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Riesz transforms on compact Lie groups, spheres and Gauss space

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Notation. For $x, y \in \mathbf{R}^n$, $x=(x_1, \dots, x_n)$, $y=(y_1, \dots, y_n)$, $|x|=(\sum_{j=1}^n x_j^2)^{1/2}$ is the Euclidean norm of x and $\langle x, y\rangle=\sum_{j=0}^n x_j y_j$ is the inner product of x and y. Sometimes, we write $x \cdot y$ instead of $\langle x, y\rangle$. If $(X, \mathcal{F}, \mu)$ is a measure space, $f: X \to \mathbf{R}^n$ is a measurable function and $p \in [1, \infty)$, the L^p norm of f is defined by $\|f\|_p = \|f\|_{L^p(X, \mathbf{R}^n)} = \left(\int_X |f|^p \, dx\right)^{1/p}$. If S is a linear operator which maps $\mathbf{R}^n$ valued L^p functions on $(X, \mathcal{F}, \mu)$ to $\mathbf{R}^m$ valued L^p functions on $(X_1, \mathcal{F}_1, \mu_1)$, that $\|S\|_p = \sup\{\|Sf\|_p : \|f\|_p = 1\}$ is the operator norm of S. If $X = X_1$ and $\mu = \mu_1$, we denote by $I \oplus S$ the operator with $(I \oplus S)f = (f, Sf)$, the latter being an $\mathbf{R}^{n+m}$ valued function.

Let $\mathcal{A}$ be a linear space of integrable functions on $(X, \mathcal{F}, \mu)$. We denote by $\mathcal{A}_0$ the subspace $\mathcal{A}_0 = \left\{f \in \mathcal{A} : \int_X f \, d\mu = 0\right\}$. If a linear operator S is only defined on $\mathcal{A}_0$, we still denote by $\|S\|_p = \sup\left\{\|Sf\|_p : f \in \mathcal{A}_0,\ \|f\|_p = 1\right\}$. For instance, $C_0^\infty(M) = \left\{f \in C^\infty(M) : \int_M f(x) \, dx = 0\right\}$, if M is a smooth Riemannian manifold and dx denotes the volume element on M. The L^p norm of a measurable vector field U on M is, by definition, the L^p norm of $|U|$, the modulus of U. Unless otherwise specified, $L^p(X)$ and $L_0^p(X)$ will denote spaces of real valued functions on X.

0. Introduction

Let M be a Riemannian manifold without boundary, ∇_M, div_M and $\Delta_M = \operatorname{div}_M \nabla_M$ be, respectively, the gradient, the divergence and the Laplacian associated with M. Then $-\Delta_M$ is a positive operator and the linear operator

$$R^M = \nabla_M \circ (-\Delta_M)^{-1/2} \tag{1}$$

is well defined on $L_0^2(M)$ and, in fact, an isometry in the L^2 norm. If f is a real valued function on M and $x \in M$, then $R^M f(x) \in T_x M$ is a vector tangent to M at x.

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The L^p norm of $R^M f$ is, by definition, the L^p norm of $x \mapsto |R^M f(x)|$, where $|\cdot|$ is the Euclidean norm induced on $T_x M$ by the Riemannian metric. The operator R^M is called the *Riesz transform* on M. In (1), $A=(-\Delta_M)^{-1/2}$ is the positive operator such that $A \circ A \circ (-\Delta_M)=I$, the identity operator.

If M is compact, as will always be the case in this article, A can first be defined for linear combinations of eigenfunctions of Δ_M, and then extended to $L^2_0(M)$ by continuity. See [GHL].

The Hilbert transform on the unit circle, $\mathcal{H}=-R^{\mathbf{S}^1}$, and the Riesz transform on $\mathbf{R}^n$, $R=R^{\mathbf{R}^n}$, are special cases of (1). The operator R^M is a singular integral operator.

The exact L^p norm of a singular integral operator is known only in a few cases. The first result of this type is Pichorides' determination of the Hilbert transform's L^p norm. For $p \in (1,\infty)$, let $p^*=\max\{p, q: 1/p+1/q=1\}$. Then

$$\|\mathcal{H}\|_p = B_p \tag{2}$$

where $B_p=\cot(\pi/2p^*)$ [Pic]. Later I. E. Verbitsky and M. Essén, [Ve], [Es], independently found that

$$\|I \oplus \mathcal{H}\|_p = E_p \tag{3}$$

where $E_p=(B_p^2+1)^{1/2}$. It has recently been proved that (2) and (3) hold with the directional Riesz transforms on $\mathbf{R}^n$, $R_j=R_j^{\mathbf{R}^n}$, instead of $\mathcal{H}$ and with the same constants. T. Iwaniec and G. Martin [IM] proved the analogue of (2), and soon after R. Bañuelos and G. Wang found a probabilistic proof for analogues of both (2) and (3) in the Euclidean context [BW].

Several authors have proved estimates of the form

$$\|R\|_p \le K_p < \infty \tag{4}$$

where R is the vector Riesz transform on $\mathbf{R}^n$ and K_p is a constant which only depends on p, $1<p<\infty$. The problem of finding the exact value of $\|R\|_p$ is still open, if $n \ge 2$. The first proof of (4) with a value of K_p that does not depend on the dimension n is due to E. M. Stein [S2], [S3]. Alternative proofs with increasingly better constants were given in [DR], [Ba], [Pis], [IM] and [BW]. [IM] has the best known constant for $p \ge 2$ and [BW] has the one for $p \le 2$.

Let now $M=G$ be a compact Lie group endowed with a biinvariant Riemannian metric and let $\mathfrak{G}$ be its Lie algebra. Let $X \in \mathfrak{G}$ be a left invariant vector field such that $|X|=1$, where $|\cdot|$ is the norm induced on $\mathfrak{G}$ by the metric of G. The operator

$$R_X = X \circ (-\Delta_G)^{-1/2} \tag{5}$$

is called the *Riesz transform in the direction* X. The operators R^G and R_X are related as follows. Let $X_1, \dots, X_n$ be an orthonormal basis for $\mathfrak{G}$ and $f\colon G \to \mathbf{R}$. Then $R^G(f)$ can be written as

$$R^G f(a) = \sum_{j=1}^{n} R_{X_j} f(a) X_j(a) \tag{6}$$

if $a \in G$, where $X_j(a)$ is the vector field X_j evaluated in a.

Let B_p and E_p be the constants in (2) and (3). In this article we prove the following theorem.

Theorem 1. *Let G be a compact Lie group endowed with a biinvariant Riemannian metric. We then have, on $L^p_0(G)$,*

$$\|R^G\|_p \le 2(p^* - 1). \tag{7}$$

If $X \in \mathfrak{G}$ and $|X| = 1$, then

$$\|R_X\|_p \le B_p \tag{8}$$

and

$$\|I \oplus R_X\|_p \le E_p. \tag{9}$$

Equality occurs in (8) *and* (9) *if $G = \mathbf{T}^n$, the n-dimensional torus with any of its invariant metrics, or if $G = SO(n)$, the orthogonal group, endowed with its standard metric.*

An estimate like (7) already appears in [S1], with a universal bound A_p that grows as p^2 as $p \to \infty$ instead of our $2(p^* - 1)$. More generally, D. Bakry [B2], [B3], [B4] showed that $\|R^M\|_p$ is universally bounded for M in the class of complete Riemannian manifolds with nonnegative Ricci curvature. See also [CL] and [B2], [B3], [B4] for related results on manifolds.

As we mentioned above, equality holds in (8) and (9) in the noncompact case $G = \mathbf{R}^n$. We conjecture that, in fact, equality should occur in (8) and (9) for all compact Lie groups. An integration by parts shows that $\|R^G\|_2 = 1$, hence (7) can not be best possible.

Let now $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$ be the unit sphere in $\mathbf{R}^n$ with the standard metric. For $1 \le l < m \le n$, consider the differential operator

$$\mathcal{T}_{lm} = x_l \partial_m - x_m \partial_l \tag{10}$$

with $\partial_m=\partial/\partial x_m$. If $x_l+ix_m=re^{i\theta}$, then $\mathcal{T}_{lm}=\partial/\partial\theta$ is the derivative with respect to the angular coordinate in the (x_l,x_m) plane, a well defined vector field on $\mathbf{S}^{n-1}$. The vector fields $\mathcal{T}_{lm}$ are connected to the spherical gradient as follows. If $f\colon\mathbf{S}^{n-1}\to\mathbf{R}$ is smooth, then

$$|\nabla_{\mathbf{S}^{n-1}}f|=\Big(\sum_{l<m}|\mathcal{T}_{lm}f|^2\Big)^{1/2}. \tag{11}$$

This follows from the fact that $\mathbf{S}^{n-1}$ is a homogeneous space of $SO(n)$. See §4. Let U be a vector field on $\mathbf{S}^{n-1}$ of the form

(12)
$$U=\sum_{l<m}\alpha_{lm}\mathcal{T}_{lm},\ \text{where the constants } \alpha_{lm} \text{ satisfy } 1=\sum_{l<m}\alpha_{lm}^2=\sup_{x\in\mathbf{S}^{n-1}}|U(x)|^2.$$

For such U, define

$$Q_U^c=U\circ(-\Delta_{\mathbf{S}^{n-1}})^{-1/2} \tag{13}$$

the *Riesz transform on* $\mathbf{S}^{n-1}$ *in the direction* U. For the relation between $R^{\mathbf{S}^{n-1}}$ and Q_U^c, see §4 below. From now on, we will denote by $R^c=R^{\mathbf{S}^{n-1}}$ the Riesz transform associated with the manifold $\mathbf{S}^{n-1}$. The superscript ^{c} stands for *cylinder*. It is meant as a reminder that R^c and Q^c naturally arise from a Neumann problem in the "cylinder" $\mathbf{S}^{n-1}\times[0,\infty)$.

Theorem 2. *The following estimates hold on* $L_0^p(\mathbf{S}^{n-1})$

$$\|R^c\|_p\le 2(p^*-1) \tag{14}$$

and, if U *is as in* (12),

$$\|Q_U^c\|_p=B_p, \tag{15}$$

$$\|I\oplus Q_U^c\|_p=E_p. \tag{16}$$

The operator R^c is only one of those to whom harmonic analysts have attached the name of *Riesz transform*, or *Riesz system*, on $\mathbf{S}^{n-1}$. See [AL] for a survey of singular integral operators on the sphere. In [KV1] and [KV2] the authors consider the operator R^b defined as

$$R^b=\nabla_{\mathbf{S}^{n-1}}\circ\Big(\frac{\partial}{\partial\nu}\Big)^{-1}$$

where $(\partial/\partial\nu)^{-1}\colon L_0^2(\mathbf{S}^{n-1})\to L_0^2(\mathbf{S}^{n-1})$ is defined on spherical harmonics Y_k of degree $k\ge1$ as $(\partial/\partial\nu)^{-1}Y_k=Y_k/k$. The operator R^b, that we call the *Riesz transform of ball type on* $\mathbf{S}^{n-1}$, is related to the Neumann problem in the unit ball of $\mathbf{R}^n$. See §3. If U is as in (12), the *Riesz transform in the direction* U is the operator

$$Q_U^b=U\circ\Big(\frac{\partial}{\partial\nu}\Big)^{-1}. \tag{17}$$

Theorem 3. *The following estimates hold on* $L_0^p(\mathbf{S}^{n-1})$

$$\|R^b\|_p \le \sqrt{n-1}\,(p^*-1), \tag{18}$$

$$\|Q_U^b\|_p = B_p, \tag{19}$$

$$\|I \oplus Q_U^b\|_p = E_p. \tag{20}$$

Sometimes, dimension free estimates "pass in the limit" to estimates for an infinite dimensional object. This heuristic principle has an application in the case of the sphere, since $\mathbf{S}^{n-1}(\sqrt{n})=\{x\in\mathbf{R}^n : |x|^2=n\}$ goes in the limit to the infinite dimensional Gauss space as n tends to infinity. See, e.g., [M]. The m dimensional Gauss space is the measure space $(\mathbf{R}^m,\gamma)$, where $\gamma(dx)=(2\pi)^{-m/2}e^{-|x|^2/2}\,dx$, $x\in\mathbf{R}^m$, is the m-dimensional Gaussian measure. Let $D=(\partial_1,\dots,\partial_m)$ be the gradient in $\mathbf{R}^m$ and D^* be its formal adjoint with respect to the measure γ. Then

$$A = D^*D = \sum_{j=1}^m \partial_{jj} - x_j\partial_j$$

is a negative operator, sometimes called the *m-dimensional Hermite operator*. The *Riesz transform for the Ornstein–Uhlenbeck process* R^O is then defined as

$$R^O = D\circ(-A)^{-1/2}. \tag{21}$$

Theorem 2 implies the L^p boundedness of R^O.

Theorem 4. *On* $L_0^p(\mathbf{R}^m,\gamma)$ *we have*

$$\|R^O\|_p \le 2(p^*-1). \tag{22}$$

The L^p boundedness of R^O was first proved by P. A. Meyer [Me3]. The best previously known constants in (22) are those in [Pis]. There, one has $\|R^O\|_p \le K_p$, with $K_p=O(p)$, as $p\to\infty$, and $K_p=O((p-1)^{-3/2})$, as $p\to 1$.

See also [G1] for a probabilistic proof. The inequality (22) follows from (15), (16) and an approximation argument that will be developed in §6.

The methods to obtain sharp estimates for singular integrals often have at their heart an argument involving a differential inequality (subharmonicity, convexity), on which one builds up by means of different tools, such as transference. In §1 we summarize some probabilistic preliminaries, including Theorems A and B from [BW], the proofs of which are based on a convexity argument in martingale theory. This is the method of differential subordination of martingales introduced by D. Burkholder [Bu1], [Bu2], [Bu3], and developed by R. Bañuelos and G. Wang [BW]. Theorems A and B will provide the main tool in the proof of Theorem 1 and Theorem 3, together with a probabilistic interpretation of some singular integral operators started by P. A. Meyer, [Me1], [Me2], and developed by R. Gundy and N. Varopoulos [GV]. See also [Va]. We exploit the flexibility of the method, working with martingales on different manifolds and making use of martingale transforms that are not of "matrix type".

§2 and §3 are devoted to the proofs of the estimates from above in Theorem 1 and Theorem 3, respectively. §3 and the proof of Theorem 3 are independent of the part of the article dealing with Theorem 1, Theorem 2 and Theorem 4, the L^p estimate for the Riesz transform in Gauss space. The estimates from above in Theorem 2 will be deduced from Theorem 1 in §4. The estimates from below in Theorem 1, Theorem 2 and Theorem 3 are deduced from analogous estimates for the Hilbert transform on the circle in §5. Their proofs are inspired by the transference method of R. Coifman and G. Weiss [CW] and by a development of this by T. Iwaniec and G. Martin [IM].

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1. Probabilistic preliminaries

In this section we collect some tools from probability theory and prove a lemma, Proposition 1.2, that we need in the proof of Theorem 1 and Theorem 3.

Here and in the following sections, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ will be a filtered probability space such that all $\mathbf{R}^N$ valued martingales $X=\{X_t\}_{t\geq 0}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ have a continuous version $\widetilde{X}$, i.e. $\widetilde{X}$ is a version of X and the map $t\mapsto\widetilde{X}_t(\omega)$ is continuous on $[0,\infty)$, almost surely (a.s.) in $\omega\in\Omega$. The martingales considered in this article are always taken to be continuous. Recall that the L^p norm of a martingale X is given by $\|X\|_p=\sup_{t\geq 0}\|X_t\|_p$, where the L^p norm on the right is with respect to the measure $\mathbf{P}$.

We will denote by $[X]$ the quadratic variation process of X. Then, $[X]_0=0$, $t\mapsto[X]_t(\omega)$ is of bounded variation on compact sets a.s. and $|X_t|^2-[X]_t$ is a real valued martingale. The covariance variation process $[X,Y]$ of two continuous, $\mathbf{R}^N$ valued martingales X and Y is defined similarly, by polarization.

Let X and Y be two continuous, $\mathbf{R}^N$ valued martingales. We say that Y is *differentially subordinate* to X (we write $Y \prec_D X$), if the process $[X]-[Y]$ is nondecreasing, a.s. We say that X and Y are *path orthogonal* ($X \perp Y$) if $[X,Y]$ vanishes identically in t, a.s. on Ω.

The following theorems provide the best constants for some inequalities involving differentially subordinate martingales.

Theorem A. ([Bu1], [Bu2], [BW]) *Let X and Y be $\mathbf{R}^N$ valued martingales such that Y is differentially subordinate to X. Then*

$$\|Y\|_p \leq (p^*-1)\|X\|_p \tag{23}$$

and p^*-1 is best possible in (23).

Theorem B. ([BW]) *Let X and Y be $\mathbf{R}$ valued, path orthogonal martingales such that Y is differentially subordinate to X. Then*

$$\|Y\|_p \leq B_p \|X\|_p \tag{24}$$

and

$$\|X \oplus Y\|_p \leq E_p \|X\|_p. \tag{25}$$

The constants in (24) *and* (25) *are best possible.*

Let now M be a Riemannian manifold of dimension n with Ricci curvature bounded from below. This condition has the purpose of ensuring that a Brownian motion on M does not explode in finite time [Em]. In this article we deal with $M=\mathbf{R}^n$ or $M=N\times\mathbf{R}$, with N a compact Riemannian manifold without boundary, so that this assumption is satisfied.

Let $\langle\cdot,\cdot\rangle$ be the inner product on TM, the tangent space to M. A *Brownian motion in M* is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process $B_t: \Omega \to M$ such that, for all smooth functions $f: M\to\mathbf{R}$,

$$f(B_t)-f(B_0)-\frac{1}{2}\int_0^t \Delta_M f(B_s)\,ds = (I_{df})_t \tag{26}$$

is an $\mathbf{R}$ valued continuous martingale, where Δ_M is the Laplacian on M. See [Em], [IW] for a full exposition of the theory.

Let Ψ be a continuous, adapted process with values in T^*M, the cotangent space of M. We say that Ψ *is above B* if $\Psi_t(\omega)\in T^*_{B_t(\omega)}M$ whenever $t\geq 0$ and

$\omega\in\Omega$. The *Itô integral* of Ψ, $(I_\Psi)_t=\int_0^t \langle\Psi_s, dB_s\rangle$, is characterized by the following properties:

(i) if $\Psi_t=df(B_t)$, with $f\colon M\to\mathbf{R}$ smooth, then $I_\Psi=I_{df}$ is defined by (26);

(ii) if K is a real valued, continuous process, then $(I_{K\Psi})_t=\int_0^t K_s\, d(I_\Psi)_s$ is the classical Itô integral of K with respect to the continuous martingale I_Ψ.

The process I_Ψ is then a continuous, real valued martingale if Ψ is above B. The covariance process of two such Itô integrals can be computed according to the formula

$$[I_\Psi, I_\Phi]_t=\int_0^t \mathrm{Trace}(\Psi_s\otimes\Phi_s)\, ds, \tag{27}$$

where $\otimes$ denotes the tensor product and $(\Psi_s\otimes\Phi_s)(\omega)=\Psi_s(\omega)\otimes\Phi_s(\omega)\in T^*_{B_s(\omega)}\otimes T^*_{B_s(\omega)}$.

Let $X\in M$ and let $\mathcal{E}nd(T^*_xM)$ be the space of all linear maps from T^*_xM to itself and define $\mathcal{E}nd(T^*M)$ as the bundle over M which is obtained by taking the union of all such $\mathcal{E}nd(T^*_xM)$ for $x\in M$. The bundle $\mathcal{E}nd(T^*M)$ can be made into a smooth manifold in the usual way.

Definition 1.1. Let B be a Brownian motion in M. A *martingale transformer with respect to* B is a bounded and continuous process A, with values in $\mathcal{E}nd(T^*M)$ above B, i.e. $A_t(\omega)\in\mathcal{E}nd(T^*_{B_t(\omega)}M)$.

Let Ψ be a continuous, bounded process with values in T^*M, above B, and let A be a martingale transformer with respect to B. The *martingale transform of* I_Ψ *by* A, $A*I_\Psi$, is the $\mathbf{R}$ valued martingale defined by

$$A*I_\Psi=I_{A\Psi}=\int_0 \langle A_s\Psi_s, dB_s\rangle. \tag{28}$$

If $\Psi=df$ for some smooth, $\mathbf{R}$ valued function f on M, we denote $A*I_{df}$ by $A*f$.

Let $\mathcal{A}=(A_1,\dots,A_l)$ be a sequence of martingale transformers above B, let $\mathcal{A}*I_\Psi=(A_1*I_\Psi,\dots,A_l*I_\Psi)$, an $\mathbf{R}^l$ valued martingale. The *norm* of $\mathcal{A}$ is defined as

$$|\!|\!|\mathcal{A}|\!|\!|=\sup_{\omega\in\Omega}\sup_{t\geq 0}\sup_{\substack{e\in T_{B_t(\omega)}M\\ |e|=1}}\left(\sum_{j=1}^{l}|A_{j,t}(\omega)e|^2\right)^{1/2}.$$

We let $|\!|\!|A|\!|\!|=|\!|\!|\mathcal{A}|\!|\!|$ if A is a martingale transformer and $\mathcal{A}=(A)$.

Proposition 1.2. *Let* Ψ *and* Φ *be bounded, continuous,* T^*M *valued processes above* B.

(i) *If* $\mathcal{A}=(A_1,\dots,A_l)$ *is a sequence of martingale transformers above* B, *then*

$$\|\mathcal{A}*I_\Psi\|_p\leq(p^*-1)|\!|\!|\mathcal{A}|\!|\!|\,\|I_\Psi\|_p. \tag{29}$$

(ii) *If A is a martingale transformer above B and $\langle A\xi,\xi\rangle=0$ identically in $t\geq 0$, $\omega\in\Omega$, $\xi\in T^*_{B_t(\omega)}M$, then*

$$\|A*I_\Psi\|_p \leq B_p |\!|\!|A|\!|\!| \|I_\Psi\|_p \tag{30}$$

and

$$\|(A*I_\Psi)\oplus I_\Psi\|_p \leq E_p |\!|\!|A|\!|\!| \|I_\Psi\|_p. \tag{31}$$

Proof. By Theorems A and B, it suffices to show that $\mathcal{A}*I_\Psi \prec_D |\!|\!|\mathcal{A}|\!|\!| I_\Psi$ in (i) and that $A*I_\Psi\prec_D |\!|\!|A|\!|\!| I_\Psi$ and $A*I_\Psi \perp |\!|\!|A|\!|\!| I_\Psi$ in (ii).

Let $x\in M$ and let $e_1,\dots,e_n$ be an orthonormal basis for T_xM, the tangent space to M at x. Suppose that $B_t(\omega)=x$. Then, for $j=1,\dots,l$,

$$\begin{aligned}\operatorname{Trace}(A_j\Psi_t(\omega)\otimes A_j\Psi_t(\omega)) &= \sum_{h=1}^{n}(A_j\Psi_t(\omega)\otimes A_j\Psi_t(\omega))(e_h,e_h)\\ &= \sum_{h=1}^{n}|\langle A_j\Psi_t(\omega),e_h\rangle|^2 = |A_j\Psi_t(\omega)|^2\end{aligned}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ here denotes the duality product $\langle\,\cdot\,,\cdot\,\rangle\colon T^*_xM\times T_xM\to\mathbf{R}$.

Thus, for $0\leq s\leq t$,

$$\begin{aligned}[\mathcal{A}*I_\Psi]_t-[\mathcal{A}*I_\Psi]_s &= \sum_{j=1}^{l}([A_j*I_\Psi]_t-[A_j*I_\Psi]_s) = \sum_{j=1}^{l}\int_s^t \operatorname{Trace}(A_j\Psi_u\otimes A_j\Psi_u)\,du\\ &= \sum_{j=1}^{l}\int_s^t |A_j\Psi_u|^2 \leq |\!|\!|\mathcal{A}|\!|\!|^2\int_s^t|\Psi_u|^2\,du = \dots = [|\!|\!|\mathcal{A}|\!|\!| I_\Psi]_t-[|\!|\!|\mathcal{A}|\!|\!| I_\Psi]_s\end{aligned}$$

which shows that $[|\!|\!|\mathcal{A}|\!|\!| I_\Psi]_t-[\mathcal{A}*I_\Psi]_t$ is nondecreasing in t. We then have that $\mathcal{A}*I_\Psi\prec_D |\!|\!|\mathcal{A}|\!|\!| I_\Psi$, which shows (i).

The same proof shows that, in (ii), $A*I_\Psi\prec_D |\!|\!|A|\!|\!| I_\Psi$, and a similar argument that $A*I_\Psi\perp |\!|\!|A|\!|\!| I_\Psi$, (ii) follows. □

2. The proof of Theorem 1

In this section we prove a variant of a theorem of R. Gundy and N. Varopoulos [GV] in which the Riesz transforms on a Lie group G are interpreted in terms of martingale transforms with respect to a Brownian motion process in $G\times\mathbf{R}$. In [Va], Varopoulos hinted at this construction. The proof of Theorem 1 will follow.

Let G be a compact Lie group of dimension n, $\mathfrak{G}$ its Lie algebra and suppose G is endowed with a Riemannian biinvariant metric. We can assume $\mathrm{Vol}(G)=1$.

Suppose that $\{X_1, \dots, X_n\}$ is an orthonormal basis for $\mathfrak{G}$. Let $\widehat{G}=G\times\mathbf{R}$, with its Lie algebra $\mathfrak{G}\oplus\mathbf{R}$ and the product Riemannian metric. We denote by $z=(x,y)\in G\times\mathbf{R}$ the elements of the product group and we identify $G\times\{0\}=G$, $(x,0)=x\in G$. An orthonormal basis for the Lie algebra $\mathfrak{G}\oplus\mathbf{R}$ is $\{X_1, \dots, X_n, X_0\}$, where $X_0=\partial/\partial y$ generates the Lie algebra of $\mathbf{R}$.

Let X be a Brownian motion in G and let Y be a Brownian motion in $\mathbf{R}$, with generator $\frac{1}{2}(d^2/dy^2)$. If we take X and Y to be independent, then $Z=(X,Y)$ is a Brownian motion in $\widehat{G}$.

Fix $\lambda>0$ and assume that the distribution of Z_0, the initial position of $Z=Z^\lambda=\{Z_t\}_{t\geq 0}$, is the product measure $\chi\otimes\delta_\lambda$, where χ is the Haar measure on G and δ_λ is a Dirac delta at $\lambda\in\mathbf{R}$, i.e., $\mathbf{P}(Z_0\in A\times(a,b))=\chi(A)$, if $\lambda\in(a,b)$, and it is equal to 0 if $\lambda\notin(a,b)$. Observe that $\chi\otimes\delta_\lambda(\widehat{G})=1$.

Let $\widehat{G}^+=G\times[0,\infty)$ and $\tau_0=\inf\{t\geq 0: Z_t\notin\widehat{G}^+\}$ the exit time of Z from $\widehat{G}^+$. Then $\{Z_{t\wedge\tau_0}\}_{t\geq 0}$ is a Brownian motion in $\widehat{G}^+$, stopped at G.

Let $A: \widehat{G}^+\to\mathcal{E}nd(T\widehat{G}^+)$ be a continuous section of the bundle $\mathcal{E}nd(T\widehat{G}^+)$ and define the process $\tilde{A}_t=A(Z_{t\wedge\tau_0})$. Then $\tilde{A}_t$ is a martingale transformer. With slight abuse of language, we will say that A itself is a *martingale transformer.*

If $f\in C_0^\infty(G)$, let F be its *Poisson integral* in $\widehat{G}^+$, i.e.,

$$0=\Delta_{\widehat{G}}F(x,y)=\Delta_G F(x,y)+\frac{\partial^2 F}{\partial y^2}(x,y),$$

if $x\in G$ and $y>0$, $F\in C^\infty(\widehat{G}^+)$, $F(x,0)=f(x)$ and F is bounded on $\widehat{G}^+$. See [S1], [Me1] and [G2] for different expositions of the theory.

Definition 2.1. If A, f and F are as above, the *A-transform* of f is

$$T_A^\lambda f=\mathbf{E}[\tilde{A}*dF\mid Z_{\tau_0}]$$

where λ is the 'starting height' of the Brownian motion and τ_0 is the exit time from $\widehat{G}^+$.

Here, $\mathbf{E}[\,\cdot\mid Z_{\tau_0}]$ is the conditional expectation with respect to the σ algebra of $\mathcal{F}$ generated by the random variable Z_{τ_0}. Observe that, being measurable with respect to the exit position, $T_A^\lambda f$ defines a function from G to $\mathbf{R}$. The following theorem gives an analytic representation of an A-transform. See [GV] for the Euclidean case.

Theorem 2.2. *Let $f,h\in C_0^\infty(G)$ and let F and H be, respectively, their Poisson integrals on $\widehat{G}^+$. Then*

$$\int_G hT_A^\lambda f\,dx=\int_{\widehat{G}^+}\langle A\,dF(x,y),dH(x,y)\rangle 2(y\wedge\lambda)\,dx\,dy. \tag{32}$$

The operator T_A^λ *can be extended to* $L_0^p(G)$ *and* $T_A = \lim_{\lambda\to\infty} T_A^\lambda$ *exists in the* L^p *operator norm,* $1<p<\infty$. *Moreover,*

$$\int_G hT_Af\,dx = \int_{\widehat{G}^+} \langle A\,dF(x,y), dH(x,y)\rangle 2y\,dx\,dy. \tag{33}$$

Proof.

$$\begin{aligned}\int_G hT_A^\lambda\,dx &= \mathbf{E}(h(Z_{\tau_0})T_A^\lambda f) = \mathbf{E}(h(Z_{\tau_0})\mathbf{E}[\tilde{A}*I_{dF} \mid Z_{\tau_0}])\\ &= \mathbf{E}\biggl(\int_0^{\tau_0} \langle \tilde{A}\,dF, dZ\rangle \int_0^{\tau_0} \langle dH, dZ\rangle\biggr)\\ &= \mathbf{E}\biggl(\int_0^{\tau_0} \langle A(Z_t)\,dF(Z_t), dH(Z_t)\rangle\,dt\biggr)\\ &= \int_{\widehat{G}^+} \langle A(x,y)\,dF(x,y), dH(x,y)\rangle (y\wedge\lambda)\,dx\,dy.\end{aligned}$$

The first equality comes from the fact that the distribution of the exit position Z_{τ_0} is dx [Mel]. The second one is the definition of T_A^λ, while the third follows from Itô's formula on manifolds [Em] applied to $H(Z)$ and the definition of the martingale transform. We use here the fact that $\int_G h(x)\,dx=0$ and the optional stopping theorem applied to the $\mathbf{R}$ valued martingale $W_t=h(Z_0)\int_0^t \langle A\,dF, dZ\rangle$ and to the stopping time τ_0. The fourth equality is a consequence of (27), while the fifth comes from the formula for the occupation time of $\widehat{G}^+$ by the Brownian motion Z^λ [Mel].

In order to prove (33), consider the *Littlewood–Paley function* of h, $G(h)$, defined by

$$G(h)(x) = \biggl(\int_0^\infty 2y|dH(x,y)|^2\,dy\biggr)^{1/2}, \quad x\in G. \tag{34}$$

Let $\lambda_1>\lambda_2$ and let q be the conjugate exponent of p. Then, by (32), Schwarz's inequality and Hölder's inequality

$$\begin{aligned}\biggl|\int_G h(T_A^{\lambda_1}-T_A^{\lambda_2})f\,dx\biggr| &= \biggl|\int_{\widehat{G}^+} \langle A\,dF(x,y), dH(x,y)\rangle 2(y\wedge\lambda_1-y\wedge\lambda_2)\,dy\,dx\biggr|\\ &\le \|\!|A|\!\|\biggl[\int_G\biggl(\int_0^\infty |dF(x,y)|^2 2(y\wedge\lambda_1-y\wedge\lambda_2)\,dy\biggr)^{p/2} dx\biggr]^{1/p}\\ &\quad\times\biggl[\int_G\biggl(\int_0^\infty |dH(x,y)|^2 2(y\wedge\lambda_1-y\wedge\lambda_2)\,dy\biggr)^{q/2} dx\biggr]^{1/q}\\ &\le \|\!|A|\!\|\,\|G(H(\,\cdot\,,\lambda_2))\|_p\|G(F(\,\cdot\,,\lambda_2))\|_q.\end{aligned}$$

By the Littlewood–Paley inequalities [S1],

$$\left|\int_G h(T_A^{\lambda_1}-T_A^{\lambda_2})f\,dx\right|\le C\|H(\cdot,\lambda_2)\|_p\|F(\cdot,\lambda_2)\|_q\le C\|F(\cdot,\lambda_2)\|_p\|h\|_q.$$

Now, $\|F(\cdot,\lambda)\|_p\le C(\lambda)\|f\|_p$, with $C(\lambda)\to 0$ as $\lambda\to\infty$. In fact, observe that the case $p=2$ follows from the spectral decomposition of $F(\cdot,\lambda)$. The case $1<p\le 2$ follows by interpolation, since $\|F(\cdot,\lambda)\|_p\le\|f\|_p$. A simple duality argument then proves the case $2\le p<\infty$. Hence, taking the supremum over $\|h\|_q=1$ in the last member of the inequality and letting $\lambda_2\to\infty$, we have $\|T_A^{\lambda_1}-T_A^{\lambda_2}\|_p\to 0$. By dominated convergence we obtain (33). □

Observe that the map $A\mapsto T_A$ is linear.

Let now $\{X_1,\dots,X_n,X_0\}$ be an orthonormal basis for $\mathfrak{G}\oplus\mathbf{R}$. Let $R_j=R_{X_j}$ be the Riesz transform on G in the direction X_j, $j=1,\dots,n$. If $l,m\in\{0,1,\dots,n\}$, we define E_{lm} to be the linear map $E_{lm}\colon\mathfrak{G}\oplus\mathbf{R}\to\mathfrak{G}\oplus\mathbf{R}$ defined by

$$E_{lm}X_j=\begin{cases} X_l, & \text{if } j=m,\\ 0, & \text{otherwise.}\end{cases}$$

Then E_{lm} defines a smooth section of $\mathcal{E}nd(T\widehat{G}^+)$. We can identify E_{lm} with a martingale transformer by means of the natural identification between $\mathfrak{G}\oplus\mathbf{R}$ and its dual, induced by the Riemannian metric. After this identification, (33) reads

$$\int_G hT_{E_{lm}}f\,dx=\int_{\widehat{G}^+}\langle E_{lm}\nabla_{\widehat{G}}F(x,y),\nabla_{\widehat{G}}H(x,y)\rangle 2y\,dx\,dy. \tag{35}$$

Theorem 2.3. *The following equalities hold.*

(i) *If* $m\ne 0$, *then* $T_{E_{0m}}=-\frac{1}{2}R_m$.

(ii) *If* $l\ne 0$, *then* $T_{E_{l0}}=\frac{1}{2}R_l$.

(iii) *If* $l,m\ne 0$, *then* $T_{E_{lm}}=-\frac{1}{2}R_lR_m$.

(iv) $T_{E_{00}}=\frac{1}{2}I$.

Proof. Consider the decomposition of $L_0^2(G)$ into eigenspaces for Δ_G, provided by the Peter–Weyl theorem [S1]. Then, $L_0^2(G)=\bigoplus_{k=1}^\infty\mathcal{H}_k$, where $\mathcal{H}_k\subset C_0^\infty(G)$, $\Delta_G\eta+\mu_k\eta=0$ if $\eta\in\mathcal{H}_k$, $0<\mu_1<\dots<\mu_k<\dots$ being the sequence of the nonnegative eigenvalues of $-\Delta_G$.

Since the metric on G is biinvariant, Δ_G commutes with all $X\in\mathfrak{G}$, hence $X\eta\in\mathcal{H}_k$ whenever $\eta\in\mathcal{H}_k$, $k\ge 1$. If $f\in\mathcal{H}_k$, then $F(x,y)=e^{-y\sqrt{\mu_k}}f(x)$ is the Poisson extension f to $\widehat{G}^+$. Suppose that $f,g\in\mathcal{H}_k$. Then, by (35),

$$\begin{aligned}\int_G hT_{E_{0m}}f\,dx&=\int_G\int_0^\infty X_m(e^{-y\sqrt{\mu_k}}f)(x)\frac{\partial}{\partial y}(e^{-y\sqrt{\mu_k}}h)(x)2y\,dy\,dx\\&=\frac{1}{2}(-\sqrt{\mu_k})^{-1}\int_G X_mf(x)h(x)\,dx=-\frac{1}{2}\int_G R_mf(x)h(x)\,dx.\end{aligned}$$

We used the definition $R_m = X_m \circ (\Delta_G)^{-1/2}$. On the other hand, if f and h belong to different eigenspaces of Δ_G, then $\int_G hT_{E_{0m}} f\,dx = 0 = -\frac{1}{2}\int_G hR_m f\,dx$. Case (i) follows by a density argument and duality between L^p spaces.

The proof of the other cases follows the same lines. □

The following corollary contains the majorizations in Theorem 1.

Corollary 2.4.
(i) $\|R^G\|_p \le 2(p^*-1)$,
(ii) $\|R_j\|_p \le B_p$,
(iii) $\|I \oplus R_j\|_p \le E_p$.

Proof. By (6), Proposition 1.2 and Theorem 2.3, it suffices to compute

$$\alpha = 2 \sup_{\substack{X\in\mathfrak{G}\\ \|X\|=1}} \left[\sum_{m=1}^{n} |E_{0m}X|^2\right].$$

The supremum is 1, hence $\alpha=2$ and (i) follows.

By linearity, (i) and (ii) in Theorem 2.3 imply that $R_j = T_{E_{j0}-E_{0j}}$. The cases (ii) and (iii) then follow from Proposition 1.2 and the facts that $\langle (E_{j0}-E_{0j})v, v\rangle = 0$, $\|E_{j0}-E_{0j}\| = 1$. □

If $X\in\mathfrak{G}$ and $|X|=1$, we can take it to be $X=X_1$, hence Corollary 2.4 implies (7), (8) and (9) in Theorem 1.

Remark. Only in the proof of Theorem 2.3 we made use of the fact that G is a Lie group. In particular, Theorem 2.2 can be shown to hold for any compact Riemannian manifold.

Also, the requirement that G be compact is used only in that G carries a biinvariant Riemannian metric. Thus, Theorem 2.3 and Theorem 1 hold, with obvious modifications, on any Lie group carrying a biinvariant Riemannian metric.

3. The proof of Theorem 3

Notation. In this section, ∇, div and Δ denote, respectively, the gradient, the divergence and the Laplacian in $\mathbf{R}^n$.

In this section, we prove an L^p estimate for a Riesz transform associated with the Neumann problem on $\mathbf{B}^n$, the unit ball of $\mathbf{R}^n$. We will prove most of Theorem 3 and (ii), (iii) in Theorem 2. We will see in §4 how these last estimates also follow from Theorem 1. This section and Theorem 3 are related to, but independent from, the others in this article.

Let $\mathcal{H}_k$ be the space of spherical harmonics of degree k and let

$$\mathcal{E}_0=\left\{f\colon \mathbf{S}^{n-1}\to\mathbf{R}: f=\sum_{k=1}^{N} f_k,\ N\in\mathbf{N},\ f_k\in\mathcal{H}_k\right\}$$

the space of harmonic polynomials with null average on $\mathbf{S}^{n-1}$ [SW]. Fix $f\in\mathcal{E}_0$ and let H be the solution in $\mathbf{B}^n$ of the Neumann problem with boundary data f, normalized so that $H(0)=0$. We will write

$$\left(\frac{\partial}{\partial\nu}\right)^{-1} f=H|_{\mathbf{S}^{n-1}}$$

where ν is the outward pointing normal vector to $\mathbf{S}^{n-1}$. The operator $(\partial/\partial v)^{-1}$ could be called the *Neumann operator* on $\mathbf{S}^{n-1}$. If $f=\sum_{k\geq 1} f_k$ is the decomposition of f into spherical harmonics, then $(\partial/\partial v)^{-1}f=\sum_{k\geq 1}(1/k)f_k$, i.e. $(\partial/\partial v)^{-1}$ has the multiplier $1/k$ on $\mathcal{E}_0$.

Let $\mathcal{T}_{lm}$ be as in (10). For $1\leq l<m\leq n$, let $Q^c_{lm}=Q^c_{\mathcal{T}_{lm}}$, where $Q^c_{\mathcal{T}_{lm}}$ is defined by (13), and $Q^b_{lm}=Q^b_{\mathcal{T}_{lm}}$, with $Q^b_{\mathcal{T}_{lm}}$ defined by (17). In this section we will work with the *auxiliary Riesz transforms on* $\mathbf{S}^{n-1}$ of *cylinder type*, Q^c, and of *ball type*, Q^b,

$$Q^c=(Q^c_{lm})_{1\leq l<m\leq n}\quad\text{and}\quad Q^b=(Q^b_{lm})_{1\leq l<m\leq n}.$$

Since, by (11), $|R^c|=|Q^c|$ and $|R^b|=|Q^b|$, estimates on the auxiliary transforms carry over to estimates for R^c and R^b.

The transform R^b and its auxiliary form Q^b were studied in [KV1], [KV2], [RW], [B1].

Since R^c is an isometry of $L^2_0(\mathbf{S}^{n-1})$, Q^c is an isometry of $L^2_0(\mathbf{S}^{n-1})$ into $L^2(\mathbf{S}^{n-1},\mathbf{R}^{n(n-1)/2})$. There is a simple relationship between Q^c and Q^b. Let S be the operator from $L^2_0(\mathbf{S}^{n-1})$ to $L^2_0(\mathbf{S}^{n-1})$ defined by the multiplier $((n-2+k)/k)^{1/2}$. Then

$$Q^b=Q^c\circ S\quad\text{and}\quad R^b=R^c\circ S. \tag{36}$$

This follows from simple considerations involving the definitions and the multipliers of $(-\Delta_{\mathbf{S}^{n-1}})^{-1/2}$ and $(\partial/\partial v)^{-1}$. If $n=2$, $-R^c=-R^b=\mathcal{H}$ the Hilbert transform on the unit circle. We only deal with $n>2$, the case $n=2$ being classical.

Let $B=(B^1,\dots,B^n)$ be a Brownian motion in $\mathbf{R}^n$, with initial position $B_0=0$. Let $\tau=\inf\{t\geq 0\colon B_t\notin\mathbf{B}^n\}$ be the time of its first exit from the unit ball $\mathbf{B}^n$. The theory presented in §1 and §2 can be extended to this setting and, in fact, it can be found in the literature [Ba], [BW], [Be1], [D], [G2], [BL].

Let $A:\overline{\mathbf{B}}^n\to\mathcal{E}nd(\mathbf{R}^n)$ be a continuous map $x\mapsto A(x)$, where $A(x)$ is an $n\times n$ matrix. Then $\tilde{A}_t=A(B_{t\wedge\tau})$ is a martingale transformer. We will identify A and $\tilde{A}$. Let $f\in C_0^\infty(\mathbf{S}^{n-1})$ and let F be its Poisson extension to $\mathbf{B}^n$. Define

$$(A*F)_t=\int_0^{t\wedge\tau}A(B_s)\nabla_{\mathbf{R}^n}F\cdot dB_s \tag{37}$$

the *martingale transform* of $F(B)$ by A, and

$$T_Af(B_\tau)=\mathbf{E}[A*F\mid B_\tau] \tag{38}$$

the A-transform of f. The operator T_A extends to a well-defined linear operator on $L_0^p(\mathbf{S}^{n-1})$, $p>1$. Let $G(0,\cdot)$ be the Green function of $\mathbf{B}^n$ for $\frac{1}{2}\Delta_{\mathbf{R}^n}$ at 0. Then

$$G(0,x)=\frac{2}{(n-2)\operatorname{Vol}(\mathbf{S}^{n-1})}(|x|^{2-n}-1),$$

for $n\geq 3$. We write $G(|x|)=G(0,x)$.

Theorem C below is the equivalent of Theorem 2.2 in our context, and its proof can be found in [Be1].

Theorem C. [Be1] *Let $f,h\in C_0^\infty(\mathbf{S}^{n-1})$, and let F,H be, respectively, their Poisson integrals in $\mathbf{B}^n$. If A is a martingale transformer, then*

$$\frac{1}{\operatorname{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}}hT_Af\,dx=\int_{\mathbf{B}^n}\langle A(x)\nabla F(x),\nabla H(x)\rangle G(|x|)\,dx. \tag{39}$$

Let now $e_1,\dots,e_n$ be the standard orthonormal basis of $\mathbf{R}^n$. Let $1\leq l<m\leq n$ and let E_{lm} be the matrix such that $E_{lm}e_k=0$ if $k\neq l,m$, $E_{lm}e_l=-e_m$ and $E_{lm}e_m=e_l$.

Let now $\varphi:[0,1]\to\mathbf{R}$, $\varphi\in C^1(0,1)\cap C([0,1])$. Define

$$\varphi^\sharp(k)=-\int_0^1 r^{2k+n-2}\frac{d}{dr}[\varphi(r^2)\operatorname{Vol}(\mathbf{S}^{n-1})G(r)]\,dr,\quad k\geq 1, \tag{40}$$

and consider the operator $S^\varphi:\mathcal{E}_0\to\mathcal{E}_0$ acting on spherical harmonics $Y_k\in\mathcal{H}_k$ as

$$S^\varphi Y_k=\varphi^\sharp(k)Y_k,\quad k\geq 1. \tag{41}$$

The following theorem shows how the auxiliary Riesz transforms Q^b and Q^c can be interpreted in terms of martingale transforms.

Theorem 3.1. *Consider $A_{lm}(x)=\varphi(|x|^2)E_{lm}$, with φ and E_{lm} as above. As operators acting on $\mathcal{E}_0$,*

$$T_{A_{lm}}=\mathcal{T}_{lm}\circ S^{\varphi} \tag{42}$$

where S^{φ} is defined by (41)*. In particular, we have the following two cases.*

(i) *If $\varphi\equiv 1$, then*

$$T_{A_{lm}}=T_{E_{lm}}=Q^b_{lm}.$$

Thus, $Q^b=(T_{E_{lm}})_{l<m}$. If U is defined by (12) *and $A=\sum_{l<m}\alpha_{lm}E_{lm}$, then $T_A=Q^b_U$.*

(ii) *Let φ be defined by*

$$\varphi(e^{-2t/(n-2)})=\frac{\int_0^t I_0(s)\,ds}{e^t-1},\quad t\geq 0, \tag{43}$$

where $I_0(z)=\sum_{l=0}^{\infty}\left(\frac{1}{2}z\right)^{2l}/(l!)^2$, $z\in\mathbf{C}$, is the modified Bessel function of order 0*, then*

$$T_{A_{lm}}=Q^c_{lm}. \tag{44}$$

Thus, $Q^c=(T_{A_{lm}})_{l<m}$ and, if U is defined as in (12) *and $A=\sum_{l<m}\alpha_{lm}E_{lm}$, then $T_A=Q^c_U$.*

Proof. In order to prove (42), it suffices to show that, for $f\in\mathcal{H}_k$, $h\in\mathcal{H}_j$,

$$\int_{\mathbf{S}^{n-1}} hT_{A_{lm}}f\,dx=\int_{\mathbf{S}^{n-1}} h\mathcal{T}_{lm}\circ S^{\varphi}f\,dx. \tag{45}$$

By Theorem C and Green's theorem,

$$\begin{aligned}\frac{1}{\operatorname{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} hT_{A_{lm}}f\,dx &= \int_{\mathbf{B}^n}\varphi(|x|^2)G(|x|)(E_{lm}\nabla F(x))\cdot\nabla H(x)\,dx\\ &= \int_{\mathbf{S}^{n-1}} h(x)(\varphi(|x|^2)G(x)E_{lm}\nabla F(x))\cdot\nu\,dx\\ &\quad -\int_{\mathbf{B}^n} H(x)\operatorname{div}(\varphi(|x|^2)G(x)E_{lm}\nabla F(x))\,dx,\end{aligned}$$

ν being the outward pointing normal vector to $\mathbf{S}^{n-1}$. Observe that in the application of Green's theorem, due to the singularity of G at 0, we should have a boundary term on $\{x\in\mathbf{B}^n:|x|=\varepsilon\}$. Since φ is bounded, this term vanishes as $\varepsilon\to 0$.

Write $x\in\mathbf{B}^n$ as $x=r\omega$, $r\in[0,1]$, $\omega\in\mathbf{S}^{n-1}$. Since G vanishes on $\mathbf{S}^{n-1}$ and since

$$\operatorname{div}(A_{lm}\nabla F(x))=2\varphi'(|x|^2)\mathcal{T}_{lm}F(x)$$

we have, for $j=k$,

$$\begin{aligned}\frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} hT_{A_{lm}}f\,dx &= -2\int_{\mathbf{B}^n} H(x)\,\frac{d}{d(r^2)}[\varphi(|x|^2)G(|x|)]\mathcal{T}_{lm}F(x)\,dx\\ &= -2\int_{\mathbf{S}^{n-1}} h(\omega)\mathcal{T}_{lm}f(\omega)\\ &\quad\times\int_0^1 r^{2k+n-1}\frac{1}{2r}\,\frac{d}{dr}[\varphi(r^2)G(r)]\,dr\,d\omega\\ &= \frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} h(\omega)\mathcal{T}_{lm}\circ S^{\varphi}f(\omega)\,d\omega.\end{aligned}$$

The second equality comes from Fubini's theorem and the fact that $H(r\omega)=r^k h(\omega)$, $F(r\omega)=r^k f(\omega)$.

In the same way, if $j\neq k$, then

$$\frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} hT_{A_{lm}}f\,dx=0=\frac{1}{\mathrm{Vol}(\mathbf{S}^{n-1})}\int_{\mathbf{S}^{n-1}} h(\omega)\mathcal{T}_{lm}\circ S^{\varphi}f(\omega)\,d\omega.$$

Equality (42) is thus proved, and (i) follows immediately.

Equalities (40), (41) and (42) show that the problem of finding a function φ such that (44) holds can be rephrased in terms of Laplace transforms. Let

$$g(r^2)=-\frac{1}{2r}\,\frac{d}{dr}[\varphi(r^2)G(r)],\quad 0<r<1,$$

and $\psi(t)=e^{-n/2}g(e^{-t})$, $t>0$. Then

$$\varphi^{\sharp}(k)=\int_0^\infty e^{-k}\psi(t)\,dt=\mathcal{L}\psi(k)$$

is the sampling on the positive integers of the Laplace transforms of ψ. An inspection of multipliers shows that $S^{\varphi}=(-\Delta_{\mathbf{S}^{n-1}})^{-1/2}$ if ψ satisfies

$$\mathcal{L}\psi(k)=\frac{1}{[k(n-2+k)]^{1/2}},\quad k\geq 1. \tag{46}$$

A solution of (46) is $\psi(t)=e^{-(n-2)/2t}I_0\big(\frac{1}{2}(n-2)t\big)$ [PBM]. The expression for φ in (ii) follows. $\square$

The fact that the function φ that allows us to represent Q^c is more complicated than the one that represents Q^b is an indication that Q^c, unlike Q^b, has no natural connection with the geometry, hence the Brownian motion, of $\mathbf{R}^n$. If we had worked with Brownian motion in the Riemannian manifold $\mathbf{S}^{n-1}\times\mathbf{R}$ we would have found, in fact, a simpler probabilistic interpretation for Q^c.

Let φ and A_{lm} be as in Theorem 3.1. Consider the sequence $\mathcal{A}=(A_{lm})_{1\leq l<m\leq n}$ and define $T_{\mathcal{A}}=(T_{A_{lm}})_{1\leq l<m\leq n}$. Then, if $f\in C_0^\infty(\mathbf{S}^{n-1})$, $T_{\mathcal{A}}f\colon\mathbf{S}^{n-1}\to\mathbf{R}^{n(n-1)/2}$.

Proposition 3.2. *If* $f\in L_0^p(\mathbf{S}^{n-1})$ *is real valued and* $\mathcal{A}$, $T_{\mathcal{A}}$ *are as above, then*
(i) $\|T_{\mathcal{A}}f\|_p\leq(p^*-1)(n-1)^{1/2}\|\varphi\|_\infty\|f\|_p$.
If U *is as in* (12) *and* $A=\sum_{l<m}\alpha_{lm}E_{lm}$, *then*
(ii) $\|T_Af\|_p\leq B_p\|\varphi\|_\infty\|f\|_p$,
(iii) $\|(I\oplus T_A)f\|_p\leq E_p\|\varphi\|_\infty\|f\|_p$.

Proof. The cases (i), (ii) and (iii) follow from Theorem 3.1, Proposition 1.2 and simple considerations involving E_{lm}, that we supply below.

A counting argument gives, for $v=(v_1,\dots,v_n)\in\mathbf{R}^n$, $\sum_{l<m}|E_{lm}v|^2=(n-1)|v|^2$. Thus, $|\!|\!|\mathcal{A}|\!|\!|=(n-1)^{1/2}\|\varphi\|_\infty$, and this proves (i).

Note that $\langle(\sum_{l<m}\alpha_{lm}E_{lm})v,v\rangle=0$. Also,

$$\left|\left(\sum_{l<m}\alpha_{lm}E_{lm}\right)v\right|^2=|v|^2-\sum_{1\leq l<m<p\leq n}(\alpha_{lm}v_p-\alpha_{lp}v_m+\alpha_{mp}v_l)^2\leq|v|^2,$$

with possible equality. Hence $|\!|\!|A|\!|\!|=\|\varphi\|_\infty$. The cases (ii) and (iii) follow. □

Corollary 3.3. *Let* U *be as in* (12). *The following inequalities hold*

$$\|Q^b\|_p\leq(p^*-1)(n-1)^{1/2},\quad \|Q_U^b\|_p\leq B_p\quad\textit{and}\quad \|I\oplus Q_U^b\|_p\leq E_p;\tag{47}$$

$$\|Q^c\|_p\leq(p^*-1)(n-1)^{1/2},\quad \|Q_U^c\|_p\leq B_p\quad\textit{and}\quad \|I\oplus Q_U^c\|_p\leq E_p.\tag{48}$$

Proof. One only has to verify that, if φ is the function defined by (43), then $\|\varphi\|_\infty=1$. From the series expansion of I_0 one easily checks that, if $s\geq0$, then $0<I_0(s)\leq e^s$ and that $I_0(0)=1$. Then $0\leq\varphi(r)\leq1$. □

As a consequence of (47), we obtain (18)–(20), with inequalities instead of equalities in (19) and (20). Similarly, (48) implies (15) and (16) with inequalities. We also obtain a constant for an inequality like (14), but with the wrong order of growth with respect to p.

Remark. A consequence of Corollary 3.3 (47), is that, if $f\in L_0^p(\mathbf{S}^{n-1})$, then

$$\|Q^bf\|_p\geq\frac{1}{(n-1)^{1/2}(p^*-1)}\|f\|_p.\tag{49}$$

To prove (49) we can use a duality argument. Define the operator $V=(V_{lm})_{l<m}$ by

$$V_{lm}=\mathcal{T}_{lm}\circ\left(\frac{\partial}{\partial\nu}\right)\circ(-\Delta_{\mathbf{S}^{n-1}})^{-1}.$$

One easily verifies on spherical harmonics that, on $L_0^2(\mathbf{S}^{n-1})$,

$$\sum_{l<m} Q_{lm}^b V_{lm} = -I, \tag{50}$$

where I is the identity operator. From (50), Hölder's inequality and the fact that the adjoint of $\mathcal{T}_{lm}$ is $-\mathcal{T}_{lm}$, we see that (49) holds if

$$\|V\|_p \le (p^*-1)(n-1)^{1/2}. \tag{51}$$

This last estimate follows from the representation formula $V_{lm}=\mathcal{T}_{lm}\circ S^\beta$, where $\beta(r^2)=r^{n-2}$, Proposition 1.2, and Proposition 3.2.

It is easy to show, however, that $\|Q^b f\|_2 \ge \|f\|_2$, hence that (49) does not exhibit the right order of growth with respect to n.

In fact, by means of a more sophisticated duality argument, one can show that there exists a universal constant $C>0$ such that

$$\|Q^b f\|_p \ge \frac{C}{\log n(p^*-1)} \|f\|_p. \tag{52}$$

The proof of (52) makes use of Brownian motion and martingale transforms on the Riemannian manifold $\mathbf{S}^{n-1}\times\mathbf{R}$. Even (52), however, does not give the right asymptotics in n for $p=2$.

4. The proof of Theorem 2

Consider $SO(n)$, the rotation group of $\mathbf{R}^n$. Its Lie algebra, $\mathfrak{so}(n)$, is the space of $n\times n$ skew symmetric matrices. The imbedding of $SO(n)$ in $\mathbf{R}^{n^2}$ induces on $SO(n)$ a biinvariant Riemannian metric, that we call the *standard metric* of $SO(n)$. We normalize this metric in such a way that an orthonormal basis for $\mathfrak{so}(n)$ is provided by $\{X_{lm}:1\le l<m\le n\}$, where $X_{lm}=[r_{jk}^{lm}]_{1\le j,k\le n}$ and $r_{lm}^{lm}=-1$, $r_{ml}^{lm}=1$ and $r_{jk}^{lm}=0$ for all other entries of X_{lm}. This corresponds to the norm $\|[a_{jk}]\|=\frac{1}{2}\sqrt{2}\,(\sum_{j,k=1}^n a_{jk}^2)^{1/2}$ on $\mathbf{R}^{n^2}$. The operator X_{lm} is the infinitesimal generator of the rotations in the (x_l,x_m) plane.

We identify $\mathbf{S}^{n-1}=SO(n)/SO(n-1)$, where $H=SO(n-1)$ is the stabilizer of the north pole e_n of $\mathbf{S}^{n-1}$, $e_n=(0,\dots,0,1)$. Let $\Pi\colon SO(n)\to\mathbf{S}^{n-1}$ be the projection $\Pi(a)=ae_n$, the image of e_n under the rotation a. By $m_{SO(n)}$ and $m_{\mathbf{S}^{n-1}}$ we denote, respectively, the Haar measure induced by the standard metric on $SO(n)$ and the Hausdorff measure on $\mathbf{S}^{n-1}$. The map $\Pi_*\colon TSO(n)\to T\mathbf{S}^{n-1}$ is the push forward map $\Pi_*(X_a)f(\Pi(a))=X_a(f\circ\Pi)$, if $X_a\in T_aSO(n)$ and $f\colon\mathbf{S}^{n-1}\to\mathbf{R}$.

The adjoint representation of $SO(n)$ is $\mathrm{Ad}(a^{-1})X=(d/dt)|_{t=0}(a^{-1}\exp(tX)a)$, $X\in\mathfrak{so}(n)$, $a\in SO(n)$, where exp denotes the exponential map $\exp\colon\mathfrak{so}(n)\to SO(n)$. Then, if $F\colon SO(n)\to\mathbf{R}$, $(\mathrm{Ad}(a^{-1})X)F(a)=-X(F\circ\varrho)(\varrho(a))$, where $\varrho(a)=a^{-1}$.

If $\mathcal{T}_{lm}$ is defined as in (10), it is easy to show that

$$\Pi_*(\mathrm{Ad}(a^{-1})X_{lm})=\mathcal{T}_{lm}(\Pi(a)), \tag{53}$$

the vector field $\mathcal{T}_{lm}$ computed at the point $\Pi(a)\in\mathbf{S}^{n-1}$.

The lemma below shows how Q^c_{lm} and $R^{\mathbf{S}^{n-1}}_{X_{lm}}$ are connected to each other. See also [AL].

Lemma 4.1. *Let U be a vector field on $\mathbf{S}^{n-1}$, of the form* (12), *and let*

$$X=\sum_{l<m}\alpha_{lm}X_{lm}\in\mathfrak{so}(n). \tag{54}$$

If $f\in C_0^\infty(\mathbf{S}^{n-1})$, then

(i) $(Q^c_{lm}f)(\Pi(a))=-R^{SO(n)}_{X_{lm}}(f\circ\Pi\circ\varrho)(\varrho(a))$,

(ii) $\|Q^c_U\|_p\le\|R^{SO(n)}_X\|_p$,

(iii) $\|I\oplus Q^c_U\|_p\le\|I\oplus R^{SO(n)}_X\|_p$,

(iv) $\|Q^c\|_p\le\|R^{SO(n)}\|_p$.

Proof. Since $\Delta_{SO(n)}(f\circ\Pi)=(\Delta_{\mathbf{S}^{n-1}}f)\circ\Pi$, we see that $\Pi^*\colon f\mapsto f\circ\Pi$ maps the eigenspace relative to the k^{th} eigenvalue $-\mu_k$ of $\mathbf{S}^{n-1}$ into the eigenspace relative to the same eigenvalue of $\Delta_{SO(n)}$. Thus

$$\begin{aligned}(Q^c_{lm}f)(\Pi(a))&=\mathcal{T}_{lm}[(-\Delta_{\mathbf{S}^{n-1}})^{-1/2}f](\Pi(a))=-X_{lm}[((-\Delta_{\mathbf{S}^{n-1}})^{-1/2}f)\circ\Pi\circ\varrho](\varrho(a))\\&=-[X_{lm}\circ(-\Delta_{SO(n)})^{-1/2}](f\circ\Pi\circ\varrho)(\varrho(a))=-R^{SO(n)}_{X_{lm}}(f\circ\Pi\circ\varrho)(\varrho(a)).\end{aligned}$$

We made use of the fact that $\Delta_{SO(n)}(F\circ\varrho)=(\Delta_{SO(n)}F)\circ\varrho$, ϱ being an isometry. The statement (i) is proved.

The statements (ii), (iii) and (iv) follow from (i) and the following fact.

There exists $\mu>0$ such that, for $f\colon\mathbf{S}^{n-1}\to\mathbf{R}$,

$$\int_{SO(n)}f\circ\Pi\,dm_{SO(n)}=\mu\int_{\mathbf{S}^{n-1}}f\,dm_{\mathbf{S}^{n-1}}. \quad\square \tag{55}$$

The first two estimates in the corollary below have already been proved, with different arguments, in Corollary 3.3.

Corollary 4.2. *Let U be as in* (12). *We have, then, the estimates*
(i) $\|Q_U^c\|_p \le B_p$,
(ii) $\|I \oplus Q_U^c\|_p \le E_p$,
(iii) $\|Q^c\|_p \le 2(p^*-1)$.

Proof. By Theorem 1, it suffices to verify that, if U and X are as in (12) and (54), then

$$|U_{\Pi(a)}| \le |X(a)| = 1 \tag{56}$$

for all $a \in SO(n)$. The equality comes from (54) and the orthogonality relations between the X_{lm}'s. The map Π is a Riemannian submersion, hence the inequality holds. □

By (iii), we have the last part of Theorem 2.

5. Transference arguments

In this section we prove several inequalities of the form $\|\mathcal{U}\|_p \ge B_p$ and the form $\|I \oplus \mathcal{U}\|_p \ge E_p$, where $\mathcal{U}$ is one of the directional Riesz transforms appearing in Theorems 1, 2 and 3. It is classical that, if $\mathcal{H}$ denotes the Hilbert transform on the unit circle, then $\|\mathcal{H}\|_p \le B_p$ and $\|I \oplus \mathcal{H}\|_p \le E_p$. In fact, we have equality in both relations [Pic], [Es]. We will see how the estimates on $\mathbf{S}^{n-1}$ can be reduced to estimates on $\mathbf{T}^k$ and the ones on $\mathbf{T}^k$ to estimates on $\mathbf{S}^1$. We have already seen in the previous section how to transfer the inequalities from $\mathbf{S}^{n-1}$ to $SO(n)$.

Let U be as in (12) and let $X \in \mathfrak{so}(n)$ be defined as

$$X = \sum_{l<m} \alpha_{lm} X_{lm}.$$

Then $|X|=1$ and $\Pi_* X = U$. After passing to some different orthogonal coordinate system in $\mathbf{R}^n$, X can be written as $\alpha_1 X_{12} + \ldots + \alpha_k X_{2k-1,2k}$, with $2k \le n$, $\sum_1^k \alpha_j^2 = 1$ and $\alpha_j \ne 0$ for $j=1, \ldots, k$. The rotation $\exp(tX)$ can then be decomposed as the commutative product of rotations by an angle of $\alpha_j t$ in the (x_{2j-1}, x_{2j}) plane, $1 \le j \le k$. Since the space of the vector fields of the form U does not depend on our choice of north pole, we can assume that we are working in such a coordinate system. Thus

$$U = \sum_1^k \alpha_j T_{2j-1,2j} \quad \text{and} \quad X = \sum_1^k \alpha_j X_{2j-1,2j}. \tag{57}$$

Let $\mathbf{T}^k$ be the the k-dimensional torus $\mathbf{T}^k=\mathbf{S}^1\times\ldots\times\mathbf{S}^1$, endowed with the Riemannian product metric, that we call the *standard metric* on $\mathbf{T}^k$. We define an operator J that extends trigonometric polynomials on $\mathbf{T}^k$ to complex valued functions on $\mathbf{S}^{n-1}$. Let $\theta=(\theta_1,\ldots,\theta_k)\in\mathbf{T}^k$ and $l=(l_1,\ldots,l_k)\in\mathbf{Z}^k$ be a multiindex.

Consider the trigonometric polynomial $f(\theta)=\sum_{|l|\le N} c_l e^{il\cdot\theta}$, where $|l|$ denotes the length of l. Define the polynomial $\tilde f$ in the variables $x_1,\ldots,x_n$ by

$$\tilde f(x)=\sum_{|l|\le N}\left(c_l\prod_{l_j\ge 0}(x_{2j-1}+ix_{2j})^{l_j}\prod_{l_h\le 0}(x_{2h-1}-ix_{2h})^{-l_h}\right),$$

$\tilde f$ only depends on the variables $x_1,\ldots,x_{2k}$ and it is harmonic in $\mathbf{R}^n$. In fact, $\tilde f$ is pluriharmonic in $\mathbf{C}^k$ in the variables $z_j=x_{2j-1}+ix_{2j}$. The restriction $Jf=\tilde f|_{\mathbf{S}^{n-1}}$ is our *extension of f to $\mathbf{S}^{n-1}$*.

Suppose that S is a multiplier operator on $\mathbf{S}^{n-1}$, acting on $\mathcal{E}_0$. Let S^1 be the multiplier operator on $\mathbf{T}^k$ defined by $S^1(e^{il\cdot\theta})=\widehat{S}(|l|)e^{il\cdot\theta}$, $l\in\mathbf{Z}^k\setminus\{0\}$, where $\widehat{S}$ is the multiplier of S. If $V=U\circ S$, set $V^1=\sum_1^k\alpha_j(\partial/\partial\theta_j)\circ S^1$, the *restriction* of V to $\mathbf{T}^k$.

Lemma 5.1.
(i) $J\circ V^1=V\circ J$.
(ii) *There exists $C>0$ such that $\int_{\mathbf{S}^{n-1}} Jf\,dm_{\mathbf{S}^{n-1}}=C\int_{\mathbf{T}^k} f\,dm_{\mathbf{T}^k}$ for all integrable f on $\mathbf{S}^{n-1}$, where $m_{\mathbf{T}^k}$ is the Haar measure on $\mathbf{T}^k$.*

Proof. The proof of (i) is a straightforward calculation. To prove (ii) it suffices to observe that the functional $f\mapsto\int_{\mathbf{S}^{n-1}} Jf\,dm_{\mathbf{S}^{n-1}}$ is invariant under translations and is continuous with respect to the L^∞ norm. $\square$

Corollary 5.2. $\|V^1\|_p\le\|V\|_p$ *and* $\|I\oplus V^1\|_p\le\|I\oplus V\|_p$.

Let $\alpha=(\alpha_1,\ldots,\alpha_k)$, $|\alpha|=1$. Consider the operators on $L^2_0(\mathbf{T}^k)$ whose action on the group characters is, for $l\neq 0$,

$$Q_U^{c1}(e^{il\cdot\theta})=\frac{il\cdot\alpha}{(|l|(|l|+n-2))^{1/2}}e^{il\cdot\theta}$$

and

$$Q_U^{b1}(e^{il\cdot\theta})=\frac{il\cdot\alpha}{|l|}e^{il\cdot\theta}.$$

Observe that $Q_U^{b1}=R_\alpha^{\mathbf{T}^k}$ is the *Riesz transform in the direction* α on $\mathbf{T}^k$ with its standard metric, where $\alpha\in\mathbf{R}^k$ can be viewed as a unit vector in the Lie algebra of $\mathbf{T}^k$. Corollary 5.2 implies the following corollary.

Corollary 5.3.
(i) $\|Q_U^{c1}\|_p \le \|Q_U^c\|_p$, $\|Q_U^{b1}\|_p \le \|Q_U^b\|_p$,
(ii) $\|I \oplus Q_U^{c1}\|_p \le \|I \oplus Q_U^c\|_p$, $\|I \oplus Q_U^{b1}\|_p \le \|I \oplus Q_U^b\|_p$.

The next step will lead us to $\mathbf{S}^1$.

In order to unify the notation, let W_α^r be the operator acting on characters as

$$W_\alpha^r(e^{il\cdot\theta}) = i\frac{l}{|l|}\cdot\alpha\left(\frac{|l|}{|l|+r}\right)^{1/2}$$

where $l \neq 0$. Then $W_\alpha^0 = Q_U^{b1}$ and $W_\alpha^{n-2} = Q_U^{c1}$.

Proposition 5.4. *If* $|\alpha|=1$ *and* $r \ge 0$, *then*

$$\|W_\alpha^r\|_p \ge B_p \quad \textit{and} \quad \|I \oplus W_\alpha^r\|_p \ge E_p. \tag{58}$$

Proof. The proof is divided into two cases.

Case 1. Suppose that $\alpha = m/|m|$, with $m=(m_1, \dots, m_k) \in \mathbf{Z}^k$. Consider the operator K that maps functions $f: \mathbf{S}^1 \to \mathbf{R}$ to functions $Kf: \mathbf{T}^k \to \mathbf{R}$ by

$$Kf(\theta) = f(\theta \cdot m).$$

For $s \ge 0$, define a multiplier operator on $\mathbf{S}^1$ by

$$\widetilde{W}_{m/|m|}^s(e^{iq\phi}) = i\,\mathrm{sign}(q)\left(\frac{|q|}{|q|+s}\right)^{1/2} e^{iq\phi}.$$

In a sense, K extends functions on $\mathbf{S}^1$ to functions on $\mathbf{T}^k$ through the foliation of $\mathbf{T}^k$ induced by the one parameter, closed subgroup $t \mapsto t\theta$, $t \in \mathbf{R}$.

Lemma 5.5.
(i) $K \circ \widetilde{W}_{m/|m|}^{r/|m|} = W_{m/|m|}^r \circ K$.
(ii) *There exists* $C>0$ *such that* $\int_{\mathbf{T}^k} Kf\, dm_{\mathbf{T}^k} = C \int_{\mathbf{S}^1} f\, dm_{\mathbf{S}^1}$ *for all integrable* f *on* $\mathbf{S}^1$.

The proof of the lemma is a simple calculation.

An immediate consequence of the lemma is that the proposition holds for $\alpha = m/|m|$ and for $r=0$, since $\widetilde{W}_{m/|m|}^0 = -\mathcal{H}$. In order to deal with the case $r>0$, we need a new idea.

Lemma 5.6.
(i) $B_p \leq \|\widetilde{W}^r_{m/|m|}\|_p$,
(ii) $E_p \leq \|I \oplus \widetilde{W}^r_{m/|m|}\|_p$.

Proof. Let f be a trigonometric polynomial on $\mathbf{S}^1$, $N>0$ a positive integer. Define $\delta_N f(e^{i\theta})=f(e^{iN\theta})$. Since $e^{i\theta}\mapsto e^{iN\theta}$ is measure preserving, $\|\delta_N f\|_p=\|f\|_p$. We have that

$$\widetilde{W}^r_{m/|m|}\circ\delta_N=\delta_N\circ\widetilde{W}^{r/N}_{m/|m|}.$$

Given $\varepsilon>0$, let f be a trigonometric polynomial such that $\|f\|_p=1$ and

$$\|\widetilde{W}^0_{m/|m|}f\|_p=\|\mathcal{H}f\|_p\geq B_p-\varepsilon.$$

There exists $s_0>0$ so that

$$\|\widetilde{W}^s_{m/|m|}f\|_p\geq B_p-2\varepsilon$$

if $0\leq s\leq s_0$. Thus,

$$\|\widetilde{W}^r_{m/|m|}\delta_N f\|_p=\|\delta_N\widetilde{W}^{r/N}_{m/|m|}f\|_p=\|\widetilde{W}^{r/N}_{m/|m|}f\|_p>B_p-2\varepsilon.$$

Since ε was arbitrary, the lemma is proved. □

From the lemma it follows that the proposition holds for $\alpha=m/|m|$ and $r>0$.

Case 2. For general α, we can find m as above such that $|1-(m/|m|)\cdot\alpha|<\varepsilon$, where $\varepsilon>0$ is any fixed positive number. Using Case 1 and an approximation argument, one proves the proposition in the general case. □

If $\mathbf{T}^k$ is given any other invariant metric, a modification of the argument in Proposition 5.4 proves that B_p and E_p are best possible in the L^p estimates for the corresponding directional Riesz transforms.

Let U be as in (12) and $X\in\mathfrak{so}(n)$ be defined by (54).

As a consequence of Proposition 5.4, Corollary 5.2, Lemma 4.1 we have the inequalities

$$B_p\leq\|W^{n-2}_\alpha\|_p=\|Q^{c1}_U\|_p\leq\|Q^c_U\|_p\leq\|R^{SO(n)}_X\|_p \tag{59}$$

and

$$B_p\leq\|W^0_\alpha\|_p=\|R^{\mathbf{T}^k}_\alpha\|_p=\|Q^{b1}_U\|_p\leq\|Q^b_U\|_p.$$

We have, as well, the corresponding inequalities with E_p instead of B_p and with $I\oplus\mathcal{U}$ instead of $\mathcal{U}$, where $\mathcal{U}$ is one of the operators in the above chain of inequalities.

This proves the cases of equality in Theorem 1 and concludes the proofs of Theorem 2 and Theorem 3.

6. Boundedness of R^O

In this section $\mathbf{S}_n=\mathbf{S}^{n-1}(\sqrt{n})$ is the $(n-1)$-dimensional sphere of radius $\sqrt{n}$. We endow $\mathbf{S}_n$ with its natural Riemannian metric and with the $SO(n)$ invariant measure μ_n normalized so that $\mu_n(\mathbf{S}_n)=1$. The L^p norms on $\mathbf{S}_n$ are taken with respect to this measure. Many geometric objects on $\mathbf{S}_n$ pass in the limit to corresponding objects on the infinite dimensional Gauss space, see [M]. In this section we prove that L^p estimates for the Riesz transform R^c on $\mathbf{S}^{n-1}$ pass in the limit to estimates for the Riesz transform associated with the Ornstein–Uhlenbeck process. In order to do this, we will see, more generally, how the spectral theory of the spherical Laplacian on $\mathbf{S}^{n-1}$ is related, as $n\to\infty$, to the spectral theory of the Hermite operator in Gauss space. See [Ma] for results of a similar flavor. As a consequence, we will have that the L^p norms of gradients and Laplacian powers on $\mathbf{S}^{n-1}$ tend to the L^p norms of gradients and Hermite operator powers in Gauss space, in a suitable way.

With $\mathcal{T}_{lm}$ as in (10), if $F\colon \mathbf{S}_n\to\mathbf{R}$ is smooth enough, we have

$$\Delta_{\mathbf{S}_n}F=\frac{1}{n}\sum_{1\le l<m\le n}\mathcal{T}_{lm}\mathcal{T}_{lm}F \tag{60}$$

and

$$|\nabla_{\mathbf{S}_n}F|^2=\frac{1}{n}\sum_{1\le l<m\le n}|\mathcal{T}_{lm}F|^2. \tag{61}$$

If F is a spherical harmonic of degree k, then,

$$\Delta_{\mathbf{S}^{n-1}}F=\frac{k(n-2+k)}{n}F.$$

Let m be a fixed positive integer. Let $\Pi_n\colon \mathbf{S}_n\to\mathbf{R}^m$ be the projection $\Pi_n(x,y)=x$, if $x\in\mathbf{R}^m$, $y\in\mathbf{R}^{n-m}$ and $|x|^2+|y|^2=n$. If $f\colon\mathbf{R}^m\to\mathbf{R}$, $f_n=f\circ\Pi_n$. Mehler's observation is that, if $E\subseteq\mathbf{R}^m$ is measurable, then

$$\int_{\mathbf{S}_n}\chi_E\circ\Pi_n\,d\mu_n=\frac{\displaystyle\int_{\substack{\mathbf{R}^m\\ |x|^2\le n}}\chi_E(x)\left(1-\frac{|x|^2}{n}\right)^{(n-m-2)/2}dx}{\displaystyle\int_{\substack{\mathbf{R}^m\\ |x|^2\le n}}\left(1-\frac{|x|^2}{n}\right)^{(n-m-2)/2}dx}\to\int_{\mathbf{R}^m}\chi_E\,d\gamma,$$

as $n\to\infty$. It is not difficult to check that, in fact, if $f\colon\mathbf{R}^m\to\mathbf{R}$ has polynomial growth, then

$$\int_{\mathbf{S}_n}f_n\,d\mu_n\to\int_{\mathbf{R}^m}f\,d\gamma. \tag{62}$$

Hence, if f is a polynomial in $x\in\mathbf{R}^m$ and $1\le p<\infty$, then

$$\lim_{n\to\infty} \|\nabla_{\mathbf{S}_n} f_n\|_{L^p(\mathbf{S}_n)} = \|\nabla_{\mathbf{R}_m} f\|_{L^p(\gamma)},$$
$$\lim_{n\to\infty} \|\Delta_{\mathbf{S}_n} f_n\|_{L^p(\mathbf{S}_n)} = \|Af\|_{L^p(\gamma)},$$

where $Af(x)=\Delta_{\mathbf{R}^m} f(x)-x\cdot\nabla_{\mathbf{R}^m} f(x)$ is the *Hermite operator*. In fact,

$$|\nabla_{\mathbf{S}_n} f_n|^2 = \sum_{j=1}^{m}(\partial_j f)^2 - \frac{1}{n}\left(\sum_{j=1}^{m} x_j\partial_j f\right)^2$$

and the error term has polynomial growth. A similar relation holds for the Laplacian, see [M].

Let $\mathcal{A}_k^m$ be the space of polynomials of degree not greater than k in $x=(x_1,\dots,x_m)$. The space $\mathcal{H}_k(\mathbf{R}^n)$ is the space of homogeneous harmonic polynomials of degree k in $\mathbf{R}^n$ and $\mathfrak{H}_k^{n,m}$ is the space of those $Y\in\mathcal{H}_k(\mathbf{R}^n)$ that are invariant under $SO(n-m)$, the subgroup of $SO(n)$ that fixes pointwise the first factor of $\mathbf{R}^n=\mathbf{R}^m\times\mathbf{R}^{n-m}$, $n>m$. Then $Y\in\mathfrak{H}_k^{n,m}$ if and only if it is a spherical harmonic of degree k on $\mathbf{R}^n$ that can be written as $Y(x_1,\dots,x_n)=\phi(x_1,\dots,x_m,x_{m+1}^2+\dots+x_n^2)$, where ϕ is a polynomial in $m+1$ variables. The space $\mathfrak{H}_k^{\infty,m}$ will be the space of the *generalized Hermite polynomials* of degree k on $\mathbf{R}^m$, i.e., the space of those $P\in\mathcal{A}_k^m$ such that $AP+kP=0$. See [Me3].

Mimicking the reasoning in Chapter IV of [SW], it is easy to verify that $\dim(\mathfrak{H}_k^{n,m})=\dim(\mathfrak{H}_k^{\infty,m})=d_k^m$ is independent of n. In fact, $d_k^m=\#\{\alpha\in\mathbf{N}^k:|\alpha|=\alpha_1+\dots+\alpha_m=k\}$.

Let now $P\in\mathfrak{H}_k^{\infty,m}$, $n>m$, and let P_n be its restriction to $\mathbf{S}_n$. Then

$$P_n = \sum_{j\le k} Q_j^{n,m}(P), \tag{63}$$

where $Q_j^{n,m}(P)$ is the $L^2(\mathbf{S}_n)$-orthogonal projection of P onto $\mathcal{H}_j(\mathbf{R}^n)$, a spherical harmonic of degree k, that we extend to a homogeneous polynomial on $\mathbf{R}^n$. Then, by $SO(n-m)$ invariance, $Q_j^{n,m}(P)\in\mathfrak{H}_k^{n,m}$. The following lemma shows how the spectral decomposition of P_n simplifies as $n\to\infty$.

Lemma 6.1. *Let $P\in\mathfrak{H}_k^{\infty,m}$ and consider its decomposition as in* (63). *Then $Q_k^{n,m}(P)$ is the leading term of P_n in the L^2 sense,*

(i) $\lim_{n\to\infty}\|Q_k^{n,m}(P)\|_{L^2(\mathbf{S}_n)}=\|P\|_{L^2(\gamma)}$ *and*

(ii) $\lim_{n\to\infty}\|Q_j^{n,m}(P)\|_{L^2(\mathbf{S}_n)}=0$, *if $j<k$.*

Proof. If Q is a spherical harmonic of degree j, then

$$(-\Delta_{\mathbf{S}_n})^{1/2}Q=\sqrt{j(n-2+j)/n}\,Q.$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=0}^{k}\frac{j(n-2+j)}{n}\|Q_j^{n,m}(P)\|^2_{L^2(\mathbf{S}_n)}&=\lim_{n\to\infty}\|(-\Delta_{\mathbf{S}_n})^{1/2}P_n\|^2_{L^2(\mathbf{S}_n)}\\&=\lim_{n\to\infty}\|\nabla_{\mathbf{S}_n}P_n\|^2_{L^2(\mathbf{S}_n)}=\|\nabla_{\mathbf{R}^m}P\|^2_{L^2(\gamma)}\\&=k\|P\|^2_{L^2(\gamma)}=\lim_{n\to\infty}\sum_{j=0}^{k}\|Q_j^{n,m}(P)\|^2_{L^2(\mathbf{S}_n)}.\end{aligned}$$

Comparing the first and the last term in the chain of equalities and taking into account that $\|Q_j^{n,m}(P)\|^2_{L^2(\mathbf{S}_n)}\le\|P\|^2_{L^2(\mathbf{S}_n)}$ is bounded, since $\|P_n\|^2_{L^2(\mathbf{S}_n)}\to\|P\|^2_{L^2(\gamma)}$, we obtain (ii) for $0<j<k$. The case $j=0$ is easier, and (i) follows. $\square$

Let now $\mathfrak{I}_k^{n,m}=\bigoplus_{j=0}^{k}\mathfrak{H}_j^{n,m}$, $m<n\le\infty$. A consequence of Lemma 6.1 is that

$$\|(-\Delta_{\mathbf{S}_n})^{1/2}P_n\|_{L^2(\mathbf{S}_n)}\to\|(-A)^{1/2}P\|_{L^2(\gamma)}, \tag{64}$$

as $n\to\infty$, if $P\in\mathfrak{I}_k^{\infty,m}$. The lemma below is the key to extend (64) to $1\le p<\infty$. The real problem is $p>2$, the case $p<2$ being easily reduced to that of $p=2$.

Lemma 6.2. *Let $1\le p<\infty$. There exist $K_p=K(p,m,k)$ and $N=N(m,k)$ such that, if $F\in\mathfrak{I}_k^{n,m}$,*

$$\|F\|_{L^p(\mathbf{S}_n)}\le K_p\|F\|_{L^2(\mathbf{S}_n)}. \tag{65}$$

Proof. If $p\le2$, (65) follows from Jensen's inequality, with $K_p=1$.

Let $p>2$. If $F\in\mathfrak{I}_k^{n,m}$, then F_n, the restriction of F to $\mathbf{S}_n$, is the restriction to $\mathbf{S}_n$ of a polynomial $\phi_n\in\mathcal{A}_k^m$ that only depends on $x=(x_1,\dots,x_m)$. By Schwarz's inequality we have

$$\begin{aligned}\|F\|^p_{L^p(\mathbf{S}_n)}&\le\frac{(2\pi)^{m/2}}{\displaystyle\int_{|x|^2\le n}\left(1-\frac{|x|^2}{n}\right)^{(n-2-m)/2}dx}C^0(n,m)\left(\int_{|x|^2\le n}|\phi_n(x)|^{2p}\,d\gamma(x)\right)^{1/2}\\&\le C^1(m)\|\phi_n\|^p_{L^{2p}(\gamma)}\le C^2(m,k,p)\|\phi_n\|^{p/2}_{L^2(\gamma)},\end{aligned}$$

where $C^j(\cdot)$ represent various positive constants dependent on the arguments in the parenthesis and, in particular,

$$C^0(n,m)=\left(\int_{|x|^2\le n}\left(1-\frac{|x|^2}{n}\right)^{n-2-m} e^{|x|^2}\,d\gamma(x)\right)^{1/2}$$

is bounded in n, for fixed m. The last inequality follows from the fact that $\mathcal{A}_k^m$ is a finite dimensional Banach space with any of its L^p norms.

Consider now on $\mathcal{A}_k^m$ the norms $[\cdot]_n$, $m<n\le\infty$, $[f]_n=\|f_n\|_{L^2(\mathbf{S}_n)}$, $[f]_\infty=\|f\|_{L^2(\gamma)}$. By (62) and simple considerations about finite dimensional Hilbert spaces, we have that

$$C^3(m,k)[f]_\infty\le[f]_n\le C^4(m,k)[f]_\infty \tag{66}$$

for $n\ge N(m,k)$. Together with the chain of inequalities above, (66) implies (65). □

Corollary 6.3. *Let $1\le p<\infty$. If P is a finite linear combination of generalized Hermite polynomials, then*

$$\|(-\Delta_{\mathbf{S}_n})^{1/2}P_n\|_{L^p(\mathbf{S}_n)}\to\|(-A)^{1/2}P\|_{L^p(\gamma)}, \tag{67}$$

as $n\to\infty$. As a consequence,

$$\|\nabla_{\mathbf{R}^m}P\|_{L^p(\gamma)}\le 2(p^*-1)\|(-A)^{1/2}P\|_{L^p(\gamma)}. \tag{68}$$

Proof. Suppose $P=P^{(1)}+...+P^{(k)}$, $P^{(l)}\in\mathfrak{H}_l^{\infty,m}$. Lemma 6.1 and Lemma 6.2 imply that the leading term, in the L^p sense, of the decomposition of P_n in spherical harmonics is $Q_1^{n,m}(P^{(1)})+...+Q_k^{n,m}(P^{(k)})$. The limit (67) can then be deduced from the Fourier multiplier's expression of $(-\Delta_{\mathbf{S}_n})^{1/2}$. The inequality (68) follows from (67) and Theorem 2. □

Proof of the L^p boundedness of R^O. Inequality (68) can be rephrased as

$$\|\nabla_{\mathbf{R}^m}\circ(-A)^{-1/2}\widetilde{P}\|_{L^p(\gamma)}\le 2(p^*-1)\|\widetilde{P}\|_{L^p(\gamma)} \tag{69}$$

where $\widetilde{P}=(-A)^{1/2}P$ is any finite, linear combination of Hermite polynomials with null average on $(\mathbf{R}^m,\gamma)$. By density, (69) extends to $L_0^p(\gamma)$ [Me3]. This proves Theorem 4. □

We believe that $\|R^O\|_p\ge B_p$, Pichorides' constant, but we do not have a proof for this. The extremal sequences used in the proof of (59) in §5, in fact, depend on the dimension of $\mathbf{S}^{n-1}$, and the limiting scheme unfolded in this section does not apply.

References

[A] Arcozzi, N., Ph. D. Thesis, Washington Univ., St. Louis, Mo., 1995.

[AL] Arcozzi, N. and Li, X., Riesz transforms on spheres, *Math. Res. Lett.* **4** (1997), 401–412.

[B1] Bakry, D., Étude probabiliste des transformées de Riesz et de l'espace H^1 sur les sphéres, in *Séminaire de Probabilités* XVIII (Azéma, J. and Yor, M., eds.), Lecture Notes in Math. **1059**, pp. 197–218, Springer-Verlag, Berlin–Heidelberg, 1984.

[B2] Bakry, D., Transformations de Riesz pour les semigroupes symétriques. I. Étude de la dimension 1, in *Séminaire de Probabilités* XIX (Azéma, J. and Yor, M., eds.), Lecture Notes in Math. **1123**, pp. 130–144, Springer-Verlag, Berlin–Heidelberg, 1985.

[B3] Bakry, D., Transformations de Riesz pour les semigroupes symétriques. II. Étude sous la condition $\Gamma_2 \geq 0$, in *Séminaire de Probabilités* XIX (Azéma, J. and Yor, M., eds.), Lecture Notes in Math. **1123**, pp. 145–174, Springer-Verlag, Berlin–Heidelberg, 1985.

[B4] Bakry, D., Étude des transformées de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, in *Séminaire de Probabilités* XXI (Azéma, J., Meyer, P.-A. and Yor, M., eds.), Lecture Notes in Math. **1247**, pp. 137–172, Springer-Verlag, Berlin–Heidelberg, 1987.

[Ba] Bañuelos, R., Martingale transforms and related singular integrals, *Trans. Amer. Math. Soc.* **293** (1986), 547–563.

[BL] Bañuelos, R. and Lindeman, A., A martingale study of the Beurling–Ahlfors transform in $\mathbf{R}^n$, *J. Funct. Anal.* **145** (1997), 224–265.

[BW] Bañuelos, R. and Wang, G., Sharp inequalities for martingales with applications to the Beurling–Ahlfors and Riesz transforms, *Duke Math. J.* **80** (1995), 575–600.

[Be1] Bennett, A., Probabilistic square functions and a priori estimates, *Trans. Amer. Math. Soc.* **291** (1985), 159–166.

[Be2] Bennett, A., Ph. D. Thesis, Princeton, N. J., 1985.

[Bu1] Burkholder, D. L., Boundary value problems and sharp inequalities for martingale transforms, *Ann. Probab.* **12** (1984), 647–702.

[Bu2] Burkholder, D. L., Sharp inequalities for martingales and stochastic integrals, *Astérisque* **157–158** (1988), 75–94.

[Bu3] Burkholder, D. L., Explorations in martingale theory and its applications, in *École d'Été de Probabilités de Saint-Flour* XIX—1989 (Hennequin, P. L., ed.), Lecture Notes in Math. **1464**, pp. 1–66, Springer-Verlag, Berlin–Heidelberg, 1991.

[CW] Coifman, R. and Weiss, G., *Transference Methods in Analysis,* CBMS Regional Conf. Ser. in Math. **31**, Amer. Math. Soc., Providence, R. I., 1977.

[CL] Coulhon, T. and Ledoux, M., Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple, *Ark. Mat.* **32** (1994), 63–77.

[DR] Duoandikoetxea, J. and Rubio de Francia, J. L., Estimations indépendantes de la dimension pour les transformées de Riesz, *C. R. Acad. Sci. Paris Sér. I Math.* **300** (1985), 193–196.

[D] DURRETT, R., *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont, Calif., 1984.

[Em] EMERY, M., *Stochastic Calculus on Manifolds*, Springer-Verlag, Berlin, 1989.

[Es] ESSÉN, M., A superharmonic proof of the M. Riesz conjugate function theorem, *Ark. Mat.* **22** (1984), 241–249.

[GHL] GALLOT, S., HULIN, D. and LAFONTAINE, J., *Riemannian Geometry,* 2nd ed., Springer-Verlag, Berlin, 1990.

[G1] GUNDY, R. F., Sur le transformation de Riesz pour le sémi-groupe d'Ornstein–Uhlenbeck, *C. R. Acad. Sci. Paris Sér. I Math.* **303** (1986), 967–970.

[G2] GUNDY, R. F., *Some Topics in Probability and Analysis,* CBMS Regional Conf. Ser. in Math. **70**, Amer. Math. Soc., Providence, R. I., 1989.

[GV] GUNDY, R. F. and VAROPOULOS, N., Les transformations de Riesz et les intégrales stochastiques, *C. R. Acad. Sci. Paris Sér. A-B* **289** (1979), A13–A16.

[IW] IKEDA, N. and WATANABE, S., *Stochastic Differential Equations and Diffusion Processes,* 2nd ed., North-Holland, Amsterdam, 1989.

[IM] IWANIEC, T. and MARTIN, G., Riesz transforms and related singular integrals, *J. Reine Angew. Math.* **473** (1996), 25–57.

[KV1] KORÁNYI, A. and VÁGI, S., Singular integrals in homogeneous spaces and some problems of classical analysis, *Ann. Scuola Norm. Sup. Pisa* **25** (1971), 575–648.

[KV2] KORÁNYI, A. and VÁGI, S., Group theoretic remarks on Riesz systems on balls, *Proc. Amer. Math. Soc.* **85** (1982), 200–205.

[Ma] MARTINI, C., Polynômes harmoniques sur un espace gaussien, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 419–422.

[M] MCKEAN, H. P., Geometry of differential space, *Ann. Probab.* **1** (1973), 197–206.

[Me1] MEYER, P.-A., Démonstrations probabilistes des inégalités de Littlewood–Paley, in *Séminaire de Probabilités* X (Meyer, P.-A., ed.), Lecture Notes in Math. **511**, pp. 125–183, Springer-Verlag, Berlin–Heidelberg, 1976.

[Me2] MEYER, P.-A., Retour sur la théorie de Littlewood–Paley, in *Séminaire de Probabilités* XV (Azéma, J. and Yor, M., eds.), Lecture Notes in Math. **850**, pp. 151–166, Springer-Verlag, Berlin–Heidelberg, 1981.

[Me3] MEYER, P.-A., Transformations de Riesz pour les lois gaussiennes, in *Séminaire de Probabilités* XVIII (Azéma, J. and Yor, M., eds.), Lecture Notes in Math. **1059**, pp. 179–193, Springer-Verlag, Berlin–Heidelberg, 1984.

[Pic] PICHORIDES, S. K., On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, *Studia Math.* **44** (1972), 165–179.

[Pis] PISIER, G., Riesz transforms: A simple analytic proof of P. A. Meyer's inequality, in *Séminaire de Probabilités* XXII (Azéma, J., Meyer, P.-A. and Yor, M., eds.), Lecture Notes in Math. **1321**, pp. 485–501, Springer-Verlag, Berlin–Heidelberg, 1988.

[PBM] PRUDNIKOV, A. P., BRYCHKOV, YU. A. and MARICHEV, O. I., *Integrals and Series.* Vol. 1. Elementary functions, Nauka, Moscow, 1981 (Russian). English transl.: Gordon and Breach, New York, 1986.

[RW] RICCI, F. and WEISS, G., A characterization of $H^1(\Sigma_{n-1})$, in *Harmonic Analysis in Euclidean Spaces* (Weiss, G. and Wainger, S., eds.), Proc. Symp. Pure Math. **35**, pp. 289–294, Amer. Math. Soc., Providence, R. I., 1979.

[S1] STEIN, E. M., *Topics in Harmonic Analysis Related to the Littlewood–Paley Theory,* Ann. of Math. Studies **63**, Princeton Univ. Press, Princeton, N. J., 1970.

[S2] STEIN, E. M., Some results in harmonic analysis in $\mathbf{R}^n$, for $n \to \infty$, *Bull. Amer. Math. Soc.* **9** (1983), 71–73.

[S3] STEIN, E. M., Three variations on the theme of maximal functions, in *Recent Progress in Fourier Analysis* (Peral, I. and Rubio de Francia, J. L., eds.), pp. 229–244, North-Holland, Amsterdam, 1985.

[SW] STEIN, E. M. and WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

[Va] VAROPOULOS, N., Aspects of probabilistic Littlewood–Paley theory, *J. Funct. Anal.* **39** (1980), 25–60.

[Ve] VERBITSKY, I. E., An estimate of the norm of a function in a Hardy space in terms of the norms of its real and imaginary parts, *Mat. Issled.* **54** (1980), 16–20 (Russian). English transl.: *Amer. Math. Soc. Transl.* **124** (1984), 11–15.

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