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Entire functions having small logarithmic sums over certain discrete subsets

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Abstract. It is known how to obtain a uniform estimate of e.g. a polynomial in terms of its logarithmic sum over the integers provided that the sum is sufficiently small. This result is generalized here and we obtain estimates in terms of logarithmic sums taken over certain discrete subsets of the real axis.

1. Introduction

Polynomials having sufficiently small logarithmic sums over the integers form a normal family in the whole complex plane. Here, the logarithmic sum of a polynomial p is

$$\sum \frac{\log^+ |p(n)|}{n^2+1}.$$

This deep result was obtained by Paul Koosis, first for polynomials of special form (see [3]) and later for general polynomials (see [4, Chapter VIII, Section B]). Koosis formulated the result in the following quantitative form.

Theorem 1.1. *There are numerical constants α_0 and k such that, for any polynomial $p(z)$ with*

$$\sum \frac{\log^+ |p(n)|}{n^2+1} = \alpha \le \alpha_0,$$

we have, for all z,

$$|p(z)| \le K_\alpha e^{3k\alpha|z|},$$

where K_α is a constant depending only on α (and not on p).

An extension to entire functions of very small exponential type was given in [6, Section 1]. Recently, a new proof of the result about polynomials has been found, see [7]. There, an extension to entire functions of exponential type less than $T_*\approx 0.44$ is obtained. In fact $T_*=\pi/M_*$, where

$$M_* = \inf_{s>0}\left\{\frac{1}{s}\int_0^{\pi/2} \exp\left((1+\sin\theta)s\right)d\theta\right\}.$$

The theorem is as follows.

Theorem 1.2. *Given $B_0<T_*$ and $\gamma>0$ there is $\eta_0>0$ such that for any $\eta\le\eta_0$ there is $C_\eta>0$ with the property that*

$$|f(z)|\le C_\eta \exp(B|y|+\gamma|z|),\quad z\in\mathbf{C},$$

for all entire functions f of exponential type $\le B\le B_0$ satisfying

$$\sum \frac{\log^+|f(n)|}{n^2+1}\le\eta.$$

In this paper we shall demonstrate how to deal with logarithmic sums over other sequences than the integers. After studying the proofs mentioned above it is not too surprising that the machinery can be made to work for more general sequences. For polynomials this is already noted in [4, p. 518]. We shall use the approach of [7] to obtain such a generalization without too much work.

Definition 1.3. Let $h>0$. We say that a sequence Λ of real numbers is relatively h-dense (in $[0,\infty)$ or the whole real line) if, outside a bounded set, any closed interval of length h contains at least one point from Λ.

We give this definition in (partial) accordance with the terminology used by Harald Bohr in his work on almost periodic functions (see [2, C 18]). He calls a set of real numbers relatively dense if there exists a positive number L such that any interval of length L contains at least one point of that set. Our generalization of Theorem 1.2 concerns symmetric relatively h-dense sequences of the real line, see Theorem 5.3. In Sections 3 and 4 the essential material is described. Section 2 contains a preliminary and elementary result.

2. A preliminary result

The result of this section is the following result.

Proposition 2.1. *Let Λ be a relatively h-dense subset of the real line and let f be an entire function of exponential type $B<\pi/h$. If*

$$\sum_{\lambda\in\Lambda}\frac{\log^+|f(\lambda)|}{\lambda^2+1}<\infty$$

then f is of zero exponential growth on the real axis.

Proof. Let $\varepsilon>0$ be given. We may suppose that ε is so small that $(h+\varepsilon)B<\pi$. We consider intervals of the form

$$I_n=[n(h+\varepsilon),n(h+\varepsilon)+h],\quad n\in\mathbf{Z}.$$

Since Λ is relatively h-dense it is possible to choose, for any $n\geq$ some n_0, an element $\lambda_n\in I_n\cap\Lambda$. The subsequence $\Lambda'=\{\lambda_n\}_{n\geq n_0}$ has the properties $|\lambda_n-\lambda_m|\geq\varepsilon|n-m|$ and $n_{\Lambda'}(r)/r\to 1/(h+\varepsilon)$ as $r\to\infty$. We next put

$$S=\{\lambda\in\Lambda'\mid\log^+|f(\lambda)|\geq\varepsilon\lambda\},$$
$$L=\{\lambda\in\Lambda'\mid\log^+|f(\lambda)|<\varepsilon\lambda\}.$$

It is easy to see (using the assumption on finite logarithmic sum) that $n_S(r)/r\to 0$ as $r\to\infty$. Therefore $L=\{l_k\}_{k\geq 1}$ is unbounded. Furthermore

$$n_{\Lambda'}(l_k)=n_S(l_k)+k.$$

We divide by l_k and let k tend to infinity; in this way we get $l_k/k\to h+\varepsilon$. Put $\varphi(z)=f(z(h+\varepsilon))$ and $l'_k=l_k/(h+\varepsilon)$. A theorem of V. Bernstein (see [1, p. 185]) yields (since φ is of exponential type $(h+\varepsilon)B<\pi$)

$$\limsup_{x\to\infty}\frac{\log|\varphi(x)|}{x}=\limsup_{k}\frac{\log|\varphi(l'_k)|}{l'_k}=\limsup_{k}\frac{\log|f(l_k)|}{l_k}(h+\varepsilon)\leq\varepsilon(h+\varepsilon),$$

so that

$$\limsup_{x\to\infty}\frac{\log|f(x)|}{x}\leq\varepsilon.$$

Since this holds for all sufficiently small ε, f is of zero growth on the positive ray. A similar argument shows that f is of zero growth on the negative ray. The proof is finished.

We shall use this result to prove that any entire function of exponential type less than T_*/h and having finite logarithmic sum over a symmetric and relatively h-dense sequence Λ of the real line belongs to the Cartwright class, that is, has finite logarithmic integral.

3. Background material

We recall here the main definitions and results needed in this exposition. Suppose that Γ is a relatively h-dense sequence in $[0,\infty)$ with no finite accumulation point. There is thus a positive constant K such that

$$[K,\infty)\subseteq\bigcup_{\gamma\in\Gamma}\left[\gamma-\tfrac{1}{2}h,\gamma+\tfrac{1}{2}h\right].$$

We now construct a subsequence $\{\lambda_n\}$ of Γ by taking $\lambda_1\in\left[K,K+\frac{1}{2}h\right]$ and

$$\lambda_{n+1}=\max\{\gamma\in\Gamma\mid\lambda_n<\gamma\leq\lambda_n+h\}.$$

We note that $\lambda_{n+2}-\lambda_n\geq h$ for all n. Indeed, if this does not hold we would have $\lambda_{n+1}\geq\lambda_{n+2}$, contradicting the choice of λ_{n+2}. This subsequence $\Lambda=\{\lambda_n\}$ is still relatively h-dense in $[0,\infty)$ and in addition the points are "almost" separated: $\lambda_{n+2}-\lambda_n\geq h$ for all n. We note that in this case

$$\sum_{n=1}^{\infty}\frac{\log\lambda_n}{\lambda_n^2+1}<\infty. \tag{1}$$

In this section and in Section 4 we shall consider relatively h-dense sequences Λ in $[0,\infty)$ having the extra property of separation.

We recall the following definition from [7].

Definition 3.1. We shall denote by $\mathcal{A}(B)$ the set of even entire functions f of Cartwright class and of exponential type $\leq B$ with $f(x)\geq 1$ for real x and with $f(0)=1$. Furthermore we put $\mathcal{A}=\bigcup_{B>0}\mathcal{A}(B)$.

For $f\in\mathcal{A}$ we put

$$\mathcal{J}(f)=\int_0^\infty\frac{\log f(t)}{t^2}\,dt,$$

and

$$\mathcal{S}(f)=\sum_{\Lambda}\frac{\log f(\lambda)}{\lambda^2}.$$

In the paper [7] extensive use was made of certain subharmonic functions called pre-multipliers. These were constructed from functions of the class $\mathcal{A}$ and a certain parameter: for $f\in\mathcal{A}$ and a parameter A greater than

$$\mathcal{J}(f)+\sqrt{2e\mathcal{J}(f)\big(\mathcal{J}(f)+\tfrac{1}{4}\pi B\big)}\,,$$

we consider the function

$$F(z)=\frac{1}{\pi}\int\frac{\log f(t)}{|z-t|^2}|y|\,dt-A|y|.$$

A result due to Koosis (see [5, p. 407]) asserts that this function has a finite superharmonic majorant in the whole complex plane. The least such superharmonic function is denoted by $\mathcal{M}F$. The subharmonic function $g\equiv-\mathcal{M}F$ is called a pre-multiplier associated with f and A. In [7, Section 3], some of the fundamental properties of g are given. It has the property

$$g(x)+\log f(x)\leq 0, \quad x\in\mathbf{R},$$

and the representation

$$g(z)=\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\,d\varrho(t),$$

where ϱ is a positive measure concentrated on the set

$$E=\{x\geq 0 \mid g(x)+\log f(x)=0\}.$$

Furthermore ϱ has the fundamental property (see [5, pp. 400–407]).

$$d\varrho(t)\leq\frac{A+B}{\pi}\,dt, \quad t\geq 0. \tag{2}$$

We sketch the main ideas behind the proof of Theorem 5.3 to be given. That theorem shall be deduced from the following comparison result: if $B<T_*/h$ and if $\{f_k\}$ is any sequence from $\mathcal{A}(B)$ for which $\mathcal{S}(f_k)\to_k 0$, then $\mathcal{J}(f_k)\to_k 0$ (Theorem 4.1). The essential ingredient in the proof of this comparison result is furnished by Theorem 3.2 below. It states that there is, in a weak form, a lower bound on $\mathcal{S}(f)$ in terms of an integral involving the pre-multiplier associated with f and the parameter A, namely

$$\int_0^\infty\frac{-g(t)}{t^2}\,d\varrho(t).$$

To a large extent the behaviour of g and ϱ depends only on the type B of f and the parameter A. One example of this is the relation (2) above and another is Theorem 5.1 in [7], stating that

$$\int_0^\infty\frac{-g(t)}{t^2}\,d\varrho(t)\geq\frac{\mathcal{J}(f)A}{2\pi}. \tag{3}$$

The logarithmic integral $\mathcal{J}(f)$ thus occurs in the lower bound on $\mathcal{S}(f)$.

Theorem 3.2. (See [7, Theorem 4.4]) *Let $B<2T_*/h$ and let $f\in\mathcal{A}(B)$. If the parameter A is sufficiently small there is a constant $C>0$, depending only on A and B, such that, for any large m,*

$$\sum_{\lambda\in\Lambda,\lambda\geq m}\frac{\log f(\lambda)}{\lambda^2}\geq-(C+c(h))\sum_{\lambda\in\Lambda,\lambda\geq m}\frac{1}{\lambda^2}+c(h)\int_m^\infty\frac{-g(t)}{t^2}\,d\varrho(t).$$

Here $c(h)$ is a positive constant, depending only on h. The bound on m depends only on Λ.

Before going into the proof of this theorem, we need the following proposition.

Proposition 3.3. *If $B<2T_*/h$ and $f\in\mathcal{A}(B)$ there is, for all sufficiently small values of the parameter A, a constant $C>0$, depending on A and B, such that*

$$\log f(x)+g(x)\geq -C$$

for all $x\in\mathbf{R}$ with $|x-E|\leq\frac{1}{2}h$.

This proposition is a scaled version of [7, Proposition 4.3] and the proof is similar.

Proof of Theorem 3.2. We write $I_\lambda=\left[\lambda-\frac{1}{2}h,\lambda+\frac{1}{2}h\right]$. Let $\lambda\in\Lambda$ and suppose that $E\cap I_\lambda\neq\emptyset$. Since $B<2T_*/h$ we have, when A is sufficiently small, a constant $C>0$ such that

$$\log f(\lambda)\geq -C+(-g(\lambda)).$$

Recalling the property (3) we get

$$\frac{\log f(\lambda)}{\lambda^2}\geq-\frac{C}{\lambda^2}+\frac{\pi}{A+B}c_1(h)\int_{I_\lambda}\frac{-g(\lambda)}{t^2}\,d\varrho(t),$$

where $c_1(h)$ is a constant depending only on h.

It is possible to "replace" $(-g(\lambda))$ by $(-g(t))$ in this integral: we have (by the representation of g)

$$-g(\lambda)\geq -g(\lambda+ih).$$

The values $Ah-g(t+ih)$, $|t-\lambda|\leq\frac{1}{2}h$ are controlled by Harnack's inequality (see [7, Proposition 3.5]) so that

$$Ah-g(\lambda+ih)\geq\tfrac{1}{3}(Ah-g(t+ih)).$$

From [7, Corollary 3.7] we have

$$g(t+ih)-g(t)\leq(A+B)h$$

and this gives

$$-g(\lambda) \geq -\left(A + \tfrac{1}{3}B\right)h - \tfrac{1}{3}g(t).$$

Thus, since $A+B \leq 2\pi/h$ (without loss of generality),

$$\frac{\log f(\lambda)}{\lambda^2} \geq -\frac{c_2(h)+C}{\lambda^2} + c_3(h)\int_{I_\lambda} \frac{-g(t)}{t^2}\,d\varrho(t),$$

with some other constants $c_2(h)$, $c_3(h)$ depending only on h. We then estimate the sum

$$\sum_{\lambda\in\Lambda,\lambda\geq m} \frac{\log f(\lambda)}{\lambda^2}$$

from below by using what we have just found on those terms for which $I_\lambda \cap E \neq \emptyset$. If $I_\lambda \cap E = \emptyset$, ϱ has no mass on I_λ and therefore, since $\bigcup I_\lambda \supseteq [m,\infty)$ for large m,

$$\sum_{\lambda\in\Lambda,\lambda\geq m} \frac{\log f(\lambda)}{\lambda^2} \geq -(C+c_2(h)) \sum_{\lambda\in\Lambda,\lambda\geq m} \frac{1}{\lambda^2} + c_3(h)\int_m^\infty \frac{-g(t)}{t^2}\,d\varrho(t).$$

4. Logarithmic integrals versus sums

The main theorem to be proved here is a generalization of [7, Theorem 1.9].

Theorem 4.1. *Let $B<2T_*/h$ and suppose that $\{f_k\}$ is a sequence from $\mathcal{A}(B)$ for which $\mathcal{S}(f_k)\to_k 0$. Then $\mathcal{J}(f_k)\to_k 0$.*

Lemma 4.2. *Suppose that φ is an entire function of exponential type $\sigma<\pi/h$. If $\varphi(\lambda)=0$ for all $\lambda\in\Lambda$ then $\varphi\equiv 0$.*

Proof. Put $f(z)=\varphi(hz)$; this function is of exponential type $\sigma h<\pi$. The number of zeros of absolute value less than or equal to r, $n_f(r)$, satisfies

$$n_f(r) \geq r - c,$$

where c is some positive constant.

Furthermore

$$\log|f(iy)f(-iy)| \leq 2\sigma h|y| + o(|y|) \leq 2\pi(|y| + (o(|y|) - \varepsilon|y|))$$

for some small positive number ε. Here $o(|y|)/|y|\to 0$ as $|y|\to\infty$.

Since

$$\limsup_{R\to\infty}\int_1^R \frac{-c+\varepsilon|y|-o(|y|)}{y^2}\,dy = \infty,$$

Theorem 9.3.4 in [1] implies that $f\equiv 0$.

Lemma 4.3. *Let $B<2T_*/h$. There is no sequence $\{f_k\}$ from $\mathcal{A}(B)$ for which $\mathcal{S}(f_k)\to_k 0$ and $\mathcal{J}(f_k)$ all take the same strictly positive value, provided that value is sufficiently small.*

Proof. Suppose we *have* such a sequence $\{f_k\}$ and say that

$$\mathcal{J}(f_k)=a>0$$

for all k. The boundedness of these integrals gives us that $\{f_k\}$ is a normal family in the whole complex plane. Furthermore any subsequence contains a further subsequence converging uniformly over compact subsets to some entire function φ of exponential type $\leq B+9a/\pi$ (see e.g. [7, Proposition 2.1]). We may assume that a is so small that $B+9a/\pi<\pi/h$. Since $\mathcal{S}(f_k)\to_k 0$ it follows that $\varphi(\lambda)=1$ for all $\lambda\in\Lambda$ and hence that (Lemma 4.2) $\varphi\equiv 1$. This means that $f_k\to_k 1$ uniformly over compact subsets so that ([7, Lemma 6.1])

$$\int_0^m \frac{\log f_k(x)}{x^2}\,dx \to_k 0$$

for any $m>0$.

We now bring in the pre-multipliers. We take

$$A=2a+\sqrt{2ea\left(a+\tfrac{1}{4}\pi B\right)}$$

and construct the pre-multipliers g_k from the f_k's and this common parameter A. Theorem 3.2 and the inequality (3) give us

$$\mathcal{S}(f_k)\geq -c_1\sum_{\lambda\in\Lambda,\lambda\geq m}\frac{1}{\lambda^2}+c_2\left(\frac{Aa}{2\pi}-\int_0^m \frac{-g_k(t)}{t^2}\,d\varrho_k(t)\right) \tag{4}$$

with suitable constants c_1, c_2 independent of k. In this relation, the integral involving the pre-multiplier g_k tends to zero as k tends to infinity. This is because ϱ_k is concentrated on the set E_k where $-g_k$ is *equal* to $\log f_k$, so that

$$\int_0^m \frac{-g_k(t)}{t^2}\,d\varrho_k(t)\leq \frac{A+B}{\pi}\int_0^m \frac{\log f_k(t)}{t^2}\,dt\to_k 0.$$

By (1), $\sum_{\lambda\in\Lambda,\lambda\geq m} 1/\lambda^2\to 0$ as $m\to\infty$, so if we choose m very large in (4) and then let k tend to infinity we obtain

$$\liminf_k \mathcal{S}(f_k)>0.$$

This is a contradiction.

Proof of Theorem 4.1. This is similar to the proof of Theorem 1.9 in [7]. For the reader's convenience we include it here. We argue by contradiction and suppose that we have a sequence $\{f_k\}$ from $\mathcal{A}(B)$, $B<2T_*/h$, for which $\mathcal{S}(f_k)\to_k 0$ and yet $\mathcal{J}(f_k)\geq$ some positive ε. We put

$$f_{k,m}(z)=1+(z/m)^2 f_k(z).$$

We take a positive number $a<\frac{1}{2}\varepsilon$, so small that there is no sequence from $\mathcal{A}(B)$ whose logarithmic sums tend to zero and whose logarithmic integrals all take the value a (Lemma 4.3). Fixing k we adjust $m=m_k$ so as to have

$$\mathcal{J}(f_{k,m_k})=a.$$

This is possible because $m\mapsto\mathcal{J}(f_{k,m})$ is a continuous function for $m>0$, tending to 0 as $m\to\infty$ (and to infinity as m tends to zero). We claim that the sequence $\{m_k\}$ must tend to infinity. Suppose that some subsequence remains bounded. We may assume (by relabelling) that $m_k\leq m$ for all k. Then $f_{k,m_k}(x)\geq f_k(x)$ for $|x|\geq m$, so that

$$\int_m^\infty \frac{\log f_k(x)}{x^2}\,dx\leq\mathcal{J}(f_{k,m_k})=a.$$

Furthermore, $f_{k,m_k}(x)\geq f_{k,m}(x)$ for real x, so, since $\{f_{k,m_k}\}$ forms a normal family in the complex plane, there is a constant C such that

$$\sup\{|f_k(x)|\mid -m\leq x\leq m\}\leq C$$

for all k. From Proposition 2.1 in [7] we obtain the following estimate

$$\log|f_k(z)|\leq B(|z|+2)+\log^+ C+(9/\pi)a(|z|+1).$$

Therefore $\{f_k\}$ is a normal family in the complex plane and a certain subsequence (which we relabel as $\{f_k\}$) tends uniformly over compact subsets to some entire function h of exponential type $\leq B+(9/\pi)a<\pi/h$, when a is small enough. Since $\mathcal{S}(f_k)\to_k 0$, h takes the value 1 at all points of Λ. By Lemma 4.2, $h\equiv 1$. By Lemma 6.1 in [7] we see that

$$\int_0^m \frac{\log f_k(t)}{t^2}\,dt\to_k 0.$$

For all large k we will then have

$$\mathcal{J}(f_k)=\int_0^m \frac{\log f_k(t)}{t^2}\,dt+\int_m^\infty \frac{\log f_k(t)}{t^2}\,dt\leq 2a<\varepsilon.$$

This contradicts the supposition that $\mathcal{J}(f_k)\geq\varepsilon$ for all k. The claim on $\{m_k\}$ follows and then Lebesgue's theorem on dominated convergence gives

$$\mathcal{S}(f_{k,m_k})\leq\sum_{\lambda\in\Lambda}\frac{\log(1+(\lambda/m_k)^2)}{\lambda^2}+\mathcal{S}(f_k)\to_k 0.$$

By construction, $\mathcal{J}(f_{k,m_k})=a$ for all k, and this contradicts, as noticed earlier, Lemma 4.3. The theorem is proved.

5. The main results

Throughout this section Λ denotes a symmetric and relatively h-dense sequence of the real line having the extra property of separation: $\lambda_{n+2}-\lambda_n\geq h$.

Before formulating the main results we state the following simple but powerful lemma, providing the link between arbitrary functions and even ones greater than or equal to 1 on the real line, taking the value 1 at the origin.

Lemma 5.1. *For any $\alpha>0$ there is $M>0$ such that for any function $f:\Lambda\to\mathbf{C}$ satisfying*

$$\sum_{\Lambda}\frac{\log^+|f(\lambda)|}{\lambda^2+1}\leq\alpha,$$

we have

$$\sum_{\Lambda\cap(0,\infty)}\lambda^{-2}\log(1+\lambda^2|f(\lambda)+f(-\lambda)|^2/M^2)\leq 6\alpha$$

and

$$\sum_{\Lambda\cap(0,\infty)}\lambda^{-2}\log(1+|f(\lambda)-f(-\lambda)|^2/M^2)\leq 6\alpha.$$

The proof of this lemma is similar to the one in [4, p. 519]. Here we need that (1) is satisfied.

Theorem 5.2. *Any entire function f of exponential type $<T_*/h$ satisfying*

$$\sum_{\Lambda}\frac{\log^+|f(\lambda)|}{\lambda^2+1}<\infty$$

belongs to the Cartwright class.

Proof. This is almost exactly like the proof of Theorem 1.6 in [7] so let us only indicate very briefly how to proceed. Suppose that we have an entire function f

of exponential type less than T_*/h with finite logarithmic sum over Λ and infinite logarithmic integral. For $f_k(z)=f(z)/k$ we thus have

$$\sum_{\lambda\in\Lambda}\frac{\log^+|f_k(\lambda)|}{\lambda^2+1}\to_k 0 \text{ and } \int\frac{\log^+|f_k(t)|}{t^2+1}\,dt\equiv\infty \quad \text{for all } k.$$

From Proposition 2.1 we see that f_k is of zero exponential growth on the real line. We can then use a result on weighted uniform approximation by sums of imaginary exponentials of exponential type D slightly larger than the type of f to construct a sequence $\{\varphi_k\}$ from $\mathcal{A}(2D)$ with the properties $\mathcal{S}(\varphi_k)\to_k 0$ and $\mathcal{J}(\varphi_k)\to_k\infty$. This contradicts the comparison theorem.

Theorem 5.3. *Given $B_0<T_*/h$ and $\gamma>0$ there is $\eta_0>0$ such that for any $\eta\le\eta_0$ there is $C_\eta>0$ with the property that*

$$|f(z)|\le C_\eta\exp(B|y|+\gamma|z|),\quad z\in\mathbf{C},$$

for all entire functions f of exponential type $\le B\le B_0$ satisfying

$$\sum_{\lambda}\frac{\log^+|f(\lambda)|}{\lambda^2+1}\le\eta.$$

Proof. Again the main job has already been done. The proof is almost the same as of Theorem 1.7 in [7]. The main ingredient is the comparison theorem. Lemma 5.1 and [7, Lemma 8.1] are also used.

Remark 5.4. The above theorems clearly hold when Λ is any sequence of the real line having a symmetric, relatively h-dense subsequence. The proofs presented here need symmetry (outside a bounded subset). It is not known how to lift this requirement.

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