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Dual spaces of dyadic Hardy spaces generated by a rearrangement invariant space X on $[0, 1]$

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Abstract. First we define the dyadic Hardy space $H_X(d)$ for an arbitrary rearrangement invariant space X on $[0,1]$. We remark that previously only a definition of $H_X(d)$ for X with the upper Boyd index $q_X<\infty$ was available. Then we get a natural description of the dual space of H_X, in the case X having the property $1\le p_X\le q_X<2$, improving an earlier result [P1].

1. Introduction

In the last 20 years many papers about Hardy spaces have been published. The interest in the topic includes Hardy spaces of analytic functions having the classical spaces $H^1(D)$ and $H^p(\mathbf{R}^n)$ as representative examples and Hardy spaces of martingales, for instance the dyadic Hardy space $H^1(d)$.

Motivated by the deep study of the rearrangement invariant spaces (r.i.s.) which was carried out by W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri in 1979 [JMST], we initiated a detailed study of dyadic Hardy spaces $H_X(d)$ generated by a r.i.s. X on $I=[0,1]$. (See [P1], [P2], [P3].)

Using the ideas of M. Frazier and B. Jawerth [FJ], very recently we improved some earlier results on this subject and proved some new results.

For the unexplained terminology we refer to [LT1] and [LT2].

We recall the notion of a *rearrangement invariant space of functions* (r.i.s.) X on $I=[0,1]$, following [LT2]. We consider without any contrary mention that all Banach spaces are real.

Now we say that X is a r.i.s. on $I=[0,1]$ if:

(1) The space X is a Banach lattice of Lebesgue measurable functions with respect to the a.e. pointwise order relation on $I=[0,1]$.

(2) The space X is an order ideal of the space $M(I)$ of all Lebesgue measurable functions on I, i.e. if $f,g\in M(I)$ with $|f|\le|g|$ and $g\in X$ it follows that $f\in X$ and $\|f\|\le\|g\|$.

(3) The space X contains all characteristic functions of all Lebesgue measurable subsets $A \subset I$.

(4) If f and g are *equimeasurable* (i.e.

$$|\{t \in I\,;|f(t)|>\lambda\}|=|\{t \in I\,;|g(t)|>\lambda\}|$$

for every $\lambda>0$, where $|A|$ is the Lebesgue measure of $A \subset I$), then for $f \in X$ it follows that $g \in X$ and $\|f\|_X=\|g\|_X$.

(5) We assume that the canonical injections $i: L_\infty \rightarrow X$ and $j: X \rightarrow L_1(I)$ have norms less than or equal to 1.

(6) We assume also that X is either a *minimal space* (i.e. the simple functions are dense in X) or a *maximal space*, i.e. $X=X''$, where $X'=\{g: I \rightarrow \mathbf{R}; \|g\|_{X'}=\sup_{\|f\|_X \leq 1}|\int fg|<\infty\}$ is the *associate Köthe space of* X.

We call a space X having properties (1), (2) and (3) a *Köthe function space on* I. The classical Orlicz and Lorentz spaces are examples of r.i. spaces.

Now we recall the useful definition of Boyd indices for a r.i.s. X. Let $0<s<\infty$ and put

$$D_s f(t)=\begin{cases} f(t/s), & \text{if } t \leq \min(1, s), \\ 0, & \text{otherwise.} \end{cases}$$

Now put

$$p_X=\sup_{s>1} \frac{\log s}{\log \|D_s\|}$$

and

$$q_X=\inf_{0<s<1} \frac{\log s}{\log \|D_s\|}.$$

Here p_X and q_X are called *the Boyd indices of* X, and they satisfy the relations $1 \leq p_X \leq q_X \leq \infty$ and $p_{X'}=(q_X)'$; $q_{X'}=(p_X)'$, where $1/p+1/p'=1$.

We now define the dyadic Hardy space $H_X(d)$. Let $Q \subset I$ be a dyadic interval. Let h_Q be the *L_2-normalized Haar function supported by* Q, i.e.,

$$h_Q=\frac{1}{|Q|^{1/2}}(1_{Q_1}-1_{Q_2}),$$

where Q_1 (resp. Q_2) is the left half (resp. the right half) of the interval Q. Then for every Lebesgue measurable function f on I with $f=\sum_Q s_Q h_Q$ a.e., put

$$S(f)=\Big(\sum_Q |s_Q|^2/|Q| 1_Q\Big)^{1/2}.$$

Now let X be a r.i.s. on I such that $q_X<\infty$. We define

$$H_X(d)=\{f\in M(I)\,;\,\|f\|_{H_X}=\|S(f)\|_X<\infty\}$$

and call $H_X(d)$ the *dyadic Hardy space generated by the r.i.s. X on I.*

If $X=L^1$, then $H_X(d)$ coincides with the dyadic Hardy space $H^1(d)$ introduced by A. Garsia [G]. It is known (see [P1]) that $H_X(d)$ is a Banach space and that $H_X(d)$ is isomorphic to a closed subspace of X itself, whenever $q_X<\infty$.

But more is true (see [A]).

Theorem 1.1. *Let X be a r.i.s. on I. The following assertions are equivalent:*

(1) *The Boyd indices of X satisfy the inequalities $1<p_X\le q_X<\infty$.*

(2) *There is a constant $C>0$ such that*

$$C^{-1}\|f\|_X\le\|S(f)\|_X\le C\|f\|_X$$

for all $f\in X$.

Therefore from the point of view of the isomorphic theory of Banach spaces, only the r.i.s. X such that either $p_X=1$ or $q_X=\infty$ are of interest.

Now we extend the previous definition to r.i.s. X with $q_X=\infty$. We use some ideas of M. Frazier and B. Jawerth (see [FJ]).

First we extend a well-known inequality of C. Fefferman and E. Stein (see [FS]).

Theorem 1.2. (Fefferman–Stein inequality) *Let $(f_i)_{i=1}^{\infty}$ be a sequence of functions on $\mathbf{R}$. Then the following inequality holds:*

$$\left\|\left(\sum_{k=1}^{\infty}|Mf_k|^r\right)^{1/r}\right\|_{L_p}\le A_{r,p}\left\|\left(\sum_{k=1}^{\infty}|f_k|^r\right)^{1/r}\right\|_{L_p}\qquad(1<r,p<\infty).\tag{1.1}$$

Here

$$Mf(x)=\sup_{Q\ni x,Q\text{ interval}}\frac{1}{|Q|}\int_Q|f(x)|\,dx,$$

for $f\in L^1(\mathbf{R})$.

We omit the proof but instead we extend the theorem to an arbitrary r.i.s. X on I such that $1<p_X\le q_X<\infty$.

2. Dyadic Hardy spaces $H_X(d)$ for r.i.s. X with $q_X=\infty$

Theorem 2.1. *Let X be a r.i.s. on I, I being either $(0,\infty)$ or $(0,1)$, such that $1<p_X\le q_X<\infty$ and let $1<r<\infty$. Then we have*

$$\left\|\left(\sum_{i=1}^{n}|Mf_i|^r\right)^{1/r}\right\|_X\le C(X,r)\left\|\left(\sum_{i=1}^{n}|f_i|^r\right)^{1/r}\right\|_X\tag{2.1}$$

for any choice of elements $f_i \in X$, $i \le n$, and every $n \in \mathbf{N}$.

Proof. We use Theorem 1.2 and interpolation techniques. (See [BS].) Let $\tau > 0$ and $1 < p < p_X \le q_X < q < \infty$. Put $f := (\sum_{k=1}^n |f_k|^r)^{1/r}$ and

$$f_k^1 = \begin{cases} f_k(x), & \text{if } f^r(x) \le \tilde{f}^r(\tau), \\ \dfrac{\tilde{f}(\tau) f_k(x)}{f(x)}, & \text{if } f^r(x) > \tilde{f}^r(\tau), \end{cases}$$

$1 \le k \le n$. Here we let

$$\tilde{f}(x) = \inf_{|E|=x} \sup_{u \in I \setminus E} |f(u)|, \quad x > 0.$$

Then

$$\left(\sum_{k=1}^n |f_k^1|^r\right)(x) = \begin{cases} f^r(x), & \text{if } f^r(x) \le \tilde{f}^r(\tau), \\ \tilde{f}^r(\tau), & \text{if } f^r(x) > \tilde{f}^r(\tau). \end{cases}$$

(See [BS, pp. 223–224].) So

$$\left(\sum_{k=1}^n |f_k^1|^r\right)^{\sim}(t) = \min(\tilde{f}^r(t), \tilde{f}^r(\tau)).$$

Put now

$$f_k^2(x) = (|f_k(x)|^r - |f_k^1(x)|^r)^{1/r} \operatorname{sgn} f_k(x), \quad k = 1, 2, \dots, n.$$

Thus it follows

$$|f_k(x)|^r = |f_k^1(x)|^r + |f_k^2(x)|^r, \quad 1 \le k \le n. \tag{2.2}$$

Moreover,

$$\left(\sum_{k=1}^n |f_k^2(x)|^r\right)(t) = \left\{ \begin{array}{ll} 0, & \text{if } f^r(t) \le \tilde{f}^r(\tau) \\ f^r(t) - \tilde{f}^r(\tau), & \text{otherwise} \end{array} \right\} = [f^r(t) - \tilde{f}^r(\tau)]^+$$

and

$$\left(\sum_{k=1}^n |f_k^2|^r\right)^{\sim}(t) = (\tilde{f}^r(t) - \tilde{f}^r(\tau))^+$$

for every $t \in I$.

Put now

$$\|f\|_{p,q}=\begin{cases}\left(\int_0^\infty [t^{1/p}\tilde{f}(t)]^q\,\frac{dt}{t}\right)^{1/q}, & 0<q<\infty,\\ \sup\limits_{0<t<\infty} t^{1/p}\tilde{f}(t), & q=\infty.\end{cases}$$

Then it follows that

$$\left\|\left(\sum_k |f_k^1|^r\right)^{1/r}\right\|_{q,1}=q\tau^{1/q}\tilde{f}(\tau)+\int_\tau^\infty t^{1/q}\tilde{f}(t)\,\frac{dt}{t} \tag{2.3}$$

and

$$\begin{aligned}\left\|\left(\sum_k |f_k^2|^r\right)^{1/r}\right\|_{p,1}&=\int_0^\tau t^{1/p}[\tilde{f}^r(t)-\tilde{f}^r(\tau)]^{1/r}\,\frac{dt}{t}\\ &\le 2\int_0^\tau t^{1/p}\tilde{f}(t)\,\frac{dt}{t}-p\tilde{f}(\tau)\tau^{1/p}.\end{aligned} \tag{2.4}$$

Since $1<p,q<\infty$, by Theorem 1.2 it follows that there is a constant $A>0$ such that we have the inequality (1.1) for $1<r<\infty$, p, q and the functions $(f_k^2)_{k=1}^n$ and $(f_k^1)_{k=1}^n$.

By Proposition 4.2 in [BS, p. 217] it follows that

$$\begin{aligned}\left\|\left(\sum_{k=1}^n |Mf_k^1|^r\right)^{1/r}\right\|_{q,\infty}&\le C\left\|\left(\sum_{k=1}^n |Mf_k^1|^r\right)^{1/r}\right\|_{q}\\ &\le C'\left\|\left(\sum_k |f_k^1|^r\right)^{1/r}\right\|_{q}\le C''\left\|\left(\sum_k |f_k^1|^r\right)^{1/r}\right\|_{q,1}\end{aligned} \tag{2.5}$$

and similarly

$$\left\|\left(\sum_k |Mf_k^2|^r\right)^{1/r}\right\|_{p,\infty}\le C''\left\|\left(\sum_k |f_k^2|^r\right)^{1/r}\right\|_{p,1}. \tag{2.6}$$

By (2.5) we get

$$\left(\left(\sum_k |Mf_k^1|^r\right)^{1/r}\right)^\sim\left(\tfrac{1}{2}\tau\right)\le C\left(\tfrac{1}{2}\tau\right)^{-1/q}\left\|\left(\sum_k |f_k^1|^r\right)^{1/r}\right\|_{q,1} \tag{2.7}$$

and (2.6) implies that

$$\left(\left(\sum_k |Mf_k^2|^r\right)^{1/r}\right)^\sim\left(\tfrac{1}{2}\tau\right)\le C\left(\tfrac{1}{2}\tau\right)^{-1/p}\left\|\left(\sum_k |f_k^2|^r\right)^{1/r}\right\|_{p,1}. \tag{2.8}$$

Using (2.3) and (2.7) we get

$$\Big(\Big(\sum_k |Mf_k^1|^r\Big)^{1/r}\Big)^{\sim}\big(\tfrac12\tau\big) \le C\Big(\tau^{-1/q}\int_\tau^\infty t^{1/q}\tilde f(t)\,\frac{dt}{t} + p\tilde f(\tau)\Big) \tag{2.9}$$

and similarly

$$\Big(\Big(\sum_k |Mf_k^2|^r\Big)^{1/r}\Big)^{\sim}\big(\tfrac12\tau\big) \le C\Big(-p\tilde f(\tau) + 2\tau^{-1/p}\int_0^\tau t^{1/p}\tilde f(t)\,\frac{dt}{t}\Big). \tag{2.10}$$

But, in view of (2.2) it follows that

$$\begin{aligned}\Big(\sum_k |Mf_k|^r\Big)^{1/r} &\le \Big(\sum_k (M[(|f_k^1|^r + |f_k^2|^r)^{1/r}])^r\Big)^{1/r}\\ &\le \Big(\sum_k [M(|f_k^1| + |f_k^2|)]^r\Big)^{1/r} \le \Big(\sum_k |Mf_k^1|^r\Big)^{1/r} + \Big(\sum_k |Mf_k^2|^r\Big)^{1/r},\end{aligned}$$

thus

$$\begin{aligned}\Big(\Big(\sum_k (Mf_k)^r\Big)^{1/r}\Big)^{\sim}(\tau) &\le \Big(\Big(\sum_k (Mf_k^1)^r\Big)^{1/r}\Big)^{\sim}\big(\tfrac12\tau\big) + \Big(\Big(\sum_k (Mf_k^2)^r\Big)^{1/r}\Big)^{\sim}\big(\tfrac12\tau\big)\\ &\le C\Big(\tau^{-1/q}\int_\tau^\infty t^{1/q}\tilde f(t)\,\frac{dt}{t} + 2\tau^{-1/p}\int_0^\tau t^{1/p}\tilde f(t)\,\frac{dt}{t}\Big)\\ &= CS_\sigma(\tilde f)(\tau),\end{aligned}$$

where σ is the interpolation interval $[(1/q, 1/q), (1/p, 1/p)]$ and S_σ is the Calderón operator. (See [BS].)

Thus, since $p<p_X$ and $q_X<q$, the proof of Theorem 5.16 in [BS, p. 153] gives us

$$\Big\|\Big(\sum_k |Mf_k|^r\Big)^{1/r}\Big\|_X \le C\Big\|\Big(\sum_k |f_k|^r\Big)^{1/r}\Big\|_X. \quad \square$$

Now we use Theorem 2.1 in order to get an equivalent norm on $H_X(d)$, whenever X is a r.i. space on $I=[0,1]$ with $q_X<\infty$.

For a dyadic interval $Q\subset I$, let $E_Q\subset Q$ be a Lebesgue measurable subset such that $|E_Q|>\frac12|Q|$, and for $f=\sum_Q s_Q h_Q$ put

$$S_E(f) = \Big(\sum_Q |s_Q|^2/|E_Q| 1_{E_Q}\Big)^{1/2}.$$

Then we have the following corollary.

Corollary 2.2. *Let X be a r.i. space on $I=(0,1)$ with $1\le p_X\le q_X<\infty$. Then*

$$\|f\|_{H_X}\sim\inf\{\|S_E(f)\|_X\,;E_Q\subset Q,\ |E_Q|>\tfrac12|Q|\}.$$

Proof. Obviously $\|S_E(f)\|_X\le 2\|S(f)\|_X=2\|f\|_{H_X}$.

Conversely, it is clear that $1_Q\le 2M(1_{E_Q})$. Therefore, for every $A>0$ and every dyadic interval Q, we have $1_Q\le 2^{1/A}[M(1_{E_Q})]^{1/A}$ and

$$S(f)\le 2^{1/A}\Big(\sum_Q\Big[M\Big(\frac{|s_Q|^A}{|E_Q|^{A/2}}1_{E_Q}\Big)\Big]^{2/A}\Big)^{1/2}. \tag{2.11}$$

We choose $A>0$ such that $1<p_X/A$ and $1<2/A$. Put $r=2/A$ and, since

$$1<\frac{p_X}{A}=p_{X^{1/A}}\le\frac{q_X}{A}=q_{X^{1/A}}<\infty,$$

(where $X^{1/A}=\{f:I\to\mathbf{R};|f|^{1/A}\in X\}$ and $\|f\|_{X^{1/A}}:=\|\,|f|^{1/A}\|_X^A$) we have by (2.11) and Theorem 2.1

$$\begin{aligned}\|f\|_{H_X}&\le 2^{1/A}\Big\|\Big(\sum_Q\Big[M\Big(\frac{|s_Q|^A}{|E_Q|^{A/2}}1_{E_Q}\Big)\Big]^{2/A}\Big)^{1/2}\Big\|_X\\ &\le 2^{1/A}\Big\|\Big(\sum_Q\frac{|s_Q|^2}{|E_Q|}1_{E_Q}\Big)^{A/2}\Big\|_{X^{1/A}}^{1/A}\\ &=2^{1/A}\Big\|\Big(\sum_Q\frac{|s_Q|^2}{|E_Q|}1_{E_Q}\Big)^{1/2}\Big\|_X=2^{1/A}\|S_E(f)\|_X.\quad\square\end{aligned}$$

Define now for $f=\sum_Q s_Qh_Q$,

$$m(f)=\sup_{Q\text{ dyadic interval}}\Big[\Big(\sum_{P\subset Q}\frac{|s_P|^2}{|P|}1_P\Big)^{1/2}\Big]^{\sim}\big(\tfrac14|Q|\big)1_Q. \tag{2.12}$$

Then we get the following theorem.

Theorem 2.3. *Let X be a r.i. space on $I=[0,1]$ with $q_X<\infty$. Then*

$$\|f\|_{H_X}\sim\inf\{\|S_E(f)\|_X\,;E_Q\subset Q,\ |E_Q|>\tfrac12|Q|\}\sim\|m(f)\|_X. \tag{2.13}$$

Proof. We use the argument of Proposition 5.5 in [FJ]. Since the operator M is of weak type (1,1) there is a constant $C>0$ such that, for each $t>0$,

$$|\{x\,;m(f)(x)>t\}|\le\big|\{x\,;M(1_{\{y:S(f)(y)>t\}})(x)>\tfrac14\}\big|\le c|\{x\,;S(f)(x)>t\}|.$$

Since X is a r.i. space it follows that

$$\|m(f)\|_X \leq c\|S(f)\|_X \tag{2.14}$$

for all $f \in H_X$.

For $x \in I$ put

$$\nu(x) = \inf\left\{\nu \in \mathbf{Z}\,;\left(\sum_{l(Q)\leq 2^{-\nu}} \frac{|s_Q|^2}{|Q|} 1_Q(x)\right)^{1/2} \leq m(f)(x)\right\},$$

where $l(Q)$ is the length of the interval Q. Put

$$E_Q = \{x \in Q\,; 2^{-\nu(x)} \geq l(Q)\} = \{x \in Q\,; S_Q(f)(x) \leq m(f)(x)\}$$

for every dyadic interval Q, where

$$S_Q(f) = \left(\sum_{P\subset Q} \frac{|s_P|^2}{|P|} 1_P\right)^{1/2}.$$

By definition of $m_Q(f) := \widetilde{S}_Q(f)\left(\frac{1}{4}|Q|\right)$, it follows that $|E_Q| \geq \frac{3}{4}|Q|$ and

$$\left(\sum_{Q} \frac{|s_Q|^2}{|E_Q|} 1_{E_Q}(x)\right)^{1/2} \leq Cm(f)(x) \tag{2.15}$$

for $x \in I$.

Therefore we have

$$\|S_E(f)\|_X \leq c\|m(f)\|_X. \tag{2.16}$$

By (2.14), (2.16) and Corollary 2.2, (2.13) follows. □

Thus, if $q_X < \infty$, we have

$$H_X(d) = \{f \in L^1(I)\,; \|m(f)\|_X < \infty\}.$$

Now for an arbitrary r.i. space X on $I = (0, 1)$ (even in the case $q_X = \infty$) we may define $H_X(d)$ as follows.

Definition 2.4. Let X be an arbitrary r.i. space on I. Then we put

$$H_X(d):=\{f\in L^1(I)\,;\, |\!|\!|f|\!|\!|_{H_X}:=\|m(f)\|_X<\infty\}. \tag{2.17}$$

The above definition permits us to improve the description of the dual space of $H_X(d)$ whenever X is a r.i.s. on I such that $1\le p_X\le q_X<2$, which was done in [P1].

In order to prove that we extend Theorem 5.9 in [FJ]. First of all we extend Proposition 5.5 in [FJ].

For $f=\sum_Q s_Q h_Q$ put

$$\begin{aligned} f^\sharp(t)&=\sup_{P\ni t}\biggl(\frac{1}{|P|}\int_P\sum_{Q\subset P}\frac{|s_Q|^2}{|Q|}1_Q(x)\,dx\biggr)^{1/2}\\ &=\sup_{P\ni t}\biggl(\frac{1}{|P|}\sum_{Q\subset P}|s_Q|^2\biggr)^{1/2}=\sup_{P\ni t}\biggl(\frac{1}{|P|}\int_P|f(u)-f_P|^2\,du\biggr)^{1/2}\end{aligned} \tag{2.18}$$

where $f_P:=(1/|P|)\int_P f(u)\,du$.

We then have the following result.

Proposition 2.5. *Let X be a r.i. space on I such that $2<p_X\le q_X\le\infty$. Then it follows*

$$\|m(f)\|_X\sim\|f^\sharp\|_X. \tag{2.19}$$

Proof. By Chebyshev's inequality we have

$$|\{x\in Q\,;S_Q(f)(x)>\varepsilon\}|\le\frac{1}{\varepsilon^2}\int_Q(S_Q(f)(x))^2\,dx\le\frac{|Q|}{\varepsilon^2}(f^\sharp(t))^2 \tag{2.20}$$

for every $t\in Q$ and $\varepsilon>0$.

If $\varepsilon>2f^\sharp(t)$, we have, by (2.20),

$$|\{x\in Q\,;S_Q(f)(x)>\varepsilon\}|<\tfrac14|Q|, \tag{2.21}$$

which in turn implies that

$$m_Q(f)\le 2f^\sharp(t),$$

where

$$m_Q(f)=\biggl(\sum_{P\subset Q}\frac{|s_P|^2}{|P|}1_P\biggr)^{\sim}\bigl(\tfrac14|Q|\bigr)$$

for all $t\in Q$.

Thus

$$m(f)(t)=\sup_Q m_Q(f)1_Q(t)\le 2f^\sharp(t)$$

and

$$\|m(f)\|_X\le 2\|f^\sharp\|_X. \tag{2.22}$$

Conversely we consider

$$\nu(x)=\inf\left\{\nu\in\mathbf{Z}\,;\left(\sum_{l(Q)\le 2^{-\nu}}\frac{|s_Q|^2}{|Q|}1_Q(x)\right)^{1/2}\le m(f)(x)\right\}. \tag{2.23}$$

Now let $E_Q=\{x\in Q;2^{-\nu(x)}\ge l(Q)\}=\{x\in Q;S_Q(f)(x)\le m(f)(x)\}$ for every dyadic interval Q. As in the proof of Theorem 2.3 it follows that $|E_Q|/|Q|\ge\frac{3}{4}$ and

$$\left(\sum_Q\frac{|s_Q|^2}{|Q|}1_{E_Q}(x)\right)^{1/2}\le Cm(f)(x) \tag{2.24}$$

for every $x\in I$.

Integrating (2.24) on the dyadic fixed interval P we have

$$\sum_{Q\subset P}|s_Q|^2\le C\int_P m^2(f)(x)\,dx$$

or

$${f^\sharp}^2(t)\le C\sup_{P\ni t}\frac{1}{|P|}\int_P m^2(f)(x)\,dx$$

for all $t\in I$.

Now, using the fact that M is a bounded (non linear) operator on Y, for every r.i. space Y such that $1<p_Y\le q_Y<\infty$, we have, denoting by X_2 the space

$$X_2=\{f\colon I\to\mathbf{R}\,;|f|^{1/2}\in X\}$$

with the quasi-norm $\|f\|_{X_2}:=\||f|^{1/2}\|_X^2$, and using the hypothesis $p_{X_2}>1$,

$$\|f^{\sharp 2}\|_{X_2}\le C\|M(m^2(f))\|_{X_2}\le C\|m^2(f)\|_{X_2}=C\|m(f)\|_X^2.$$

Thus

$$\|f^\sharp\|_X\le C\|m(f)\|_X, \tag{2.25}$$

and (2.22) and (2.25) prove Proposition 2.5. □

Theorem 2.6. *Let X be a r.i. space on I such that $1=p_X\le q_X<2$. Then the dual space of H_X may be identified with $H_{X'}$, by the map carrying $l\in(H_X)^*$ onto $t=\sum_Q t_Q h_Q\in H_{X'}$, where $t_Q=l(h_Q)$ for every dyadic interval Q. Moreover,*

$$\|l\|_{(H_X)^*}\sim\|t\|_{H_{X'}}.$$

Proof. Let $t=\sum_Q t_Q h_Q$ and $s=\sum_Q s_Q h_Q$. Using the notation of Proposition 2.5 we have, by the Cauchy–Schwarz inequality and (2.24),

$$\begin{aligned}\Big|\sum_Q s_Q t_Q\Big| &\le c\int\sum_Q \frac{|s_Q|}{|Q|^{1/2}}1_Q\frac{|t_Q|}{|E_Q|^{1/2}}1_{E_Q}\\ &\le c\int\Big(\sum_Q\frac{|s_Q|^2}{|Q|}1_Q\Big)^{1/2}\Big(\sum_Q\frac{|t_Q|^2}{|E_Q|}1_{E_Q}\Big)^{1/2}\\ &\le c\|s\|_{H_X}\Big\|\Big(\sum_Q\frac{|t_Q|^2}{|E_Q|}1_{E_Q}\Big)^{1/2}\Big\|_{X'}\le c\|s\|_{H_X}\|m(t)\|_{X'}=c\|s\|_{H_X}\|t\|_{H_{X'}},\end{aligned}$$

i.e.,

$$\|l\|_{(H_X)^*}\le c\|t\|_{H_{X'}}. \tag{2.26}$$

Conversely, let $l\in(H_X)^*$, $t_Q=l(h_Q)$ and $s=\sum_Q s_Q h_Q\in H_X$. Now fix a dyadic interval P and let us consider the space $X_1=\{Q;Q\subset P\}$ endowed with the measure $\mu(Q)=|Q|/|P|$.

Then

$$\begin{aligned}\Big(\frac{1}{|P|}\sum_{Q\subset P}|t_Q|^2\Big)^{1/2}=\Big\|\Big(\frac{t_Q}{|Q|^{1/2}}\Big)_Q\Big\|_{l^2(X_1,d\mu)} &= \sup_{\|s\|_{l^2(X_1,d\mu)}\le 1}\Big|\frac{1}{|P|}\sum_{Q\subset P}s_Q t_Q|Q|^{1/2}\Big|\\ &\le\|l\|_{(H_X)^*}\sup_{\|s\|_{l^2(X_1,d\mu)}\le 1}\Big\|\sum_{Q\subset P.}\frac{s_Q|Q|^{1/2}}{|P|}h_Q\Big\|_{H_X}.\end{aligned}$$

But

$$\begin{aligned}\Big\|\sum_{Q\subset P}\frac{s_Q|Q|^{1/2}}{|P|}h_Q\Big\|_{H_X} &= \Big\|\Big(\sum_{Q\subset P}\frac{|s_Q|^2}{|P|^2}1_Q\Big)^{1/2}\Big\|_X\\ &= \sup_{\substack{\|h\|_{X'}\le 1\\ h\text{ decreasing}}}\int\Big[\Big(\sum_{Q\subset P}\frac{|s_Q|^2}{|P|^2}1_Q\Big)^{1/2}\Big]^{\sim}h,\end{aligned}$$

since X is a r.i. space.

On the other hand, for a fixed $\varepsilon>0$ there is an $s\in l^2(X_1,d\mu)$ such that

$$\|s\|_{l^2(X_1,d\mu)}\leq 1$$

and an $h\in X'$, $\|h\|_{X'}\leq 1$, h decreasing, such that

$$\sup_{\|s\|_{l^2(X_1,d\mu)}\leq 1}\Big\|\sum_{Q\subset P}\frac{s_Q|Q|^{1/2}}{|P|}h_Q\Big\|_{H_X}\leq\Big\|\sum_{Q\subset P}\frac{s_Q|Q|^{1/2}}{|P|}h_Q\Big\|_{H_X}+\tfrac{1}{2}\varepsilon$$
$$\leq\int\Big[\Big(\sum_{Q\subset P}\frac{|s_Q|^2}{|P|^2}1_Q\Big)^{1/2}\Big]^{\sim}h+\varepsilon.$$

Consequently,

$$\Big(\frac{1}{|P|}\sum_{Q\subset P}|t_Q|^2\Big)^{1/2}\leq\|l\|\Big[\int\Big[\Big(\sum_{Q\subset P}\frac{|s_Q|^2}{|P|^2}1_Q\Big)^{1/2}\Big]^{\sim}h+\varepsilon\Big]$$
$$\leq\|l\|\Big[\Big(\frac{1}{|P|}\int_P h^2\Big)^{1/2}\Big(\int_P\sum_{Q\subset P}\frac{|s_Q|^2}{|P|}1_Q\Big)^{1/2}+\varepsilon\Big]$$
$$\leq\|l\|\Big[\Big(\frac{1}{|P|}\int_P h^2\Big)^{1/2}+\varepsilon\Big].$$

Now

$$t^\sharp\leq\|l\|([M(h^2)]^{1/2}+\varepsilon)$$

and, since $p_{X'_2}=\frac{1}{2}p_{X'}>1$,

$$\|t^\sharp\|_{X'}\leq\|l\|\big[\|[M(h^2)]^{1/2}\|_{X'}+\varepsilon\big]=\|l\|\big[\|M(h^2)\|_{X'_2}^{1/2}+\varepsilon\big]$$
$$\leq\|l\|\big[\|h^2\|_{X'_2}^{1/2}+\varepsilon\big]=\|l\|\big[\|h\|_{X'}+\varepsilon\big]\leq\|l\|(1+\varepsilon).$$

Since $\varepsilon>0$ is arbitrarily small we have

$$\|t^\sharp\|_{X'}\leq\|l\|_{(H_X)^*}. \tag{2.27}$$

Thus (2.26) and (2.27) prove the theorem. □

It is clear by Theorem 2.6 that for $X=L^1$, $H_{X'}$ coincides with the classical space

$$\mathrm{BMO}(d):=\{f:I\to\mathbf{R}\,;\,\|f^\sharp\|_{L_\infty}<\infty\}.$$

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