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Maximal invariant subspaces in the Bergman space

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1. Introduction

Let A^2 be the space of all complex-valued holomorphic functions f on the open unit disk $\mathbf{D}$ that are subject to the boundedness condition

$$\|f\|_{A^2} = \left(\int_{\mathbf{D}} |f(z)|^2\, dS(z)\right)^{1/2} < +\infty;$$

here, $dS(z)=dx\,dy$ is area measure in the plane ($z=x+iy$). This space is usually called the *Bergman space*. A closed subspace $\mathcal{M}$ of A^2 is said to be *invariant* (or z-invariant) if $z\mathcal{M}$ is contained in $\mathcal{M}$. Since the operator of multiplication by z is bounded below on A^2, $z\mathcal{M}$ is a closed subspace of $\mathcal{M}$. We define the *index* of the invariant subspace $\mathcal{M}$ to be the dimension of the quotient space $\mathcal{M}/z\mathcal{M}$, with values in the set $\{0, 1, 2, \dots, +\infty\}$. We will at times refer to this number as $\operatorname{ind}(\mathcal{M})$. The index of $\mathcal{M}$ can only equal 0 if $\mathcal{M}$ is the zero subspace. There are invariant subspaces of arbitrary index [4], [6]. Let $z[\mathcal{M}]$ denote the operator on $A^2/\mathcal{M}$ induced by z,

$$z[\mathcal{M}](f+\mathcal{M}) = zf+\mathcal{M}.$$

The invariant subspaces of A^2 are ordered with respect to inclusion. Thus an invariant subspace $\mathcal{M}$ is said to be *maximal* if the only invariant subspace strictly containing it is the whole space A^2. For each $\alpha\in\mathbf{D}$, the invariant subspace

$$\mathcal{M}_\alpha = \{f\in A^2 : f(\alpha)=0\}$$

is maximal, because its codimension is 1. The question is: are there any other maximal invariant subspaces? One reason why maximal invariant subspaces are

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of interest comes from the analogy with the Gelfand theory of maximal ideals in commutative Banach algebras. Another is the following: an invariant subspace $\mathcal{M}$ is maximal if and only if the operator $z[\mathcal{M}]$ lacks nontrivial invariant subspaces. So, if in some way a maximal invariant subspace $\mathcal{M}$ could be found with infinite codimension, we would have an example of an operator on an infinite-dimensional separable Hilbert space which lacks nontrivial invariant subspaces.

The following result is proved in this note.

Theorem 1.1. *If $\mathcal{M}$ is a maximal invariant subspace, then $\mathcal{M}=\mathcal{M}_\alpha$ for some $\alpha\in\mathbf{D}$, where as above*

$$\mathcal{M}_\alpha=\{f\in A^2 : f(\alpha)=0\}.$$

Let $\mathcal{N}$ be a given invariant subspace in A^2, and suppose the invariant subspace $\mathcal{M}$ is maximal with respect to containment in $\mathcal{N}$. If in this situation the dimension of $\mathcal{N}/\mathcal{M}$ always equals 1, then every operator on infinite-dimensional separable Hilbert space possesses nontrivial invariant subspaces [6]. So, in a sense, the main theorem provides weak support for this conjecture. It is conceivable that the methods used here can be extended so far as to prove that in the above situation $\dim\mathcal{N}/\mathcal{M}=1$ whenever $\mathcal{N}$ has finite index. However, handling the then remaining case when $\operatorname{ind}(\mathcal{N})=+\infty$ apparently is at par with the invariant subspace problem.

2. The proof of the main theorem

We suppose $\mathcal{M}$ is a maximal invariant subspace in A^2, which is not of the type $\mathcal{M}_\alpha$, and shall try to obtain a contradiction. Introduce the zero set of $\mathcal{M}$,

$$Z(\mathcal{M})=\{z\in\mathbf{D} : f(z)=0 \text{ for all } f\in\mathcal{M}\},$$

and note that by maximality, we must have $Z(\mathcal{M})=\emptyset$, for otherwise $\mathcal{M}$ would indeed be contained in one of the $\mathcal{M}_\alpha$.

Lemma 2.1. *The index of $\mathcal{M}$ equals* 1.

Proof. Suppose, for the sake of argument, that $\operatorname{ind}(\mathcal{M})>1$. Then $z\mathcal{M}$ has codimension larger than 1 in $\mathcal{M}$, and thus cannot coincide with the larger invariant subspace $\mathcal{M}\cap zA^2$, which has codimension 1 in $\mathcal{M}$. Consider the invariant subspace $\mathcal{N}$ with

$$z\mathcal{N}=\mathcal{M}\cap zA^2,$$

which thus contains $\mathcal{M}$ as a strictly smaller subspace. By the maximality of $\mathcal{M}$, we should have $\mathcal{N}=A^2$, but this is impossible, because then $\mathcal{M}$ would have to contain

zA^2, and thus coincide with zA^2. The contradiction obtained proves the assertion of the lemma. □

We proceed to look at the spectrum $\sigma(z[\mathcal{M}])$ of the operator $z[\mathcal{M}]$: by definition, $\lambda \in \mathbf{C}$ is in $\sigma(z[\mathcal{M}])$ if and only if the the operator $\lambda - z[\mathcal{M}]$ fails to be invertible. By Lemma 2.1, $Z(\mathcal{M})=\emptyset$, and so by [5], $\sigma(z[\mathcal{M}])$ is a closed subset of the unit circle $\mathbf{T}$. It is well known that if the spectrum of an operator is disconnected, then the operator has nontrivial invariant subspaces; one can show this using the Shilov idempotent theorem. Hence the only case that concerns us is when $\sigma(z[\mathcal{M}])$ is a connected subset of $\mathbf{T}$, and this can happen in three ways: a point, an arc, and the whole circle. In order to discard the two latter possibilities, we shall estimate the norm of the resolvent $(\lambda - z[\mathcal{M}])^{-1}$ for $\lambda \in \mathbf{C} \setminus \mathbf{T}$. Let us form the extremal function $G_{\mathcal{M}}$ for the invariant subspace $\mathcal{M}$, which is the function which among all functions in $\mathcal{M}$ of norm 1 has the largest value (in modulus) at the origin. Since $\operatorname{ind}(\mathcal{M})=1$ and $Z(\mathcal{M})=\emptyset$, the function $G_{\mathcal{M}}$ has no zeros in $\mathbf{D}$ [3]. We can now use the construction in [5] to estimate the norm of the resolvent inside the unit disk, and the result is

$$\|(\lambda - z[\mathcal{M}])^{-1}\| \le \exp\left(\frac{C}{1-|\lambda|}\right), \quad \lambda \in \mathbf{D},$$

for some constant C. Outside the unit disk, we shall be happy with the unsophisticated estimate

$$\|(\lambda - z[\mathcal{M}])^{-1}\| \le \|(\lambda - z)^{-1}\| \le \frac{1}{1-|\lambda|}, \quad \lambda \in \mathbf{C} \setminus \bar{\mathbf{D}}.$$

These estimates are radial, and if we take the log-log of the bounding function, it is integrable. This means that there is a non-quasianalytic functional calculus operating on $z[\mathcal{M}]$, so that if $\sigma(z[\mathcal{M}])$ is an arc or the circle, then $z[\mathcal{M}]$ has nontrivial invariant subspaces (see [7], [2]). The only remaining case is when the spectrum is a point, which by rotational symmetry can be assumed to be 1.

Proposition 2.2. *If $\mathcal{N}$ is an invariant subspace with $\sigma(z[\mathcal{N}])=\{1\}$, then, for some $\beta \in]0,+\infty[$, $\mathcal{N}$ coincides with $\mathcal{N}_\beta$, the invariant subspace of all functions f in A^2 with*

$$\limsup_{t \to 1^-} (1-t) \log |f(t)| \le -\beta.$$

The proof of the proposition is standard; let me nevertheless supply some hints on how to construct a proof. For $\beta \ge 0$, $\mathcal{N}_\beta$ is a closed invariant subspace, as can be seen by applying H^p theory to a smaller tangential disk at 1 (for $\beta=0$, $\mathcal{N}_\beta$ is the whole space). Now select β by requesting that it be the smallest such that $\mathcal{N} \subset \mathcal{N}_\beta$.

The extremal function [3] $G_{\mathcal{N}}$ for $\mathcal{N}$ can be shown to have an analytic extension to $\mathbf{C}\backslash\{1\}$ [5], and since it generates $\mathcal{N}$ [1], its logarithmic residue at 1 is

$$\lim_{t\to 1^-}(1-t)\log|G_{\mathcal{N}}(t)|=-\beta.$$

One shows that $G_{\mathcal{N}}$ can be multiplied by an H^∞ function to obtain the classical inner function

$$u_\beta(z)=\exp\biggl(-\frac{\beta}{2}\,\frac{1+z}{1-z}\biggr),$$

which is known to generate $\mathcal{N}_\beta$, whence the conclusion $\mathcal{N}=\mathcal{N}_\beta$ follows.

It is now clear that no invariant subspace $\mathcal{N}$ with $\sigma(z[\mathcal{N}])=\{1\}$ can be maximal, which concludes the proof of Theorem 1.1.

Remark. It should perhaps be pointed out that the result obtained here is in no way peculiar to the unweighted Bergman space. In fact, there is a large class of weighted Bergman spaces where the same holds true. It should be mentioned in connection with this that Aharon Atzmon recently found a one-parameter family of invariant subspaces with spectrum $\sigma(z[\mathcal{M}])$ equal to $\{1\}$ for radially weighted Bergman spaces where the spaces contain holomorphic functions of very rapid growth in the disk.

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