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On the dynamics of composite entire functions

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Abstract. Let f and g be nonlinear entire functions. The relations between the dynamics of $f\circ g$ and $g\circ f$ are discussed. Denote by $\mathcal{J}(\cdot)$ and $\mathcal{F}(\cdot)$ the Julia and Fatou sets. It is proved that if $z\in\mathbf{C}$, then $z\in\mathcal{J}(f\circ g)$ if and only if $g(z)\in\mathcal{J}(g\circ f)$; if U is a component of $\mathcal{F}(f\circ g)$ and V is the component of $\mathcal{F}(g\circ f)$ that contains $g(U)$, then U is wandering if and only if V is wandering; if U is periodic, then so is V and moreover, V is of the same type according to the classification of periodic components as U. These results are used to show that certain new classes of entire functions do not have wandering domains.

1. Introduction and main results

The *Fatou set* $\mathcal{F}(f)$ of a nonlinear entire (or rational) function f is the subset of the complex plane (or Riemann sphere) where the iterates f^n of f form a normal family. The complement of $\mathcal{F}(f)$ is called the *Julia set* and denoted by $\mathcal{J}(f)$. The Fatou set is open and completely invariant; that is, $z\in\mathcal{F}(f)$ if and only if $f(z)\in\mathcal{F}(f)$. The Julia set is closed and also completely invariant. It is also known to be the closure of the set of repelling periodic points. If U_0 is a component of $\mathcal{F}(f)$, then $f^n(U_0)$ lies in some component U_n of $\mathcal{F}(f)$ and $U_n\setminus f^n(U_0)$ is either empty or contains exactly one point by a result of Heins [22]. If $U_n\neq U_m$ for all $n\neq m$, then U_0 is called a *wandering domain* of f. Otherwise U_0 is called *preperiodic* and if $U_n=U_0$ for some $n\in\mathbf{N}$, then U_0 is called *periodic*. Sullivan [30] proved that rational functions do not have wandering domains. Transcendental entire functions, however, may have wandering domains, see [2], [3], [4], [16], [30], but various classes of entire functions without wandering domains are known [3], [6], [7], [9], [12], [13], [18], [21], [28].

Already before Sullivan's work a classification of periodic components of $\mathcal{F}(f)$ was known. Let f be an entire function and U_0 a periodic component of $\mathcal{F}(f)$, say

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$U_n=U_0$. Then one of the following possibilities holds:

- There exists $z_0\in U_0$ such that $f^{nm}|_{U_0}\to z_0$ as $m\to\infty$, $f^n(z_0)=z_0$ and $|(f^n)'(z_0)|<1$. Then U_0 is called an *attracting domain* and z_0 is called an *attracting periodic point.*
- There exists $z_0\in\partial U_0$ such that $f^{nm}|_{U_0}\to z_0$ as $m\to\infty$, $f^n(z_0)=z_0$ and $(f^n)'(z_0)=1$. Then U_0 is called a *parabolic domain* and z_0 is called a *parabolic periodic point.*
- There exists a conformal map $\phi\colon\{z\in\mathbf{C}:|z|<1\}\to U_0$ and $\alpha\in\mathbf{R}\backslash\mathbf{Q}$ such that $\phi^{-1}(f^n(\phi(z)))=e^{2\pi i\alpha}z$ for $|z|<1$. With $z_0=\phi(0)$ we have $f^n(z_0)=z_0$ and $(f^n)'(z_0)=e^{2\pi i\alpha}$. Then U_0 is called a *Siegel disc.*
- The sequence $f^{nm}|_{U_0}\to\infty$ as $m\to\infty$. Then U_0 is called a *Baker domain.*

We note here that in the case of a Siegel disc U_0 the limit functions of the family $\{f^n|_{U_0}\}$ are all non-constant, while in the other cases they are all constant. We also note that if U_0 is periodic of period $n\geq 2$, then the components $U_1,\dots,U_{n-1}$ of the periodic cycle which U_0 belongs to are of the same type according to the above classification.

There is a similar classification for rational functions. Here Baker domains do not play a special role, but there is the additional possibility of a *Herman ring.* As an introduction to iteration theory, we recommend Beardon's [8], Carleson and Gamelin's [14], and Steinmetz's [29] books as well as Milnor's [25] lecture notes for rational functions and the survey articles of Baker [5] and Erëmenko and Lyubich [17] for rational and entire functions. The iteration theory of transcendental meromorphic functions is surveyed in [10]. The classical references are Fatou [19] and Julia [23] for rational and Fatou [20] for transcendental entire functions.

Baker and Singh [7] proved that if $g(z)=a+b\exp(2\pi iz/c)$ and if f is entire, then $f\circ g$ has no wandering domains if $g\circ f$ has no wandering domains. They used this to show that $\exp(\exp z)-\exp z$ does not have wandering domains. Here we compare the dynamics of $f\circ g$ and $g\circ f$ without assuming that g has the special form above. Our main results are as follows.

Theorem 1. *Let f and g be nonlinear entire functions and $z\in\mathbf{C}$. Then $z\in\mathcal{J}(f\circ g)$ if and only if $g(z)\in\mathcal{J}(g\circ f)$.*

It follows that if U_0 is a component of $\mathcal{F}(f\circ g)$, then $g(U_0)$ is contained in a component V_0 of $\mathcal{F}(g\circ f)$. The result of Heins [22] already mentioned implies that $V_0\backslash g(U_0)$ contains at most one point.

Theorem 2. *Let f and g be nonlinear entire functions. Let U_0 be a component of $\mathcal{F}(f\circ g)$ and let V_0 be the component of $\mathcal{F}(g\circ f)$ that contains $g(U_0)$. Then*

(i) *U_0 is wandering if and only if V_0 is wandering,*

(ii) *if U_0 is periodic, then so is V_0, moreover, V_0 is of the same type according to the classification of periodic components as U_0.*

In particular it follows that $f\circ g$ has wandering domains if and only if $g\circ f$ has wandering domains. We use Theorem 2 to show that certain new classes of entire functions do not have wandering domains.

Theorem 3. *Let $F=\{e^{iz}\pm z,\ i(e^z\pm z),\ \sin z\pm z,\ \cos z\pm z\}$ and $G=\{g_1\circ g_2\circ\ldots\circ g_m;\ g_j=\sin z$ or $\cos z,\ j=1,2,\ldots,m,\ m\in\mathbf{N}\}$. Then for any $f\in F$ and $g\in G$, $f\circ g$ has no wandering domains.*

For an entire function f, we denote by $A(f)$ the set of asymptotic values of f, by $C(f)$ the set of critical values of f, and by $\operatorname{sing}(f^{-1})$ the set of singularities of the inverse function of f. Then $\operatorname{sing}(f^{-1})=A(f)\cup C(f)$.

Theorem 4. *Let f be a real entire function satisfying $|f(x)|\le|x|$ for $-1\le x\le 1$. Suppose that $\operatorname{sing}(f^{-1})\subset\mathbf{R}$. Then $f(\sin z)$ does not have wandering domains.*

Here an entire function f is called real if $f(\mathbf{R})\subset\mathbf{R}$. To give specific examples of entire functions which Theorem 4 applies to we recall that the Pólya–Laguerre class LP consists of all entire functions f which have a representation

$$f(z)=\exp(-az^2+bz+c)z^n\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left(\frac{z}{z_k}\right),$$

where $a,b,c\in\mathbf{R}$, $a\ge 0$, $n\in\mathbf{N}_0$, $z_k\in\mathbf{R}\setminus\{0\}$ for all $k\in\mathbf{N}$, and $\sum_{k=1}^{\infty}|z_k|^{-2}<\infty$. In particular, real entire functions of order less than two with only real zeros are in LP. Pólya [27] and Laguerre [24] proved that an entire function f is in LP if and only if there is a sequence of real polynomials with only real zeros which converges locally uniformly to f.

Proposition 1. *Let $f=f_1\circ f_2\circ\ldots\circ f_n$ where $f_1,f_2,\ldots,f_n\in$ LP. Then we have $\operatorname{sing}(f^{-1})\subset\mathbf{R}$. In particular, $\operatorname{sing}(f^{-1})\subset\mathbf{R}$ if $f\in$ LP.*

We note that $f(z)=z\cos z\in$ LP and obtain from Theorem 4 and Proposition 1 that $\sin z\cos(\sin z)$ does not have wandering domains. More generally, an odd function f is in LP if and only if it has the form

$$f(z)=\exp(-az^2+c)z^n\prod_{k=1}^{\infty}\left(1-\frac{z^2}{z_k^2}\right)$$

with $n\in\{1,3,5,\ldots\}$ and a,c,z_k as above. It is easy to see that the hypothesis of Theorem 4 are satisfied if $c\le 0$ and $|z_k|\ge 1/\sqrt{2}$ for all $k\in\mathbf{N}$.

We remark that Theorems 3 and 4 are just examples of applications of Theorem 2 and that we have not tried to state these results in their most general forms. Using Theorem 2 one can find more classes of entire functions without wandering domains.

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2. Proof of the theorems

We need the following lemma.

Lemma 1. *Let f and g be nonlinear entire functions and $z_0 \in \mathbf{C}$. If z_0 is a periodic point of $f \circ g$, then $g(z_0)$ is a periodic point of $g \circ f$.*

Proof. Let $h = f \circ g$ and $k = g \circ f$. Suppose $h^n(z_0) = z_0$ where $n \in \mathbf{N}$. Then $g(z_0) = g(h^n(z_0)) = k^n(g(z_0))$.

Proof of Theorem 1. Let $z_0 \in \mathcal{J}(f \circ g)$. Since the Julia set is the closure of the set of repelling periodic points, there are periodic points z_j of $f \circ g$ such that $z_j \to z_0$. By Lemma 1, $g(z_j)$ are periodic points of $g \circ f$ and hence $g(z_0)$ is a limit of periodic points of $g \circ f$ because $g(z_j) \to g(z_0)$. It follows that $g(z_0) \in \mathcal{J}(g \circ f)$.

Interchanging the role of f and g we see that if $w_0 \in \mathcal{J}(g \circ f)$, then $f(w_0) \in \mathcal{J}(g \circ f)$. Suppose now that $z_0 \in \mathbf{C}$ and $g(z_0) \in \mathcal{J}(g \circ f)$. Then $f(g(z_0)) \in \mathcal{J}(f \circ g)$. Because of the complete invariance of the Julia set we conclude that $z_0 \in \mathcal{J}(f \circ g)$. The proof is complete.

Proof of Theorem 2. Let $h = f \circ g$ and $k = g \circ f$. For $n \in \mathbf{N}$, let U_n be the component of $\mathcal{J}(h)$ containing $h^n(U_0)$ and let V_n be the component of $\mathcal{J}(k)$ containing $k^n(V_0)$. Since $g(h^n(U_0)) = k^n(g(U_0))$ for all $n \in \mathbf{N}$ we see that $g(U_n) \subset V_n$ and analogously $f(V_n) \subset U_{n+1}$. We conclude that if $U_m = U_n$, then $V_m = V_n$ and if $V_m = V_n$ then $U_{m+1} = U_{n+1}$. In particular, if $U_0 = U_n$, then $V_0 = V_n$.

Let now $U_0 = U_n$. Suppose that $h^{n_j}|_{U_0} \to \phi$ as $j \to \infty$ where $\phi \not\equiv \infty$. Take a domain V^* in V_0 such that a branch $g^* \colon V^* \to U^* \subset U_0$ of the inverse function of g is defined. Then $k^n|_{V^*} = g \circ h^n \circ g^*|_{V^*}$ and hence $k^{n_j}|_{V^*} \to \psi := g \circ \phi \circ g^*$. If U_0 is a Siegel disc, then ϕ is nonconstant, hence ψ is also nonconstant and thus V_0 is a Siegel disc. If U_0 is an attracting domain, then ϕ is a constant lying in $\mathcal{F}(h)$, hence ψ is a constant in $\mathcal{F}(k)$ and thus V_0 is an attracting domain. The case of a parabolic domain is analogous, except that ϕ and ψ are in $\mathcal{J}(h)$ and $\mathcal{J}(k)$ now.

The arguments show that if V_0 is an attracting domain, parabolic domain or Siegel disc, then so is U_1 and hence U_0. It follows that if U_0 is a Baker domain, then so is V_0. This completes the proof.

Remark. The above proof also shows that if V_0 is periodic, then U_1 is periodic. We note that U_0 need not be periodic. To see this simply take $f=g$ such that $\mathcal{F}(f)$ has an invariant component V_0 which is not completely invariant. Then take U_0 as a component of $f^{-1}(V_0)\backslash V_0$.

To prove Theorems 3 and 4, we also need the following results.

Lemma 2. *Let f and g be two entire functions. Then*

$$C(f\circ g)\subset C(f)\cup f(C(g)),$$
$$A(f\circ g)\subset A(f)\cup f(A(g)),$$

and

$$\operatorname{sing}((f\circ g)^{-1})\subset \operatorname{sing}(f^{-1})\cup f(\operatorname{sing}(g^{-1})).$$

Proof. We have $(f\circ g)'=f'(g)g'$ and thus $C(f\circ g)\subset C(f)\cup f(C(g))$. If $f\circ g$ tends to $\alpha\in\mathbf{C}$ along a path γ tending to ∞, then along γ either g tends to ∞ or g tends to a point β satisfying $f(\beta)=\alpha$ (see [7] for details). We have $\alpha\in A(f)$ in the first case and $\alpha\in f(A(g))$ in the second case. Now the second and the last conclusion follow.

Lemma 3. (Denjoy–Carleman–Ahlfors theorem [26, §XI.4]) *If the inverse function of a meromorphic function f has n direct singularities, $n\geq 2$, then*

$$\liminf_{r\to\infty}\frac{T(r,f)}{r^{n/2}}>0.$$

Consequently, the inverse function to a meromorphic function of finite order ϱ has at most $\max\{2\varrho,1\}$ *direct singularities. Moreover, an entire function of finite order ϱ has at most 2ϱ finite asymptotic values.*

Proof of Theorem 3. The functions $\sin z$ and $\cos z$ have the critical values ± 1 and no asymptotic values. And any $f\in F$ has at most finitely many asymptotic values by Lemma 3. Thus $g(f)$ has only finitely many asymptotic values by Lemma 2. (In fact, it is not difficult to see that functions in F and hence the function $g(f)$ have no asymptotic values at all.) Since all the critical values of $g(f)$ are among the finitely many values ± 1, $g_1(\pm 1)$, $g_1\circ g_2(\pm 1), \dots$, $g_1\circ g_2\circ\dots\circ g_m(\pm 1)$, $g_1\circ g_2\circ\dots\circ g_m\left(\pm\left(\frac{1}{2}\pi+i\right)\right)$, $g_1\circ g_2\circ\dots\circ g_{m-1}(\pm g_m(i))$, $g_1\circ g_2\circ\dots\circ g_{m-1}(\pm g_m(0))$, and $g_1\circ g_2\circ\dots\circ g_{m-1}\left(\pm g_m\left(\frac{1}{2}\pi\right)\right)$ again by Lemma 2, $g(f)$ has only finite many critical values. Hence $g(f)$ is of finite type (i.e. the inverse function to $g(f)$ has only a finite number of singularities) and thus $g(f)$ has no wandering domains by [18] or [21]. We now apply Theorem 2 to conclude that $f(g)$ has no wandering domains. This completes the proof.

Remark. It is not hard to see that such $f\circ g$ has infinitely many different critical values and is not of finite type.

Proof of Theorem 4. We define $h(z)=\sin f(z)$. Then

$$\operatorname{sing}(h^{-1})\subset\{-1,1\}\cup\sin(\operatorname{sing}(f^{-1}))\subset[-1,1]$$

by Lemma 2. It now follows from a result of Erëmenko and Lyubich [18] that there is no component U_0 of $\mathcal{F}(h)$ such that $h^n|_{U_0}\to\infty$ as $n\to\infty$. Thus if h has a wandering domain U_0, then there is a sequence (n_k) of positive integers and $a\in\mathbf{C}$ such that $h^{n_k}|_{U_0}\to a$ as $k\to\infty$. Clearly we have $a\in\mathcal{J}(h)$. Let $P(h)=\overline{\bigcup_{n=0}^{\infty}h^n(\operatorname{sing}(h^{-1}))}$. It follows from a result of Baker [1] that $a\in P(h)$. (Actually we even have that a is a limit point of $P(h)$, but we do not need this result proved in [12] here.) Our hypotheses imply that $|h(x)|<|x|$ for $0<|x|\leq 1$. We conclude that $P(h)\subset[-1,1]$, that $h^n|_{[-1,1]}\to 0$ as $n\to\infty$ and that 0 is an attracting or parabolic fixed point of h. If 0 is attracting, then $[-1,1]\subset\mathcal{F}(h)$ and thus $P(h)\cap\mathcal{J}(h)=\emptyset$, contradicting $a\in P(h)\cap\mathcal{J}(h)$. If 0 is parabolic, then $[-1,0)$ and $(0,1]$ are contained in the parabolic domains associated to 0. We conclude that $[-1,1]\cap\mathcal{J}(h)=\{0\}$, so that $a=0$. The dynamics near parabolic fixed points are well understood. In particular, it is known and not difficult to see that a parabolic fixed point cannot be a limit function of a sequence of iterates in a wandering domain. Thus we again have a contradiction. Hence h has no wandering domains. Theorem 2 now implies that $f(\sin z)$ does not have wandering domains.

3. Proof of Proposition 1

Lemma 4. ([11]) *Let f be a meromorphic function of finite order. If a is an asymptotic value of f, then a is a limit of critical values $a_k\neq a$ or all singularities of f^{-1} over a are logarithmic.*

Proof of Proposition 1. Let $f\in\mathrm{LP}$. It follows from the characterization of LP mentioned in the introduction that $f'\in\mathrm{LP}$. Hence all critical values of f are real.

We now assume that f has an asymptotic value $\alpha\in\mathbf{C}\setminus\mathbf{R}$ and seek a contradiction. Clearly $\bar{\alpha}$ is also an asymptotic value of f. It follows from Lemma 4 that f^{-1} has logarithmic (and hence direct) singularities over α and $\bar{\alpha}$. From a theorem of Lindelöf [26, §III.7.3] we deduce that between the paths where f tends to α and $\bar{\alpha}$, there must be paths where f tends to ∞ and thus there are also two direct singularities over ∞. Thus f^{-1} has at least four direct singularities. By Lemma 3 we thus have

$$\liminf_{r\to\infty}\frac{\log M(r,f)}{r^2}>0. \tag{1}$$

We may write f in the form $f(z)=e^{-az^2}p(z)$, where p is an entire function of genus 0 or 1 and $a\geq 0$. It follows that $\log M(r,p)=o(r^2)$ as $r\to\infty$ and hence $a>0$ by (1). This implies that $f(z)\to 0$ as $z\to\infty$ along the positive or negative real axis. Using Lindelöf's theorem again we conclude that between the real axis and the paths where f tends to α and $\bar{\alpha}$ there must be paths where f tends to ∞. This leads to four direct singularities of f^{-1} over ∞ and thus altogether to six direct singularities of f^{-1}. Hence

$$\liminf_{r\to\infty} \frac{\log M(r,f)}{r^3} > 0$$

by Lemma 3. On the other hand, we have $\log M(r,f)=O(r^2)$ as $r\to\infty$ by the form of f. This is a contradiction. Thus all asymptotic values of f are real.

Altogether we see that $\operatorname{sing}(f^{-1})\subset\mathbf{R}$ if $f\in\mathrm{LP}$. The case that f has the form $f=f_1\circ f_2\circ\ldots\circ f_n$ with $f_1, f_2, \ldots, f_n\in\mathrm{LP}$ now follows from Lemma 2.

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