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A bounded domain in $\mathbf{C}^N$ which embeds holomorphically into $\mathbf{C}^{N+1}$

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1. Introduction and the results

We say that a map from a complex manifold M to $\mathbf{C}^N$ is a holomorphic embedding if it is a holomorphic immersion which is one to one and proper, i.e. the preimage of every compact set is compact. If $f: M \to \mathbf{C}^N$ is a holomorphic embedding then $f(M)$ is a closed submanifold of $\mathbf{C}^N$.

By a result of Eliashberg and Gromov [EG] every p-dimensional Stein manifold can be holomorphically embedded into $\mathbf{C}^q$ where $q > \frac{1}{2}(3p+1)$, and this is sharp for even p. It is a natural question whether there are large classes of p-dimensional Stein manifolds which embed holomorphically into $\mathbf{C}^q$ where $q \leq \frac{1}{2}(3p+1)$. For instance, does every pseudoconvex domain in $\mathbf{C}^N$ embed holomorphically into $\mathbf{C}^{N+1}$? This does not seem to be an easy question. While there are trivial examples of such domains, e.g. $\mathbf{C}^N$, $\mathbf{C}^{N-1} \times \Delta$ where Δ is the open unit disc in $\mathbf{C}$ it seems already difficult to provide bounded domains in $\mathbf{C}^N$ which embed holomorphically into $\mathbf{C}^{N+1}$. There are such domains.

Theorem 1.1. *For every $N \in \mathbf{N}$ there are arbitrarily small $\mathcal{C}^1$-perturbations of the polydisc Δ^N which embed holomorphically into $\mathbf{C}^{N+1}$.*

This means that for each j, $1 \leq j \leq N$, there is an arbitrarily small $\mathcal{C}^1$-perturbation Γ_j of the surface $\{z \in \mathbf{C}^N : |z_j| = 1\}$ such that the domain Ω bounded by $\Gamma_1, \dots, \Gamma_N$ embeds holomorphically into $\mathbf{C}^{N+1}$.

We describe the idea of the proof. Denote by $|\cdot|$ the Euclidean norm on $\mathbf{C}^N$ and by $\|\cdot\|$ the sup norm on $\mathbf{C}^N$. We write $\mathbf{B} = \{z \in \mathbf{C}^N : |z| < 1\}$.

One of two essential ingredients of the proof of the embedding theorem for finitely connected domains [GS] is to use a sequence of holomorphic automorphisms of $\mathbf{C}^2$ in the following way: For each $n \in \mathbf{N}$ let $\Phi_j(z, w) = (z, w + T_j z^{N_j})$, $\Psi_j(z, w) = (z + S_j w^{M_j}, w)$ where T_j, S_j are large positive numbers and M_j, N_j are large positive integers. Write $\Theta_j = \Psi_j \circ \Phi_j$ and $F_n = \Theta_n \circ \Theta_{n-1} \circ \dots \circ \Theta_1$, $n \in \mathbf{N}$ and look at the limit

F of the sequence F_n. One can show, for instance, that if T_j, S_j, M_j, N_j are chosen in the right way then $F(\zeta,0)$ converges for all ζ belonging to a domain Ω which is a slight perturbation of Δ and, moreover, $\zeta\mapsto F(\zeta,0)$ embeds Ω holomorphically into $\mathbf{C}^2$.

In the first known way of embedding Δ holomorphically into $\mathbf{C}^2$, [St], [KN] one takes a Fatou–Bieberbach domain D (i.e. a proper subset of $\mathbf{C}^2$ for which there is a biholomorphic map F mapping D onto $\mathbf{C}^2$) which is a Runge domain. One intersects D with a complex line Λ which meets D and $\mathbf{C}^2\setminus D$. A component P of $\Lambda\cap D$ is simply connected and thus biholomorphically equivalent to Δ, and F embeds P holomorphically into $\mathbf{C}^2$.

Write $\Lambda=\mathbf{C}\times\{0\}$. In the construction above F_n converges uniformly on compacta in $\Omega\times\{0\}$. It is natural to ask whether the construction can be performed in such a way that $\Omega\times\{0\}=\Lambda\cap D$ where D is the domain of (uniform on compacta) convergence of F_n or, more generally, in such a way that F embeds D holomorphically into $\mathbf{C}^2$, that is, so that F maps D biholomorphically onto C^2. Berit Stensønes proved that if one chooses S_n, T_n, M_n and N_n in the right way then D is a Fatou–Bieberbach domain with boundary of class $\mathcal{C}^\infty$ which F maps biholomorphically onto $\mathbf{C}^2$ [S].

It is also natural to ask whether the method of embedding a domain close to Δ holomorphically into $\mathbf{C}^2$ with the limit of a sequence of composition maps F_n as above can be generalized to several variables to get holomorphic embeddings of bounded domains in $\mathbf{C}^N$ into $\mathbf{C}^{N+1}$. This is possible and the result is Theorem 1.1. In fact, it turns out that to prove Theorem 1.1 is not much easier than to prove the following, more general theorem.

Theorem 1.2. *Let $N\geq 2$ and let $R>0$ be arbitrarily large. There are domains $G_1, G_2, \dots, G_{N-1}$ with boundaries of class $\mathcal{C}^1$ such that for each j, $1\leq j\leq N-1$, $G_j\cap R\mathbf{B}$ is an arbitrarily small $\mathcal{C}^1$-perturbation of $\{z\in\mathbf{C}^N:|z_j|<1\}\cap R\mathbf{B}$ and such that there is a map F mapping $G=\bigcap_{j=1}^{N-1} G_j$ biholomorphically onto $\mathbf{C}^N$*

This means that for each j, $1\leq j\leq N-1$, $(bG_j)\cap(R\mathbf{B})$ is an arbitrarily small $\mathcal{C}^1$-perturbation of the surface $\{z\in\mathbf{C}^N:|z_j|=1\}\cap R\mathbf{B}$.

Let F map G biholomorphically onto $\mathbf{C}^N$. Let $H=\{z\in\mathbf{C}^N:z_N=0\}$. Then the component of $G\cap H$ that contains the origin is a domain in $\mathbf{C}^{N-1}$ that is a slight perturbation of the $(N-1)$-disc Δ^{N-1} and which F embeds holomorphically into $\mathbf{C}^N$. Thus Theorem 1.1 follows from Theorem 1.2.

Let us describe briefly the maps F_n that we will use to prove Theorem 1.2. Given $n\in\mathbf{N}$, let $p_{n1}, \dots, p_{n,N-1}$, $q_{n2}, \dots, q_{nN}$ be positive numbers and let $N_{n1}, \dots$,

$N_{n,N-1}$, $M_{n2}, \dots, M_{nN}$ be positive integers. Let

$$\varphi_{nj}(\zeta)=q_{nj}\left(\frac{\zeta}{p_{n,j-1}}\right)^{M_{nj}} \quad (2\le j\le N), \qquad \psi_{nj}(\zeta)=p_{nj}\left(\frac{\zeta}{q_{n,j+1}}\right)^{N_{nj}} \quad (1\le j\le N-1)$$

and, for $n\in\mathbf{N}$,

$$\Phi_n(z)=(z_1, z_2+\varphi_{n2}(z_1), \dots, z_n+\varphi_{nN}(z_{N-1})),$$
$$\Psi_n(z)=(z_1+\psi_{n1}(z_2), \dots, z_{N-1}+\psi_{n,N-1}(z_N), z_N).$$

Clearly Φ_n and Ψ_n are holomorphic automorphisms of $\mathbf{C}^N$. Put $\Theta_n=\Psi_n\circ\Phi_n$ and let $F_n=\Theta_n\circ\Theta_{n-1}\circ\dots\circ\Theta_1$ $(n\in\mathbf{N})$. We shall show that if the numbers p_{nj}, q_{nj}, N_{nj}, M_{nj} are chosen in the right way then the domain of (uniform on compacta) convergence of the sequence F_n is a domain G of the form described in Theorem 1.2, and the map $F=\lim F_n$ maps G biholomorphically onto $\mathbf{C}^N$.

2. Two perturbation lemmas

Let $D\subset\mathbf{R}^n$ be a bounded open set and let $K=b\Delta\times\overline{D}$. We denote by $\mathcal{C}^1(K)$ the Banach space of all real continuous functions on K such that the partial derivatives of the function $(\theta,x)\mapsto f(e^{i\theta},x)$ exist on $\mathbf{R}\times D$ and extend continously to $\mathbf{R}\times\overline{D}$ with norm

$$\|f\|=\sup_{w\in K}|f(w)|+\sup_{\theta\in\mathbf{R},x\in D}\left|\frac{\partial}{\partial\theta}f(e^{i\theta},x)\right|+\sum_{k=1}^{n}\sup_{w\in b\Delta\times D}\left|\frac{\partial f}{\partial x_k}(w)\right|.$$

If $r>0$ we write $D(r)=\{z\in\mathbf{C}^N:|z_k|\le r\ (1\le k\le N)\}$ and, given j, $1\le j\le N$, we write $P_j(r)=\{z\in\mathbf{C}^N:|z_j|=1,\ |z_k|\le r\ (1\le k\le N,\ k\ne j)\}$. Suppose that $\varphi\in\mathcal{C}^1(P_j(r))$ is a positive function. Then we call the domain

$$\{z\in\mathbf{C}^N: z_j=ts,\ 0\le t<\varphi(z_1,\dots,z_{j-1},s,z_{j+1},\dots,z_N),\ s\in b\Delta,\ |z_k|<r \quad (1\le k\le N,\ k\ne j)\}$$

the standard domain over $P_j(r)$ given by φ or the standard domain over $P_j(r)$ bounded by Γ_φ; we also say that φ is associated with this domain. Here

$$\Gamma_\varphi=\{z\in\mathbf{C}^N: z_j=\varphi(z_1,\dots,z_{j-1},s,z_{j+1},\dots,z_N)s,\ s\in b\Delta,\ |z_k|\le r \quad (1\le k\le N,\ k\ne j)\}$$

is called *the smooth graph* over $P_j(r)$ given by φ. If $\varepsilon>0$ then the ε-neighbourhood of Γ_φ is the set of all smooth graphs Γ_ψ over $P_j(r)$ given by functions $\psi\in\mathcal{C}^1(P_j(r))$ such that $\|\psi-\varphi\|<\varepsilon$. Let D_φ and D_ψ be two standard domains over $P_j(r)$ given by φ, ψ respectively. We shall write $D_\varphi<D_\psi$ if $\varphi(w)<\psi(w)$ for each $w\in P_j(r)$.

The following lemma is elementary. It is not stated in full generality but rather in a form suitable for our applications.

Lemma 2.1. *Let $r>0$. Let Φ be a holomorphic automorphism of $\mathbf{C}^N$ and let $R>0$ be so large that $\Phi(D(r))\subset\subset D(R)$. Let $1\le k,j\le N$. Let $0<\alpha<R$, let $S=\{z\in D(R):|z_j|=\alpha\}$ and assume that $\Phi^{-1}(S)\cap D(r)$ is a graph over $P_k(r)$. Given $\varepsilon>0$ there is a $\delta>0$ such that if T is a graph over $P_j(R)$ in the δ-neighbourhood of S then $\Phi^{-1}(T)\cap D(r)$ is a graph over $P_k(r)$ belonging to the ε-neighbourhood of $\Phi^{-1}(S)\cap D(r)$.*

Proof. The conditions imply that for each $(z_1,\dots,z_{k-1},s,z_{k+1},\dots,z_N)$ such that $|s|=1$ and $|z_l|=r$ for at least one $l\neq k$, $1\le l\le N$, $\Phi^{-1}(S\cap\operatorname{Int}D(r))$, a closed submanifold of $\Phi^{-1}(\operatorname{Int}D(R))$, is transverse to the ray $\{(z_1,\dots,z_{k-1},ts,z_{k+1},\dots,z_N): t>0\}$. By compactness it follows that there is a slightly larger $r'>r$, $\Phi(D(r'))\subset\subset D(R)$, such that $\Phi^{-1}(S)\cap D(r')$ is still a smooth graph over $P_k(r')$. Thus a sufficiently small $\mathcal{C}^1$-perturbation of $\Phi^{-1}(S)$ will intersect $D(r)$ in a set of the form $L\cap D(r)$ where L is a small $\mathcal{C}^1$-perturbation of $\Phi^{-1}(S)\cap D(r')$. Provided that this perturbation is small enough $L\cap D(r)$ is a graph over $P_k(r)$ arbitrarily close to $\Phi^{-1}(S)\cap D(r)$. The details are left to the reader.

Lemma 2.2. *Let $0<R<\infty$. Given $\varepsilon>0$ there is an $\alpha_0>0$ such that for every $m\in\mathbf{N}$ and for every α, $0<\alpha<\alpha_0$, there is a function $\phi_{m,\alpha}\in\mathcal{C}^1(b\Delta\times(R\bar\Delta))$ such that*

$$\{(z,w)\in\mathbf{C}^2:|z^m+\alpha w|=1,\ |w|\le R\}=\{(\phi_{m,\alpha}(\zeta,w)\zeta,w):\zeta\in b\Delta,\ |w|\le R\}$$

and such that $\|\phi_{m,\alpha}-1\|<\varepsilon$.

Note that the equality in the lemma implies that $\{(z,w)\in\mathbf{C}^2:|z^m+\alpha w|<1, |w|<R\}=\{(t\zeta,w):0\le t<\varphi_{m,\alpha}(\zeta,w),\ \zeta\in b\Delta,\ |w|<R\}$. It will be important that α_0 depends only on R and ε and not on m.

Proof. Fix $m\in\mathbf{N}$. Assume that $\alpha R<\tau$ where $1>3\tau^2$. This implies that for each w, $|w|<R$, and each $\varphi\in\mathbf{R}$ there is a unique $r=r(\varphi,w)>0$ such that $z=re^{i\varphi}$ satisfies $|z^m+\alpha w|=1$. Indeed, $|r^me^{im\varphi}+\alpha w|^2=1$ gives $(r^m)^2+Ar^m+B=0$ where $A=e^{im\varphi}\alpha\bar w+e^{-im\varphi}\alpha w$, $B=\alpha^2w\bar w-1$. If $|w|\le R$ then

$$|A|<2\tau,\quad A^2-4B>4(1-2\tau^2)>0 \tag{2.1}$$

so $(A^2-4B)^{1/2}-|A|\ge2(1-2\tau^2)^{1/2}-2\tau>0$. Since we must have $2r^m=-A\pm(A^2-4B)^{1/2}$ it follows that $r=[(-A+(A^2-4B)^{1/2})/2]^{1/m}$ is the unique $r>0$ such that $z=re^{i\varphi}$ satisfies $|z^m+\alpha w|=1$. Now,

$$2mr^{m-1}\frac{\partial r}{\partial\varphi}=-\frac{\partial A}{\partial\varphi}+(A^2-4B)^{-1/2}\left(A\frac{\partial A}{\partial\varphi}-2\frac{\partial B}{\partial\varphi}\right),$$
$$2mr^{m-1}\frac{\partial r}{\partial w}=-\frac{\partial A}{\partial w}+(A^2-4B)^{-1/2}\left(A\frac{\partial A}{\partial w}-2\frac{\partial B}{\partial w}\right)$$

and

$$\frac{\partial A}{\partial\varphi}=m(i\alpha\bar{w}e^{im\varphi}-i\alpha we^{-im\varphi}),\quad \frac{\partial B}{\partial\varphi}=0,\quad \frac{\partial A}{\partial w}=\alpha e^{-im\varphi},\quad \frac{\partial B}{\partial w}=\alpha^2\bar{w}.$$

Let $|w|<R$. We have

$$\left|\frac{\partial A}{\partial\varphi}\right|\le 2m\tau,\quad \left|\frac{\partial A}{\partial w}\right|\le\alpha,\quad \left|\frac{\partial B}{\partial w}\right|\le\alpha\tau. \tag{2.2}$$

Moreover, $|z^m+\alpha w|=1$ implies that $1-\tau<r^m<1+\tau$ so

$$1-\tau<r^{m-1}<1+\tau. \tag{2.3}$$

Now, (2.1), (2.2) and (2.3) imply that

$$\left|\frac{\partial r}{\partial\varphi}\right|\le\frac{\tau}{1-\tau}(1+\tau(1-2\tau^2)^{-1/2}), \tag{2.4}$$

$$\left|\frac{\partial r}{\partial\bar{w}}\right|=\left|\frac{\partial r}{\partial w}\right|\le\frac{1}{2(1-\tau)}\left(\alpha+\frac{2\tau\alpha+2\tau\alpha}{2(1-2\tau^2)^{1/2}}\right). \tag{2.5}$$

Moreover, (2.3) implies that $1-\tau<r<1+\tau$. This, together with (2.4) and (2.5) proves that if $r=\phi_{m,\alpha}$ then the C^1-norm of $\phi_{m,\alpha}-1$ is arbitrarily small provided that τ and α are small enough. This estimate is independent of m. This completes the proof.

3. The induction lemma

If $\Theta:\mathbf{C}^N\to\mathbf{C}^N$ is a holomorphic map then we denote by $J\Theta$ the Jacobian matrix of Θ.

Lemma 3.1. *Let $0<q+3\tau<p_j<Q<R$ $(1\le j\le N)$ where $\tau>0$. Let $\varepsilon>0$, $\gamma>0$. Define $K=\{z\in\mathbf{C}^N:|z_j|\le p_j-\tau\ (1\le j\le N)\}$. There are $s_j>Q$ $(1\le j\le N-1)$ and a holomorphic automorphism $\Theta=(\Theta_1,\dots,\Theta_N)$ of $\mathbf{C}^N$ such that*

(i) *for each $j=1,2,\dots,N-1$ the set $D(R)\cap\{z\in\mathbf{C}^N:|\Theta_j(z)|=s_j\}$ is a smooth graph over $P_j(R)$ which belongs to the γ-neighbourhood of $D(R)\cap\{z\in\mathbf{C}^N:|z_j|=p_j\}$,*

(ii) *$|\Theta(z)-z|<\varepsilon$ $(z\in K)$,*

(iii) *$\|\Theta(z)\|\ge q$ $(z\notin K)$,*

(iv) *$\det((J\Theta)(z))=1$ $(z\in\mathbf{C}^N)$,*

(v) *$K\subset\{z\in\mathbf{C}^N:|\Theta_j(z)|<s_j\ (1\le j\le N-1)\}$.*

Proof. Part 1. With no loss of generality assume that $\varepsilon<\tau$. We show that there are $q_2,\dots,q_N>Q$ and $M_2,\dots,M_N\in\mathbf{N}$ such that if $\Phi=(\Phi_1,\dots,\Phi_N)$ is the holomorphic automorphism of $\mathbf{C}^N$ given by

$$\Phi_1(z)=z_1,\quad \Phi_j(z)=z_j+q_j\left(\frac{z_{j-1}}{p_{j-1}}\right)^{M_j}\quad (2\le j\le N)$$

then

(a) for each $j=2,\dots,N$ the set $D(R)\cap\{z\in\mathbf{C}^N:|\Phi_j(z)|=q_j\}$ is a smooth graph over $P_{j-1}(R)$ which belongs to the $\frac{1}{2}\gamma$-neighbourhood of $D(R)\cap\{z\in\mathbf{C}^N:|z_{j-1}|=p_{j-1}\}$,

(b) $|\Phi(z)-z|<\frac{1}{2}\varepsilon$ $(z\in K)$,

(c) if $z\notin K$ then at least one of the inequalities $|\Phi_j(z)|\ge p_j-2\tau$, $1\le j\le N$, holds,

(d) $K\subset\{z\in\mathbf{C}^N:|\Phi_j(z)|<q_j\ (2\le j\le N)\}$.

Part 2. Assume that we have proved Part 1. Choose $P>q_j$ $(2\le j\le N)$ and then $R'>0$ so large that $\Phi(D(R))\subset\subset D(R')$. Let $\eta>0$ be very small. Choose $q_1>Q$ so large that $|\Phi_1(z)|<q_1$ $(z\in K)$ and then choose $\tilde\tau$, $0<\tilde\tau<\tau$ so small that $\Phi(K)\subset\widetilde{K}$ where

$$\widetilde{K}=\{z:|z_j|\le q_j-\tilde\tau\ (1\le j\le N)\}.$$

By Part 1 there are $s_1,\dots,s_{N-1}>P$ and $N_1,\dots,N_{N-1}\in\mathbf{N}$ such that if $\Psi=(\Psi_1,\dots,\Psi_N)$ is the holomorphic automorphism of $\mathbf{C}^N$ given by

$$\Psi_N(z)=z_N,\quad \Psi_j(z)=z_j+s_j\left(\frac{z_{j+1}}{q_{j+1}}\right)^{N_j}\quad (1\le j\le N-1)$$

then

(a′) for each $j=1,\dots,N-1$ the set $D(R')\cap\{z\in\mathbf{C}^N:|\Psi_j(z)|=s_j\}$ is a graph over $P_{j+1}(R')$ which belongs to the η-neighbourhood of $D(R')\cap\{z\in\mathbf{C}^N:|z_{j+1}|=q_{j+1}\}$,

(b′) $|\Psi(z)-z|<\frac{1}{2}\varepsilon$ if $z\in\widetilde{K}$,

(c′) if $z\notin\widetilde{K}$ then at least one of the inequalities $|\Psi_j(z)|\ge q_j-2\tilde\tau$ $(1\le j\le N)$ holds,

(d′) $\widetilde{K}\subset\{z\in\mathbf{C}^N:|\Psi_j(z)|<s_j\ (1\le j\le N-1)\}$.

Put $\Theta=\Psi\circ\Phi$. By Lemma 2.1 one can choose $\eta>0$ so small that (i) will hold whenever Ψ satisfies (a′). Let $z\in K$. Since $\Phi(K)\subset\widetilde{K}$ it follows by (b) and (b′) that $|\Theta(z)-z|=|\Psi(\Phi(z))-\Phi(z)+\Phi(z)-z|<\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon$ which proves (ii). Further, if $z\in K$ then $\Phi(z)\in\widetilde{K}$ and by (d′) (v) follows. Part (iv) is obviously satisfied. Finally, let $z\notin K$. By (c), $\|\Phi(z)\|>q+\tau$. If $\Phi(z)\in\widetilde{K}$ then (b′) implies that $|\Psi(\Phi(z))-\Phi(z)|<$

$\varepsilon<\tau$ so $\|\Theta(z)\|>q$. If $\Phi(z)\notin\widetilde{K}$ then (c′) implies that $\|\Psi(\Phi(z))\|>Q-2\tilde{\tau}>q$ since $\tilde{\tau}<\tau$ which proves (iii).

Part 3. It remains to prove Part 1. For each j, $1\leq j\leq N-1$, one uses Lemma 2.2 to choose $q_j>Q$ so large that for any $M_j\in\mathbf{N}$

$$D(R)\cap\left\{z\in\mathbf{C}^N:\left|\frac{z_j}{q_j}+\left(\frac{z_{j-1}}{p_{j-1}}\right)^{M_j}\right|=1\right\}$$

is a graph over $P_{j-1}(R)$ which belongs to the $\frac{1}{2}\gamma$-neighbourhood of $D(R)\cap\{z\in\mathbf{C}^N: |z_{j-1}|=p_{j-1}\}$. Now, choose $M_2,\dots,M_N\in\mathbf{N}$ so large that

$$q_j\left(\frac{p_{j-1}-\tau}{p_{j-1}}\right)^{M_j}<\min\{\varepsilon/2N,\tau\}. \tag{3.1}$$

Define Φ as in Part 1. Part (a) is clearly satisfied. Since $\Phi_1(z)=z_1$ and $\Phi_j(z)-z_j=q_j(z_{j-1}/p_{j-1})^{M_j}$ (b) follows by (3.1). Let $z\notin K$. Then $|z_j|>p_j-\tau$ for at least one j, $1\leq j\leq N$. If $|z_1|>p_1-\tau$ then we have $|\Phi_1(z)|=|z_1|>p_1-2\tau$. Suppose that $|z_1|\leq p_1-\tau$. Then there is some j, $2\leq j\leq N$, such that $|z_{j-1}|\leq p_{j-1}-\tau$ and $|z_j|>p_j-\tau$. It follows that

$$|\Phi_j(z)|=|z_j+q_j(z_{j-1}/p_{j-1})^{M_j}|\geq p_j-\tau-q_j((p_{j-1}-\tau)/p_{j-1})^{M_j}>p_j-2\tau.$$

This proves (c). Finally, let $z\in K$. Then, if $2\leq j\leq N$ we have

$$\begin{aligned}|\Phi_j(z)|&\leq|z_j|+|q_j(z_{j-1}/p_{j-1})^{M_j}|\leq p_j-\tau+q_j((p_{j-1}-\tau)/p_{j-1})^{M_j}\\&<p_j-\tau+\tau=p_j<q_j\end{aligned}$$

which proves (d). This completes the proof.

4. Proof of Theorem 1.2

Part 1. Suppose that Q_n is an increasing sequence of positive numbers converging to $+\infty$ and that $s_{n1},\dots,s_{nN}$ are sequences, $s_{0j}=1$ $(1\leq j\leq N-1)$ such that

$$Q_n<s_{nj}<Q_{n+1}-1\quad(n\in\mathbf{N},\ 1\leq j\leq N), \tag{A1}$$

and that Θ_n is a sequence of holomorphic automorphisms of $\mathbf{C}^N$. Put $F_0=\mathrm{Id}$, $F_n=\Theta_n\circ\Theta_{n-1}\circ\dots\circ\Theta_1=(F_{n1},F_{n2},\dots,F_{nN})$ $(n\in\mathbf{N})$. For each $n\in\mathbf{N}$ and each j, $1\leq j\leq N-1$, consider the unbounded domain

$$G_{nj}=\{z\in\mathbf{C}^N:|F_{nj}(z)|<s_{nj}\}$$

whose boundary is the real hypersurface

$$\Gamma_{nj} = \{z \in \mathbf{C}^N : |F_{nj}(z)| = s_{nj}\}.$$

Our goal is to choose the constants s_{nj} and the automorphisms Θ_n in such a way that, as $n \to \infty$, for each j, $1 \le j \le N-1$, the domains G_{nj} converge to a domain G_j and the surfaces Γ_{nj} converge to a surface Γ_j of class $\mathcal{C}^1$ such that $\Gamma_j = bG_j$; moreover, we want to do this in such a way that $F = \lim F_n$ maps $\bigcap_{n=1}^{N-1} G_j$ biholomorphically onto $\mathbf{C}^N$.

Part 2. Note that in the definition of G_{nj} and Γ_{nj} the constants s_{nN} play no role. They come into play when we define, for each $n \in \mathbf{N} \cup \{0\}$, the compact sets

$$\widetilde{K}_n = \{z \in \mathbf{C}^N : |z_j| \le s_{nj} \ (1 \le j \le N)\},$$
$$K_n = F_n^{-1}(\widetilde{K}_n) = \{z \in \mathbf{C}^N : |F_{nj}(z)| \le s_{nj} \ (1 \le j \le N)\}.$$

Our wish is to choose s_{nj} and Θ_n in such a way that $f = \lim F_n$ maps $\bigcup_{n=1}^{\infty} K_n$ biholomorphically onto $\mathbf{C}^N$ and that $\bigcup_{n=1}^{\infty} K_n = \bigcap_{j=1}^{N-1} G_j$. To this end we assume first that for each $n \in \mathbf{N}$

(B1) $K_{n-1} \subset\subset K_n$,

(B2) $|\Theta_n(w) - w| < 2^{-n}$ $(w \in \widetilde{K}_{n-1})$,

(B3) $\|\Theta_n(w)\| \ge Q_{n-1}$ $(w \notin \widetilde{K}_{n-1})$,

(B4) $\det((J\Theta)(z)) = 1$ $(z \in \mathbf{C}^N)$.

Define $G = \bigcup_{n=1}^{\infty} K_n$. We show that (B1)–(B4) imply that F_n converges uniformly on compacta in G to a limit F which maps G biholomorphically onto $\mathbf{C}^N$.

If $z \in K_n$ then $F_n(z) \in \widetilde{K}_n$ so by (B2), $|\Theta_{n+1}(F_n(z)) - F_n(z)| < 2^{-(n+1)}$ which implies that

$$|F_{n+1}(z) - F_n(z)| < 2^{-(n+1)} \quad (z \in K_n) \tag{4.1}$$

so by (B1), F_n converges uniformly on compacta in G to a map F. By (B4), $\det((JF)(z)) = 1$ $(z \in G)$ so F is regular on G. To show that F is one to one on G we use a classical argument also used by Stensønes in [S]: It is easy to see that given $r > 0$ and $M < \infty$ there is a $\varrho > 0$ such that whenever $\Phi: r\mathbf{B} \to \Phi(r\mathbf{B})$ is a biholomorphic map such that $\Phi(0) = 0$, $\det((J\Phi)(0)) = 1$ and $|\Phi(z)| < M$ $(z \in r\mathbf{B})$ we have $\varrho\mathbf{B} \subset \Phi(r\mathbf{B})$ [BM, p. 51]. Suppose that there are $u, w \in G$, $u \ne w$, such that $F(u) = F(w)$. Let $U, W \subset G$ be disjoint closed balls centered at u, w, respectively. The preceding discussion implies that there is a $\varrho > 0$ such that for each $n \in \mathbf{N}$, $F_n(u) + \varrho\mathbf{B} \subset F_n(U)$ and $F_n(w) + \varrho\mathbf{B} \subset F_n(W)$. Since $F_n(u)$ and $F_n(w)$ have the same limit it follows that for n large enough $F_n(U) \cap F_n(W)$ is nonempty which contradicts the fact that F_n is one to one. This proves that F is one to one on G.

Suppose that $z \in K_{n+1} \setminus K_n$. Then $F_n(z) \notin \widetilde{K}_n$ so by (B3), $\|\Theta_{n+1}(F_n(z))\| > Q_n$, that is, $\|F_{n+1}(z)\| > Q_n$. By (B1), $z \in K_{n+1} \cap K_{n+2} \cap \ldots$ so (B2) implies that $\|F(z)\| \geq \|F_{n+1}(z)\| - \|F_{n+2}(z) - F_{n+1}(z)\| - \ldots > Q_n - 2^{-(n+2)} - 2^{-(n+3)} - \ldots > Q_n - 1$. Since $Q_n \to +\infty$ it follows that $F: G \to \mathbf{C}^N$ is a proper map. Thus $F: G \to C^N$ is an embedding and so $F(G)$ is a closed submanifold of $\mathbf{C}^N$. It follows that $F(G) = \mathbf{C}^N$, that is, F maps G biholomorphically onto $\mathbf{C}^N$.

Part 3. To get the convergence of G_{nj} to G_j, of Γ_{nj} to Γ_j, and to show that $G = \bigcap_{j=1}^{N-1} G_j$ we first describe what we mean by convergence.

We shall choose a sequence of compact sets $D_0 \subset\subset D_1 \subset\subset \ldots$, $\bigcup_{k=1}^{\infty} D_k = \mathbf{C}^N$, and we will, for each k, study the convergence of $D_k \cap G_{nj}$, $D_k \cap \Gamma_{nj}$. To simplify the geometry, each D_k will be a polydisc, however, not in the original coordinate system on $\mathbf{C}^N$ but in the coordinate system given by the automorphism F_k. Let us describe this.

Fix $n \in \mathbf{N}$. The map F_n defines new coordinates on $\mathbf{C}^N$ that we denote by $z_1^n, \ldots, z_N^n$:

$$z_j^n(z) = F_{nj}(z) \quad (1 \leq j \leq N).$$

In this coordinate system, for $\varrho > 0$, we define

$$D^n(\varrho) = \{(z_1^n, \ldots, z_N^n) : |z_j^n| \leq \varrho \ (1 \leq j \leq N)\}$$

and, for $1 \leq j \leq N$,

$$P_j^n(\varrho) = \{(z_1^n, \ldots, z_N^n) : |z_j^n| = 1, \ |z_k^n| \leq \varrho \ (1 \leq k \leq N, \ k \neq j)\}.$$

If $\varphi \in C^1(P_j^n(\varrho))$ is a positive function then, again, we call the set

$$\Gamma_\varphi = \{(z_1^n, \ldots, z_N^n) : z_j^n = \varphi(z_1^n, \ldots, z_{j-1}^n, s, z_{j+1}^n, \ldots, z_N^n)s, \ s \in b\Delta, \ |z_k^n| \leq \varrho \\ (1 \leq k \leq N, \ k \neq j)\}$$

the graph over $P_j^n(\varrho)$ given by φ. It is clear what we mean by the γ-neighbourhood of such a graph. Further, we again call the domain

$$\{(z_1^n, \ldots, z_N^n) : z_j^n = ts, \ 0 \leq t < \varphi(z_1^n, \ldots, z_{j-1}^n, s, z_{j+1}^n, \ldots, z_N^n), \ s \in b\Delta, \ |z_k^n| < \varrho \\ (1 \leq k \leq N, \ k \neq j)\}$$

the standard domain over $P_j^n(\varrho)$ given by φ or bounded by Γ_φ.

Assume that there is a sequence R_n, $R_0 = R$, such that for each $n \in \mathbf{N}$,

(C1) $D^{n-1}(R_{n-1}) \subset\subset D^n(R_n)$, $n\mathbf{B} \subset D^n(R_n)$, (C2) for each j, $1 \leq j \leq N-1$, and for each k, $0 \leq k \leq n$, $G_{nj} \cap \operatorname{Int} D^k(R_k)$ is a standard domain over $P_j^k(R_k)$ bounded by $\Gamma_{nj} \cap D^k(R_k)$, a smooth graph over $P_j^k(R_k)$,

(C3) for each j, $1 \leq j \leq N-1$, and for each k, $0 \leq k \leq n-1$, $G_{n-1,j} \cap \operatorname{Int} D^k(R_k) < G_{nj} \cap \operatorname{Int} D^k(R_k)$.

Our next assumption tells, for each j, $1 \leq j \leq N-1$, how far apart $\Gamma_{n-1,j}$ and Γ_{nj} are as graphs over $P_j^k(R_k)$, $0 \leq k \leq n-1$. It implies, in particular, that each of the sequences of associated functions converges in $\mathcal{C}^1$-sense.

Assume that there is a decreasing sequence γ_n of positive numbers such that for each $n \in \mathbf{N}$

(C4) for each j, $1 \leq j \leq N-1$,

(C4.1) $\Gamma_{nj} \cap D^{n-1}(R_{n-1})$ is contained in the $(\gamma_{n-1}/2)$-neighbourhood of $\Gamma_{n-1,j} \cap D^{n-1}(R_{n-1})$,

(C4.2) $\Gamma_{nj} \cap D^{n-2}(R_{n-2})$ is contained in the $(\gamma_{n-2}/2^2)$-neighbourhood of $\Gamma_{n-1,j} \cap D^{n-2}(R_{n-2})$,

...

(C4.(n-1)) $\Gamma_{nj} \cap D^1(R_1)$ is in the $(\gamma_1/2^{n-1})$-neighbourhood of $\Gamma_{n-1,j} \cap D^1(R_1)$,

(C4.n) $\Gamma_{nj} \cap D^0(R_0)$ is in the $(\gamma_0/2^n)$-neighbourhood of $\Gamma_{n-1,j} \cap D^0(R_0)$.

Finally, we assume that for each $n \in \mathbf{N} \cup \{0\}$ the constant s_{nN} is so large that K_n covers the part of $\bigcap_{j=1}^{N-1} G_{nj}$ contained in $D^n(R_n)$:

(C5) for each $n \in \mathbf{N} \cup \{0\}$

$$(\operatorname{Int} K_n) \cap \operatorname{Int} D^n(R_n) = \left(\bigcap_{j=1}^{N-1} G_{nj} \right) \cap \operatorname{Int} D^n(R_n).$$

Part 4. Parts (C3) and (C4) imply that for each $k \in \mathbf{N} \cup \{0\}$ and each j, $1 \leq j \leq N-1$,

$$E_{kj} = \bigcup_{n=k}^{\infty} [G_{nj} \cap \operatorname{Int} D^k(R_k)]$$

is a standard domain over $P_j^k(R_k)$ bounded by S_{kj}, a graph over $P_j^k(R_k)$ which is the limit of the sequence $\Gamma_{nj} \cap D^k(R_k)$, $n \geq k$, as $n \to \infty$. Suppose that $l \leq k$. Note that $D^l(R_l) \subset D^k(R_k)$. So for each j, $1 \leq j \leq N-1$, we have

$$\begin{aligned} E_{lj} &= \bigcup_{n=l}^{\infty} [G_{nj} \cap \operatorname{Int} D^l(R_l)] = \bigcup_{n=k}^{\infty} [G_{nj} \cap \operatorname{Int} D^l(R_l)] \\ &= \bigcup_{n=k}^{\infty} [G_{nj} \cap \operatorname{Int} D^l(R_l) \cap \operatorname{Int} D^k(R_k)] = E_{kj} \cap \operatorname{Int} D^l(R_l). \end{aligned}$$

In particular, $E_{lj}\subset E_{kj}$ $(l\leq k)$. For each j, $1\leq j\leq N-1$, define $G_j=\bigcup_{l=0}^{\infty} E_{lj}$. Then

$$G_j\cap \operatorname{Int} D^k(R_k)=\bigcup_{l=k}^{\infty} G_{lj}\cap \operatorname{Int} D^k(R_k)=\bigcup_{l=k}^{\infty} E_{kj}=E_{kj}.$$

Further, for each $k\in\mathbf{N}$ (C5) implies that

$$\begin{aligned}\left(\bigcup_{n=k}^{\infty} \operatorname{Int} K_n\right)\cap \operatorname{Int} D^k(R_k) &= \bigcup_{n=k}^{\infty}\left(\bigcap_{j=1}^{N-1} G_{nj}\cap \operatorname{Int} D^k(R_k)\right)\\ &= \bigcap_{j=1}^{N-1}\bigcup_{n=k}^{\infty} G_{nj}\cap \operatorname{Int} D^k(R_k)\\ &= \bigcap_{j=1}^{N-1} E_{kj}=\bigcap_{j=1}^{N-1} G_j\cap \operatorname{Int} D^k(R_k).\end{aligned}$$

By (C1) $\bigcup_{k=0}^{\infty} \operatorname{Int} D^k(R_k)=\mathbf{C}^N$ so it follows by (B1) that $\bigcup_{n=1}^{\infty} K_n=\bigcap_{j=1}^{N-1} G_j$. Since $\bigcup_{k=0}^{\infty} \operatorname{Int} D^k(R_k)=\mathbf{C}^N$ it follows also that for each j, $1\leq j\leq N-1$, $bG_j=\bigcup_{k=0}^{\infty}[(\operatorname{Int} D^k(R_k))\cap bG_j]$. Since $\operatorname{Int} D^k(R_k)$ is open it follows that $(\operatorname{Int} D^k(R_k))\cap bG_j$ consists of those boundary points of $(\operatorname{Int} D^k(R_k))\cap G_j=E_{kj}$ which are contained in $\operatorname{Int} D^k(R_k)$, that is, $(\operatorname{Int} D^k(R_k))\cap bG_j=(\operatorname{Int} D^k(R_k))\cap S_{kj}$. This shows that for each j, $1\leq j\leq N-1$, bG_j is of class $\mathcal{C}^1$ and such that for each $k\in\mathbf{N}\cup\{0\}$, $bG_j\cap D^k(R_k)$ is a smooth graph over $P_j^k(R_k)$. Moreover, by (C4), $bG_j\cap D^0(R_0)$ is in the γ_0-neighbourhood of $\{z\in\mathbf{C}^N:|z_j|=1\}\cap D^0(R_0)$. Provided that γ_0 is small enough this completes the proof of Theorem 1.2.

5. Proof of the induction step and the completion of the proof of Theorem 1.2

To complete the proof of Theorem 1.2 we shall show that there are

- an increasing sequence Q_n converging to $+\infty$,
- sequences s_{nj},
- a sequence R_n,
- a decreasing sequence γ_n,

such that (A1), (B1)–(B4) and (C1)–(C5) are satisfied. We shall do this by induction, using Lemma 3.1. We first make a simple remark. Suppose that f is a holomorphic function on $\mathbf{C}^N$ whose gradient is everywhere different from zero. Assume that $\Gamma=\{z\in\mathbf{C}^N:|f(z)|=\alpha\}\cap D(R)$ is a graph over $P_j(R)$ for some j, $1\leq j\leq N$.

Then $\{z\in\mathbf{C}^N:|f(z)|<\alpha\}\cap\operatorname{Int}D(R)$ is the standard domain over $P_j(R)$ bounded by Γ. This is a simple consequence of the maximum principle.

To begin the induction let $s_{01}=...=s_{0,N-1}=1$. Let $R_0=R$ and choose $s_{0N}>1$ so large that (C5) holds for $n=0$. Let $Q_0=\frac{1}{2}$ and choose $\gamma_0>0$ small. Choose Q_1 with $Q_1-1>s_{01},...,s_{0N}$. Set $\nu=\frac{1}{4}\gamma_0$. Lemma 3.1 gives $s_{11},...,s_{1,N-1}>Q_1$ and Θ_1, a holomorphic automorphism of $\mathbf{C}^N$ which satisfies (B2), (B3), (B4) for $n=1$, such that if $F_1=\Theta_1$ then for each j, $1\le j\le N-1$, the set $\Gamma_{1j}\cap D^0(R_0)$ is a smooth graph over $P_j^0(R_0)$ which is in the $\frac{1}{2}\nu$-neighbourhood of $\{z\in\mathbf{C}^N:|z_j|=s_{0j}+\nu\}\cap D^0(R_0)$. This implies that (C2), (C3) and (C4) are satisfied for $n=1$. Choose $R_1>0$ so that (C1) holds for $n=1$ and then $s_{1N}>0$ so large that (C5) and (B1) hold for $n=1$. Finally, choose Q_2 with $Q_2-1>s_{11},s_{12},...,s_{1N}$ so that (A1) holds for $n=1$.

Let $m\in\mathbf{N}$. Suppose that we have already constructed $Q_1,...,Q_{m+1}$ and s_{nj}, $0\le n\le m$, $1\le j\le N$ such that (A1) holds for $1\le n\le m$, automorphisms $\Theta_1,...,\Theta_m$ of $\mathbf{C}^N$ such that (B1)–(B4) hold for $1\le n\le m$, positive numbers $R_0=R,R_1,...,R_m$ such that (C1) holds for $1\le n\le m$, and $\gamma_0,...,\gamma_{m-1}$, $\gamma_0>\gamma_1>...>\gamma_{m-1}>0$, such that (C2)–(C5) hold for $n=m$.

Choose γ_m, $0<\gamma_m<\gamma_{m-1}$, and then choose $\nu>0$ so small that for each j, $1\le j\le N-1$, $\{z\in\mathbf{C}^N:|F_{mj}(z)|=s_{mj}+\nu\}\cap D^k(R_k)$ is in the $\frac{1}{2}(\gamma_k/2^{n-k+1})$-neighbourhood of $\Gamma_{mj}\cap D^k(R_k)$ $(0\le k\le m)$. This is possible by Lemma 2.1. Now, Lemma 3.1 together with Lemma 2.1 produces $s_{m+1,1},...,s_{m+1,N-1}>Q_{m+1}$ and a holomorphic automorphism Θ_{m+1} such that (B2), (B3) and (B4) hold for $n=m+1$ and such that, if $F_{m+1}=\Theta_{m+1}\circ F_m$, (C2) holds for $n=m+1$, and for each k, $0\le k\le m$, and j, $1\le j\le N-1$, $\Gamma_{m+1,j}\cap D^k(R_k)$ is in the $\frac{1}{2}(\gamma_k/2^{n-k+1})$-neighbourhood of $\{z\in\mathbf{C}^N: |F_{mj}(z)|=s_{mj}+\nu\}\cap D^k(R_k)$ and so close to $\{z\in\mathbf{C}^N:|F_{mj}(z)|=s_{mj}+\nu\}\cap D^k(R_k)$ that (C3) is satisfied for $n=m+1$ (this choice implies also that (C4) is satisfied for $n=m+1$) and that $K_m\subset\bigcap_{j=1}^{N-1}G_{m+1,j}$.

Choose $R_{n+1}>0$ so large that (C1) holds for $n=m+1$ and then choose $s_{m+1,N}$ so large that (C5) and (B1) hold for $n=m+1$. Finally, choose Q_{m+2} with $Q_{m+2}-1>s_{m+1,1},...,s_{m+1,N}$ so that (A1) holds for $n=m+1$. This completes the proof of the induction step. Theorem 1.2 is proved.

6. Remarks

In the induction process we were able to choose the decreasing sequence γ_n arbitrarily. With a little extra care one can choose γ_n decreasing to zero fast enough to achieve that the surfaces $\Gamma_1,...,\Gamma_{N-1}$ meet transversely in the sense that if $w\in\Gamma_{j_1}\cap...\Gamma_{j_l}$ then the (nonzero) perpendicular vectors to $\Gamma_{j_1},...,\Gamma_{j_l}$ at w are linearly independent. In particular, in this case Γ_j and Γ_k intersect transversely for every j, k, $1\leq j,k\leq N$, $j\neq k$.

Once we know that F maps G onto $\mathbf{C}^N$ then G must be the domain of (uniform on compacta) convergence of F_n. If there were an open set W, $G\subset W$, such that F_n would converge uniformly on compacta in W then by the argument used in Section 4, Part 2, F would be one to one on W which, if $W\neq G$, would contradict the fact that $F(G)=\mathbf{C}^N$.

The author has no doubt that one could prove $\mathcal{C}^\infty$-versions of Theorems 1.1 and 1.2. This would complicate things considerably and we shall not attempt to do this.

It does not seem that using the same method one could prove that Δ^N embeds holomorphically into $\mathbf{C}^{N+1}$.

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