xi$ vanishes on the 0-dimensional spheres $\{\pm k\}_{k=1,2,\ldots}$. On the other hand, for $n>1$, the Fourier transform $\widehat{\chi}_Q$ does not vanish identically on any sphere. In a sense, the theorem says that the zeros of the Fourier transform influence the norm in $L^p(\Sigma_{n-1})$, $1<p\leq\infty$, but not the norm in $L^1(\Sigma_{n-1})$.

For simplicity of exposition we split the proof of the theorem into some lemmas.

Lemma 2.5. *There exist two constants* a *and* b *such that for any positive* ϱ,

$$a\frac{\log^{n-1}(2+\varrho)}{(2+\varrho)^n}\leq\|\widehat{\chi}_Q(\varrho\cdot)\|_{L^1(\Sigma_{n-1})}\leq b\frac{\log^{n-1}(2+\varrho)}{(2+\varrho)^n}.$$

Proof. The estimate from above has been proved in Theorem 2.1. We start proving the estimate from below in the case $n=2$ and again we assume ϱ large.

The Fourier transform of the characteristic function of the unit square is

$$\widehat{\chi}_Q(\varrho\cos(\phi),\varrho\sin(\phi))=\frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)}\frac{\sin(\pi\varrho\sin(\phi))}{\pi\varrho\sin(\phi)}.$$

Let $\left\{\phi\in\left[0,\frac{1}{4}\pi\right]:|\sin(\pi\varrho\cos(\phi))|\geq\frac{1}{2}\right\}=\bigcup_{j\geq 1}[a_j,b_j]$. Since

$$\sin(\pi\varrho\cos(\phi))=\sin\left(\pi\varrho-\tfrac{1}{2}\pi\varrho\phi^2+\ldots\right)$$

we have $a_j \approx \sqrt{j/\varrho}$, $b_j \approx \sqrt{(j+1)/\varrho}$, with $1 \le j \le c\varrho$. Hence

$$\begin{aligned}\int_0^{2\pi} \left| \frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)} \frac{\sin(\pi\varrho\sin(\phi))}{\pi\varrho\sin(\phi)} \right| d\phi &\ge c\varrho^{-2} \int_0^{\pi/4} \left| \frac{\sin(\pi\varrho\cos(\phi))\sin(\pi\varrho\sin(\phi))}{\sin(\phi)} \right| d\phi \\ &\ge c\varrho^{-2} \sum_{j=1}^{c\varrho} \frac{1}{\sin(b_j)} \int_{a_j}^{b_j} |\sin(\pi\varrho\sin(\phi))|\, d\phi \\ &\ge c\varrho^{-2} \sum_{j=1}^{c\varrho} \frac{b_j - a_j}{\sin(b_j)} \ge c\frac{\log(\varrho)}{\varrho^2}.\end{aligned}$$

The proof of the case $n>2$ is by induction on the dimension n. Using the same notation as in the proof of Theorem 2.1, we have $Q=Q_n=\left[-\frac{1}{2}, \frac{1}{2}\right] \times Q_{n-1}$ and also

$$\widehat{\chi}_{Q_n}(t,y) = \frac{\sin(\pi t)}{\pi t} \widehat{\chi}_{Q_{n-1}}(y).$$

Hence

$$\begin{aligned}\int_{\Sigma_{n-1}} |\widehat{\chi}_{Q_n}(\varrho\sigma)|\, d\sigma &= \int_0^{\pi} \left| \frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)} \right| \int_{\Sigma_{n-2}} |\widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta)| \sin^{n-2}(\phi)\, d\eta\, d\phi \\ &\ge c \int_{\varrho^{-1/2}}^{\pi/4} \left| \frac{\sin(\pi\varrho\cos(\phi))}{\varrho} \right| \frac{\log^{n-2}(\varrho\sin(\phi))}{(\varrho\sin(\phi))^{n-1}} \sin^{n-2}(\phi)\, d\phi \\ &\ge c\frac{\log^{n-1}(\varrho)}{\varrho^n}. \quad \square\end{aligned}$$

Lemma 2.6. *If $1<p\le+\infty$ and if $|\sin(\pi\varrho)|>\varepsilon$, then there exist two constants a and b such that*

$$a\varrho^{-1-(n-1)/p} \le \|\widehat{\chi}_Q(\varrho\cdot)\|_{L^p(\Sigma_{n-1})} \le b\varrho^{-1-(n-1)/p}.$$

Proof. Since the estimate from above has been proved in Theorem 2.1, we only need to prove the estimate from below.

When $0<\phi<1/\varrho$ we have $\varrho\sin(\phi)=O(1)$ and $\pi\varrho\cos(\phi)=\pi\varrho+O(\varrho^{-1})$, so that, for large ϱ, $\sin(\pi\varrho\cos(\phi))>\frac{1}{2}\varepsilon$. Therefore

$$\begin{aligned}\|\widehat{\chi}_Q(\varrho\cdot)\|^p_{L^p(\Sigma_{n-1})} &= \int_0^{\pi} \int_{\Sigma_{n-2}} \left| \frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)} \widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta) \right|^p \sin^{n-2}(\phi)\, d\eta\, d\phi \\ &\ge \int_0^{1/\varrho} \int_{\Sigma_{n-2}} \left| \frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)} \widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta) \right|^p \sin^{n-2}(\phi)\, d\eta\, d\phi \\ &\ge c\varepsilon^p \varrho^{-p} \int_0^{1/\varrho} \sin^{n-2}(\phi)\, d\phi \ge a\varrho^{-p-n+1}. \quad \square\end{aligned}$$

Lemma 2.7. *If* $1<p\le\infty$ *then*

$$\|\widehat{\chi}_Q(\varrho\cdot)\|_{L^p(\Sigma_{n-1})}\ge a\varrho^{-3/2-(2n-3)/2p}.$$

Proof. When $\alpha/\sqrt{\varrho}<\phi<\beta/\sqrt{\varrho}$ we have $\pi\varrho\cos(\phi)=\pi\varrho-\frac{1}{2}\pi t^2+O(\varrho^{-1})$, with $\alpha<t<\beta$. Therefore, if α and β are suitably chosen, then $|\sin(\pi\varrho\cos(\phi))|>\varepsilon$, and

$$\begin{aligned}\|\widehat{\chi}_Q(\varrho\cdot)\|^p_{L^p(\Sigma_{n-1})}&\ge\int_{\alpha/\sqrt{\varrho}}^{\beta/\sqrt{\varrho}}\int_{\Sigma_{n-2}}\left|\frac{\sin(\pi\varrho\cos(\phi))}{\pi\varrho\cos(\phi)}\widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta)\right|^p\sin^{n-2}(\phi)\,d\eta\,d\phi\\&\ge c\int_{\alpha/\sqrt{\varrho}}^{\beta/\sqrt{\varrho}}\int_{\Sigma_{n-2}}\frac{\varepsilon^p}{\varrho^p}\left|\widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta)\right|^p\varrho^{-(n-2)/2}\,d\eta\,d\phi\\&\ge c\varepsilon^p\varrho^{2-n-3p/2}|\{\phi\in(\alpha/\sqrt{\varrho},\beta/\sqrt{\varrho}):|\sin(\pi\varrho\sin(\phi))|>\varepsilon\}|,\end{aligned}$$

where we have used the inequality, contained in Lemma 2.6,

$$\int_{\Sigma_{n-2}}|\widehat{\chi}_{Q_{n-1}}(\varrho\sin(\phi)\eta)|^p d\eta\ge c|\varrho\sin(\phi)|^{2-n-p}\ge c\varrho^{1-n/2-p/2},$$

which holds whenever $|\sin(\pi\varrho\sin(\phi))|>\varepsilon$ and $\alpha/\sqrt{\varrho}<\phi<\beta/\sqrt{\varrho}$. Since

$$|\{\phi\in(\alpha/\sqrt{\varrho},\beta/\sqrt{\varrho}):|\sin(\pi\varrho\sin(\phi))|>\varepsilon\}|\ge c\varrho^{-1/2}$$

the desired estimate follows. □

Lemma 2.8. *If* $1<p\le+\infty$ *and if* ϱ *is a positive integer,* $\varrho=k$, *then*

$$ak^{-3/2-(2n-3)/2p}\le\|\widehat{\chi}_Q(k\cdot)\|_{L^p(\Sigma_{n-1})}\le bk^{-3/2-(2n-3)/2p}.$$

Proof. We only need to prove the estimate from above. By Theorem 2.1, with $n-1$ in place of n, we have

$$\begin{aligned}\|\widehat{\chi}_Q(k\cdot)\|^p_{L^p(\Sigma_{n-1})}&=\int_0^\pi\int_{\Sigma_{n-2}}\left|\frac{\sin(\pi k\cos(\phi))}{\pi k\cos(\phi)}\widehat{\chi}_{Q_{n-1}}(k\sin(\phi)\eta)\right|^p\sin^{n-2}(\phi)\,d\eta\,d\phi\\&\le ck^{-p-(n-2)}\int_0^\pi\left|\frac{\sin(\pi k\cos(\phi))}{k\cos(\phi)}\right|^p\sin^{-p}(\phi)\,d\phi\\&\le ck^{-2p-(n-2)}\int_0^{\pi/4}|\sin(2\pi k\sin^2(\phi/2))|^p\phi^{-p}\,d\phi\\&\le ck^{-2p-(n-2)}\left(\int_0^{k^{-1/2}}k^p\phi^p\,d\phi+\int_{k^{-1/2}}^{\pi/4}\phi^{-p}\,d\phi\right)\\&\le ck^{-3p/2-n+3/2}.\quad\square\end{aligned}$$

We end this section by briefly considering domains more general than polyhedra, namely domains B whose boundaries ∂B have finite Minkowski measure.

Theorem 2.9. *Let B be a domain in $\mathbf{R}^n$ and assume that, for every $\varepsilon>0$,*

$$|\{x\in\mathbf{R}^n : d(x,\partial B)<\varepsilon\}|<c\varepsilon.$$

Then

$$\left(\frac{1}{|\{\varrho\le|\xi|\le 2\varrho\}|}\int_{\{\varrho\le|\xi|\le 2\varrho\}}|\widehat{\chi}_B(\xi)|^p\,d\xi\right)^{1/p}\le\begin{cases} c(2+\varrho)^{-(n+1)/2}, & 1\le p\le 2,\\ c(2+\varrho)^{-1-(n-1)/p}, & 2\le p\le+\infty.\end{cases}$$

For the class of domains with boundaries of finite Minkowski measure these estimates are sharp.

As we said, the case $p=2$ of this theorem is a consequence of Jackson's approximation theorem, however here we like to present a short direct proof.

Lemma 2.10. *Let ϕ be a function in $L^2(\mathbf{R}^n)$ and assume that*

$$\left(\int_{\mathbf{R}^n}|\phi(x+h)-\phi(x)|^2\,dx\right)^{1/2}\le c|h|^{1/2}.$$

Then

$$\left(\int_{\{|\xi|\ge\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi\right)^{1/2}\le c(2+\varrho)^{-1/2}.$$

Proof. It is enough to show that for every nonnegative integer k,

$$\int_{\{2^k\le|\xi|\le 2^{k+1}\}}|\hat{\phi}(\xi)|^2\,d\xi\le c2^{-k}.$$

By Plancherel's formula

$$\int_{\mathbf{R}^n}|\phi(x+h)-\phi(x)|^2\,dx=\int_{\mathbf{R}^n}|e^{2\pi i\xi\cdot h}-1|^2|\hat{\phi}(\xi)|^2\,d\xi.$$

The lemma now follows by splitting the set $\{2^k\le|\xi|\le 2^{k+1}\}$ into a finite number of pieces where $|e^{2\pi i\xi\cdot h}-1|\ge c$, for suitable h's with $|h|\approx 2^{-k}$. $\square$

Proof of Theorem 2.9. Since

$$\begin{aligned}\int_{\mathbf{R}^n}|\chi_B(x+h)-\chi_B(x)|^2\,dx&=|((B-h)\setminus B)\cup(B\setminus(B-h))|\\&\le|\{x\in\mathbf{R}^n : d(x,\partial B)<h\}|,\end{aligned}$$

the case $p=2$ of the theorem follows from the above lemma. Assuming this case, the other cases follow easily. Indeed when $p=+\infty$ it suffices to bound the decay of $\widehat{\chi}_B$

with the L^1 modulus of continuity of χ_B as in the proof of Theorem 2.1. The case $2<p<+\infty$ follows by interpolation between 2 and $+\infty$. When $p<2$ the estimate follows since the L^p norm is not greater than the L^2 norm.

We already know by Theorem 2.3 that when $2\le p\le+\infty$ the above estimates are sharp for simplices. When $1\le p\le 2$ the estimates are sharp for domains with smooth boundaries with strictly positive curvature. See [9] and [7]. See also [20] for mean square estimates without curvature assumptions, or [13] and [11] for two-dimensional results proved under mild regularity assumptions. The sharpness of the estimates when $1\le p\le 2$ can also be checked directly when B is the unit ball in $\mathbf{R}^n$, since in this case

$$\widehat{\chi}_B(\xi)=|\xi|^{-n/2}J_{n/2}(2\pi|\xi|)\sim\pi^{-1}|\xi|^{-(n+1)/2}\cos\big(2\pi|\xi|-\tfrac{1}{4}\pi(n+1)\big).\quad\square$$

3. Integer points in polyhedra

Theorem 3.1. *Let χ_P be the characteristic function of a compact polyhedron in $\mathbf{R}^n$, $n\ge2$, and let*

$$\mathcal{D}(\varrho,\theta,t)=\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}P-t}(m)-\varrho^n|P|,$$

with $\varrho\ge0$, $\theta\in\mathrm{SO}(n)$, $t\in\mathbf{T}^n$. Then

(i) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{1,\infty}(\mathrm{SO}(n)\times\mathbf{T}^n)}\le c\log^{n-1}(2+\varrho)$,

(ii) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{1}(\mathrm{SO}(n)\times\mathbf{T}^n)}\le c\log^{n}(2+\varrho)$,

(iii) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{p}(\mathrm{SO}(n)\times\mathbf{T}^n)}\le c(2+\varrho)^{(n-1)(1-1/p)}$, $1<p\le+\infty$.

We split the proof of the theorem into several lemmas.

Lemma 3.2. *Let X and Y be finite measure spaces and let $L^{1,\infty}(X\times Y)$ be the weak space of measurable functions on $X\times Y$ with*

$$\|F\|_{L^{1,\infty}(X\times Y)}=\sup_{\lambda>0}\lambda|\{(x,y)\in X\times Y:|F(x,y)|>\lambda\}|<+\infty.$$

Let also $L^{1,\infty}(X,L^2(Y))$ be the mixed norm space of measurable functions on $X\times Y$ with

$$\|F\|_{L^{1,\infty}(X,L^2(Y))}=\sup_{\lambda>0}\lambda|\{x\in X:\|F(x,\cdot)\|_{L^2(Y)}>\lambda\}|<+\infty.$$

Then $L^{1,\infty}(X,L^2(Y))$ is contained into $L^{1,\infty}(X\times Y)$ and

$$\|F\|_{L^{1,\infty}(X\times Y)}\le c\|F\|_{L^{1,\infty}(X,L^2(Y))}.$$

Proof. Let $F\in L^{1,\infty}(X,L^2(Y))$. Since the statement is rearrangement invariant, we can assume that $X=Y=(0,1)$ equipped with the Lebesgue measure and also $\left(\int_0^1 |F(x,y)|^2\,dy\right)^{1/2}\le x^{-1}$. Then

$$\begin{aligned}
|\{(x,y):0<x<1,\ 0<y<1,\ |F(x,y)|>\lambda\}| \\
&\le \lambda^{-1}+|\{(x,y):\lambda^{-1}<x<1,\ 0<y<1,\ |F(x,y)|>\lambda\}| \\
&\le \lambda^{-1}+\int_{\lambda^{-1}}^1 |\{y:0<y<1,\ |F(x,y)|>\lambda\}|\,dx \\
&\le \lambda^{-1}+\int_{\lambda^{-1}}^1 \left(\lambda^{-2}\int_0^1 |F(x,y)|^2\,dy\right)dx \\
&\le \lambda^{-1}+\lambda^{-2}\int_{\lambda^{-1}}^1 x^{-2}\,dx\le 2\lambda^{-1}. \quad \square
\end{aligned}$$

Lemma 3.3. *Let $\{f_m\}$ be a sequence of functions in $L^{1,\infty}(X)$. Then*

$$\left\|\left(\sum_m |f_m|^2\right)^{1/2}\right\|_{L^{1,\infty}(X)} \le c\sum_m \|f_m\|_{L^{1,\infty}(X)}.$$

Proof. Note that the inequality $\|\sum_m |f_m|\|_{L^{1,\infty}(X)}\le c\sum_m \|f_m\|_{L^{1,\infty}(X)}$ may fail, since $L^{1,\infty}$ is not normable, but the lemma holds since $(\sum_m |f_m|^2)^{1/2}$ can be much smaller than $\sum_m |f_m|$. Recall that for every $\alpha>0$ and $p>0$,

$$\begin{aligned}
\|\,|g|^\alpha\|_{L^{p,\infty}(X)}^p &= \sup_{\lambda>0} \lambda^p|\{x\in X:|g(x)|^\alpha>\lambda\}| \\
&= \sup_{\lambda>0} \lambda^{\alpha p}|\{x\in X:|g(x)|>\lambda\}| = \|g\|_{L^{\alpha p,\infty}(X)}^{\alpha p}.
\end{aligned}$$

Also, if $0<q<1$ one has the q-triangular inequality

$$\left\|\sum_m g_m\right\|_{L^{q,\infty}(X)}^q \le c\sum_m \|g_m\|_{L^{q,\infty}(X)}^q,$$

see e.g. [17, Lemma 1.8]. Hence

$$\begin{aligned}
\left\|\left(\sum_m |f_m|^2\right)^{1/2}\right\|_{L^{1,\infty}(X)} &= \left\|\sum_m |f_m|^2\right\|_{L^{1/2,\infty}(X)}^{1/2} \\
&\le c\sum_m \|\,|f_m|^2\|_{L^{1/2,\infty}(X)}^{1/2} = c\sum_m \|f_m\|_{L^{1,\infty}(X)}. \quad \square
\end{aligned}$$

Lemma 3.4. *The following identity holds in the L^2-sense,*

$$\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}P-t}(m)=\varrho^n\sum_{m\in\mathbf{Z}^n}\widehat{\chi}_P(\varrho\theta m)\,e^{2\pi i m\cdot t}.$$

Proof.

$$\int_{\mathbf{T}^n}\Bigl(\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}P-t}(m)\Bigr)e^{-2\pi i k\cdot t}\,dt=\sum_{m\in\mathbf{Z}^n}\int_{\mathbf{T}^n}\chi_{\varrho\theta^{-1}P}(m+t)e^{-2\pi i k\cdot t}\,dt$$
$$=\int_{\mathbf{R}^n}\chi_{\varrho\theta^{-1}P}(t)e^{-2\pi i k\cdot t}\,dt=\varrho^n\widehat{\chi}_P(\varrho\theta k).\qquad\square$$

Lemma 3.5. $\|\mathcal{D}(\varrho,\cdot\,,\cdot)\|_{L^{1,\infty}(\mathrm{SO}(n),L^2(\mathbf{T}^n))}\le c\,\log^{n-1}(2+\varrho)$.

Proof. First observe that, by the previous lemma and Parseval's equality,

$$\left(\int_{\mathbf{T}^n}\left|\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}P-t}(m)-\varrho^n|P|\right|^2dt\right)^{1/2}=\varrho^n\Bigl(\sum_{m\neq 0}|\widehat{\chi}_P(\varrho\theta m)|^2\Bigr)^{1/2}.$$

Let us split the series in $0<|m|\le\varrho^{n-1}$ and $|m|>\varrho^{n-1}$. By Lemma 3.3 and Theorem 2.1(i),

$$\left\|\varrho^n\Bigl(\sum_{0<|m|\le\varrho^{n-1}}|\widehat{\chi}_P(\varrho\theta m)|^2\Bigr)^{1/2}\right\|_{L^{1,\infty}(\mathrm{SO}(n))}$$
$$\le c\varrho^n\sum_{0<|m|\le\varrho^{n-1}}\|\widehat{\chi}_P(\varrho\theta m)\|_{L^{1,\infty}(\mathrm{SO}(n))}$$
$$\le c\varrho^n\sum_{0<|m|\le\varrho^{n-1}}\frac{\log^{n-2}(2+|\varrho m|)}{(2+|\varrho m|)^n}\le c\,\log^{n-1}(2+\varrho).$$

Also, by Theorem 2.1(iii) with $p=2$,

$$\left\|\varrho^n\Bigl(\sum_{|m|>\varrho^{n-1}}|\widehat{\chi}_P(\varrho\theta m)|^2\Bigr)^{1/2}\right\|_{L^{1,\infty}(\mathrm{SO}(n))}$$
$$\le\varrho^n\left\|\Bigl(\sum_{|m|>\varrho^{n-1}}|\widehat{\chi}_P(\varrho\theta m)|^2\Bigr)^{1/2}\right\|_{L^2(\mathrm{SO}(n))}$$
$$=\varrho^n\Bigl(\sum_{|m|>\varrho^{n-1}}\|\widehat{\chi}_P(\varrho\theta m)\|^2_{L^2(\mathrm{SO}(n))}\Bigr)^{1/2}$$
$$\le c\varrho^n\Bigl(\sum_{|m|>\varrho^{n-1}}|\varrho m|^{-n-1}\Bigr)^{1/2}\le c.\qquad\square$$

Lemma 3.6. *Let $L^p(\mathrm{SO}(n), L^2(\mathbf{T}^n))$ be the mixed norm space of measurable functions on $\mathrm{SO}(n)\times\mathbf{T}^n$ with*

$$\|F\|_{L^p(\mathrm{SO}(n),L^2(\mathbf{T}^n))}=\left(\int_{\mathrm{SO}(n)}\left[\int_{\mathbf{T}^n}|F(\theta,t)|^2\,dt\right]^{p/2}d\theta\right)^{1/p}<+\infty.$$

Then

$$\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^p(\mathrm{SO}(n),L^2(\mathbf{T}^n))}\le\begin{cases} c\log^n(2+\varrho) & \text{if } p=1,\\ c(2+\varrho)^{(n-1)(1-1/p)} & \text{if } 1<p\le+\infty.\end{cases}$$

Proof. Let $1<p\le 2$. Arguing as in the previous lemma, by Theorem 2.1(iii), we have

$$\begin{aligned}\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^p(\mathrm{SO}(n),L^2(\mathbf{T}^n))}&=\varrho^n\left(\int_{\mathrm{SO}(n)}\left[\sum_{m\neq0}|\widehat{\chi}_P(\varrho\theta m)|^2\right]^{p/2}d\theta\right)^{1/p}\\ &\le\varrho^n\left(\int_{\mathrm{SO}(n)}\left[\sum_{m\neq0}|\widehat{\chi}_P(\varrho\theta m)|^p\right]d\theta\right)^{1/p}\\ &\le c\varrho^n\left(\sum_{m\neq0}|\varrho m|^{1-n-p}\right)^{1/p}\le c\varrho^{(n-1)(1-1/p)}.\end{aligned}$$

When $p=+\infty$ we have

$$\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^\infty(\mathrm{SO}(n),L^2(\mathbf{T}^n))}\le\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^\infty(\mathrm{SO}(n)\times\mathbf{T}^n)}\le c\varrho^{n-1}.$$

The estimate for $2<p<+\infty$ can be obtained by interpolation between 2 and $+\infty$. Finally, the estimate when $p=1$ can be obtained by splitting the series $(\sum_{m\neq0}|\widehat{\chi}_P(\varrho\theta m)|^2)^{1/2}$ in $0<|m|\le\varrho^{n-1}$ and $|m|>\varrho^{n-1}$, as in the proof of Lemma 3.5. □

Proof of Theorem 3.1. The estimate in (i) follows from Lemmas 3.2 and 3.5. If $1\le p\le 2$ we have

$$\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^p(\mathrm{SO}(n)\times\mathbf{T}^n)}\le\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^p(\mathrm{SO}(n),L^2(\mathbf{T}^n))},$$

therefore, when $1\le p\le 2$, the estimates in (ii) and (iii) follow from Lemma 3.6.

The case $p=+\infty$ follows from the inequality

$$|\mathcal{D}(\varrho,\theta,t)|\le\left|\left\{x\in\mathbf{R}^n: d(x,\partial(\varrho\theta^{-1}P-t))<\tfrac12\sqrt{n}\,\right\}\right|.$$

The case $2<p<+\infty$ follows by interpolation between 2 and $+\infty$. □

The estimate (iii) in Theorem 3.1 is sharp. We suspect that also (i) is best possible since the $\log^{n-1}(\varrho)$ result matches with related results in [5], [6], [15] and [16]. The following theorem summarizes what we know on this subject.

Theorem 3.7. *Let S be a simplex in $\mathbf{R}^n$ with $n\geq 2$. Then*
(i) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{1,\infty}(\mathrm{SO}(n)\times\mathbf{T}^n)}\geq c\log^{n-2}(2+\varrho)$,
(ii) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{1}(\mathrm{SO}(n)\times\mathbf{T}^n)}\geq c\log^{n-1}(2+\varrho)$,
(iii) $\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{p}(\mathrm{SO}(n)\times\mathbf{T}^n)}\geq c(2+\varrho)^{(n-1)(1-1/p)}$, $1<p\leq+\infty$.

Proof. Observe that for every $k\neq 0$,

$$\|\mathcal{D}(\varrho,\cdot,\cdot)\|_{L^{p}(\mathrm{SO}(n)\times\mathbf{T}^n)}=\varrho^n\left(\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}\left|\sum_{m\neq 0}\widehat{\chi}_P(\varrho\theta m)e^{2\pi i m\cdot t}\right|^p dt\,d\theta\right)^{1/p}$$
$$\geq\varrho^n\left(\int_{\mathrm{SO}(n)}|\widehat{\chi}_P(\varrho\theta k)|^p\,d\theta\right)^{1/p}.$$

Then (ii) and (iii) are immediate consequences of the corresponding estimates (ii) and (iii) in Theorem 2.3. The case (i) follows from (ii) via an interpolation argument similar to the one used in the proof of (i) in Theorem 2.3. □

For the discrepancies associated to domains more general than polyhedra we have the following result, which is a companion of Theorem 2.9.

Theorem 3.8. *Assume that the domain B satisfies, for every $\varepsilon>0$,*

$$|\{x\in\mathbf{R}^n : d(x,\partial B)<\varepsilon\}|<c\varepsilon.$$

Then the discrepancy associated to B satisfies

$$\left(\frac{1}{\varrho}\int_{\varrho}^{2\varrho}\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\tau,\theta,t)|^p\,dt\,d\theta\,d\tau\right)^{1/p}\leq\begin{cases}c(2+\varrho)^{(n-1)/2}, & 1\leq p\leq 2,\\ c(2+\varrho)^{(n-1)(1-1/p)}, & 2\leq p\leq+\infty.\end{cases}$$

For the class of domains with boundaries of finite Minkowski measure these estimates are sharp.

Proof. This result is contained in [3]. However the proof is similar to the one of Theorem 3.1. One only has to use the estimate for the decay of the Fourier transform provided by Theorem 2.9. □

We end this section with the following remarks.

For $n=2$ Tarnopolska-Weiss [19], improving a previous result of Randol [14], showed that, for every $\varepsilon>0$, the discrepancy associated to a polygon satisfies

$$\int_{\mathrm{SO}(2)}|\mathcal{D}(\varrho,\theta,t)|\,d\theta\leq c_\varepsilon\log^{2+\varepsilon}(2+\varrho)$$

with c_ε independent of t. This result has been stated for any dimension, but, for $n>2$, the proof contains a minor mistake that, when corrected, gives the bound $\log^{n+\varepsilon}(2+\varrho)$. Combining our estimates for the average decay of $\widehat{\chi}_P$ with the arguments in [19] one can prove

$$\left(\int_{SO(n)} |\mathcal{D}(\varrho,\theta,t)|^p\,d\theta\right)^{1/p} \leq \begin{cases} c\log^n(2+\varrho) & \text{when } p=1, \\ c(2+\varrho)^{(n-1)(1-1/p)} & \text{when } 1<p\leq+\infty. \end{cases}$$

Observe that these estimates give a different proof of the statements (ii) and (iii) in Theorem 3.1.

Our second remark is of a somewhat different nature. We have seen that in the case $p=2$ the discrepancies are essentially the same for a large class of domains. On the other hand when $1\leq p<2$ the discrepancy associated to polyhedra is much smaller than the one associated to domains with smooth boundary with strictly positive curvature. The situation reverses when $2<p\leq+\infty$, the discrepancy of domains with smooth boundary with strictly positive curvature is much smaller than the one of polyhedra. It is therefore natural to ask for the existence of intermediate discrepancies between polyhedra and convex domains with smooth boundary of strictly positive curvature. The answer is that when $p\neq2$ the situation may be chaotic. Indeed, if $\varepsilon>0$ and $\varrho_k\to+\infty$, then for most convex sets A the associated discrepancy

$$\mathcal{D}(\varrho,\theta,t)=\sum_{m\in\mathbf{Z}^n}\chi_{\varrho\theta^{-1}A-t}(m)-\varrho^n|A|$$

satisfies

$$\int_{SO(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varrho_k,\theta,t)|\,dt\,d\theta \quad \begin{cases} <\log^{\varepsilon+n-1}(\varrho_k) & \text{for infinitely many } k\text{'s}, \\ >\varrho_k^{-\varepsilon-(n-1)/2} & \text{for infinitely many } k\text{'s}. \end{cases}$$

This result follows from the estimates for the discrepancies associated to polyhedra and to domains with smooth boundary with strictly positive curvature, through a category argument of Gruber in [4].

4. Irregularities of distributions

In this section we briefly revisit some results of Montgomery [11] and Beck [1] on the irregularities of distributions of finite sets of points in the torus. We start observing that the results of the previous section still hold true rescaling the problem in the following way. Instead of fixing the lattice $\mathbf{Z}^n$ and dilating the polyhedron, one can fix the polyhedron and shrink the lattice. This is a particular case of the following.

Assume that B is a domain contained in the torus $\mathbf{T}^n$ with diameter smaller than 1 and let $\{z_j\}_{j=1}^M$ be a distribution of M points in $\mathbf{T}^n$. Generalizing the definitions we have been using throughout this paper, we now define the discrepancy as

$$\mathcal{D}(\varepsilon,\theta,t)=\sum_{j=1}^{M}\chi_{\varepsilon\theta^{-1}B-t}(z_j)-M\varepsilon^n|B|,$$

where now $\varepsilon\leq 1$ and the rotation θ and the translation t are in $\mathbf{T}^n$. This means to dilate, rotate and translate in $\mathbf{R}^n$ and then take the quotient with respect to the lattice $\mathbf{Z}^n$. Indeed, assuming that the diameter of B is smaller than 1, the projection $\mathbf{R}^n\to\mathbf{T}^n$ is injective on $\varepsilon\theta^{-1}B-t$.

If the points $\{z_j\}_{j=1}^M$ are chosen at random, then the mean square value of the discrepancy is proportional to $\sqrt{M}$. Indeed, since the discrepancy has the Fourier expansion

$$\mathcal{D}(\varepsilon,\theta,t)=\sum_{m\neq 0}\sum_{j=1}^{M}e^{2\pi i z_j\cdot m}\varepsilon^n\widehat{\chi}_B(\varepsilon\theta m)e^{2\pi i m\cdot t},$$

a repeated application of Parseval's formula yields

$$\left(\int_{\mathbf{T}^n}\cdots\int_{\mathbf{T}^n}\left[\int_0^1\int_{SO(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon\right]dz_1\ldots dz_M\right)^{1/2}$$
$$=\sqrt{M}\left(\sum_{m\neq 0}\int_0^1\int_{SO(n)}|\varepsilon^n\widehat{\chi}_B(\varepsilon\theta m)|^2\,d\theta\,d\varepsilon\right)^{1/2}.$$

On the other hand we have implicitly seen in the previous sections that the discrepancy of points evenly distributed on a lattice is smaller, since it is of the order of $M^{(n-1)/2n}$. The following result shows that this is a lower bound for the discrepancy of M points.

Theorem 4.1. *Assume that B is a domain satisfying*

$$a|h|\leq|((B-h)\setminus B)\cup(B\setminus(B-h))|\leq b|h|$$

for sufficiently small $|h|\leq 1$. Then there exist constants $0<q<1$ and $c>0$ such that for every distribution of points $\{z_j\}_{j=1}^M$,

$$\left(\int_q^1\int_{SO(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon\right)^{1/2}\geq cM^{(n-1)/2n}.$$

The proof of this theorem needs a refined version of Lemma 2.10.

Lemma 4.2. *Let ϕ be a function in $L^2(\mathbf{R}^n)$ and assume that for some positive constants a, b and every $h\in\mathbf{R}^n$ with $|h|\le 1$,*

$$a|h|\le\int_{\mathbf{R}^n}|\phi(x+h)-\phi(x)|^2\,dx\le b|h|.$$

Then there exist positive constants α, β, γ, δ, such that, for every $\varrho\ge 1$,

$$\frac{\alpha}{\varrho}\le\int_{\{\gamma\varrho\le|\xi|\le\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi\le\frac{\beta}{\varrho}.$$

Proof. Lemma 2.10 directly implies the estimate $\int_{\{\gamma\varrho\le|\xi|\le\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi\le\beta/\varrho$. This lemma also implies the estimate $\int_{\{|\xi|\le\gamma\varrho\}}|\xi|^2|\hat{\phi}(\xi)|^2\,d\xi\le c\gamma\varrho$. Then we have

$$\begin{aligned}a|h|&\le\int_{\mathbf{R}^n}|\phi(x+h)-\phi(x)|^2\,dx=\int_{\mathbf{R}^n}|e^{2\pi i\xi\cdot h}-1|^2|\hat{\phi}(\xi)|^2\,d\xi\\&\le 4\pi^2|h|^2\int_{\{|\xi|\le\gamma\varrho\}}|\xi|^2|\hat{\phi}(\xi)|^2\,d\xi+4\int_{\{\gamma\varrho\le|\xi|\le\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi+4\int_{\{|\xi|\ge\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi\\&\le c(\gamma\varrho|h|^2+\delta^{-1}\varrho^{-1})+4\int_{\{\gamma\varrho\le|\xi|\le\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi.\end{aligned}$$

Hence, if $|h|=\varrho^{-1}$, γ is suitably small and δ suitably large,

$$\int_{\{\gamma\varrho\le|\xi|\le\delta\varrho\}}|\hat{\phi}(\xi)|^2\,d\xi\ge\frac{a}{4}|h|-\frac{c}{4}(\gamma\varrho|h|^2+\delta^{-1}\varrho^{-1})\ge\frac{\alpha}{\varrho}.\quad\square$$

Proof of Theorem 4.1. When $\eta\le M\varepsilon^n|B|\le 1-\eta$ one obviously has $|\mathcal{D}(\varepsilon,\theta,t)|\ge\eta$. Hence, if $R=\omega M^{-1/n}$ for a suitable ω,

$$\frac{1}{R}\int_{qR}^{R}\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon>c>0.$$

Following [1], the proof will consist in blowing up this trivial estimate. As in the proof of Lemma 3.4 we have

$$\sum_{j=1}^{M}\chi_{\varepsilon\theta^{-1}B-t}(z_j)-M\varepsilon^n|B|=\sum_{m\neq 0}\sum_{j=1}^{M}e^{2\pi iz_j\cdot m}\varepsilon^n\widehat{\chi}_B(\varepsilon\theta m)e^{2\pi im\cdot t}.$$

Hence, by Lemma 4.2, if $0<R<r\leq 1$,

$$
\begin{aligned}
&\frac{1}{r}\int_{qr}^{r}\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon\\
&=\sum_{m\neq 0}\left|\sum_{j=1}^{M}e^{2\pi i z_j\cdot m}\right|^2\frac{1}{r}\int_{qr}^{r}\int_{\mathrm{SO}(n)}|\varepsilon^n\widehat{\chi}_B(\varepsilon\theta m)|^2\,d\theta\,d\varepsilon\\
&\approx\sum_{m\neq 0}\left|\sum_{j=1}^{M}e^{2\pi i z_j\cdot m}\right|^2\left(\frac{r}{|m|}\right)^n\int_{\{qr|m|\leq|\xi|\leq r|m|\}}|\widehat{\chi}_B(\xi)|^2\,d\xi\\
&\approx r^{2n}\sum_{m\neq 0}\left|\sum_{j=1}^{M}e^{2\pi i z_j\cdot m}\right|^2(1+r|m|)^{-n-1}\\
&\geq\inf_{m\neq 0}\left(\left(\frac{r}{R}\right)^{2n}\left(\frac{1+R|m|}{1+r|m|}\right)^{n+1}\right)\left(R^{2n}\sum_{m\neq 0}\left|\sum_{j=1}^{M}e^{2\pi i z_j\cdot m}\right|^2(1+R|m|)^{-n-1}\right)\\
&\approx\left(\frac{r}{R}\right)^{n-1}\frac{1}{R}\int_{qR}^{R}\int_{\mathrm{SO}(n)}\int_{\mathbf{T}^n}|\mathcal{D}(\varepsilon,\theta,t)|^2\,dt\,d\theta\,d\varepsilon.
\end{aligned}
$$

The desired estimate follows taking $R=\omega M^{-1/n}$ and $r=1$. $\square$

References

1. BECK, J., Irregularities of distribution I, *Acta Math.* **159** (1987), 1–49.
2. BRANDOLINI, L., RIGOLI, M. and TRAVAGLINI, G., Average decay of Fourier transforms and geometry of convex sets, Preprint.
3. COLZANI, L., Approximation of Lebesgue integrals by Riemann sums and lattice points in domains with fractal boundary, to appear in *Monatsh. Math.*
4. GRUBER, P. M., In most cases approximation is irregular, *Rend. Sem. Mat. Univ. Politec. Torino* **41** (1983), 19–33.
5. HARDY, G. H. and LITTLEWOOD, J. E., Some problems of diophantine approximation: the lattice points of a right-angled triangle, *Proc. London Math. Soc.* **20** (1922), 15–36.
6. HARDY, G. H. and LITTLEWOOD, J. E., Some problems of diophantine approximation: the lattice points of a right-angled triangle (second memoir), *Abh. Math. Sem. Univ. Hamburg* **1** (1922), 212–249.
7. HERZ, C. S., Fourier transforms related to convex sets, *Ann. of Math.* **75** (1962), 81–92.
8. HERZ, C. S., On the number of lattice points in a convex set, *Amer. J. Math.* **84** (1962), 126–133.
9. HLAWKA, E., Über Integrale auf convexen Körpern, I & II, *Monatsh. Math.* **54** (1950), 1–36, 81–99.
10. KENDALL, D. G., On the number of lattice points in a random oval, *Quart. J. Math. Oxford Ser.* (1) **19** (1948), 1–26.
11. MONTGOMERY, H. L., *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis,* CBMS Regional Conf. Ser. in Math. **84**, Amer. Math. Soc., Providence, R. I., 1994.
12. NIKOL'SKIĬ, S. M., *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, Berlin–Heidelberg, 1975.
13. PODKORYTOV, A. N., The asymptotic of a Fourier transform on a convex curve, *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **1991**:2, 50–57, 125 (Russian). English transl.: *Vestnik Leningrad. Univ. Math.* **24**:2 (1991), 57–65.
14. RANDOL, B., On the Fourier transform of the indicator function of a planar set, *Trans. Amer. Math. Soc.* **139** (1969), 271–278.
15. SKRIGANOV, M. M., On integer points in polygons, *Ann. Inst. Fourier* (*Grenoble*) **43** (1993), 313-323.
16. SKRIGANOV, M. M., Constructions of uniform distributions in terms of geometry of numbers, *Algebra i Analiz* **6** (1994), 200–230.
17. STEIN, E. M., TAIBLESON, M. H. and WEISS, G., Weak type estimates for maximal operators on certain H^p classes, *Rend. Circ. Mat. Palermo* **1981**, *suppl.* 1, 81–97.
18. STEIN, E. M. and WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton, N. J., 1971.
19. TARNOPOLSKA-WEISS, M., On the number of lattice points in a compact n-dimensional polyhedron, *Proc. Amer. Math. Soc.* **74** (1979), 124–127.

20. VARCHENKO, A. N., Number of lattice points in families of homothetic domains in $\mathbf{R}^n$, *Funktsional Anal. i Prilozhen* **17**:2 (1983), 1–6 (Russian). English transl.: *Functional Anal. Appl.* **17** (1983), 79–83.

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