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Singular measures with small $H(p,q)$-projections

Evgueni Doubtsov(1)

Abstract. We construct a singular probability measure μ on the complex sphere such that the Poisson integral of μ is a pluriharmonic function in the ball and the Fourier transform of μ is $\mathcal{O}(1/\sqrt{p})$ as $p\to\infty$.

1. Introduction

Let $\mathbf{T}$ denote the unit circle and $\mu\in M(\mathbf{T})$ be a measure. Recall the following classical observation.

Heuristic uncertainty principle. If the Fourier transform $\hat{\mu}$ is small (in a certain sense), then μ is regular.

For example, by the classical F. and M. Riesz theorem, if $\hat{\mu}=0$ on $\mathbf{Z}_+$, then μ is absolutely continuous with respect to Lebesgue measure m.

We are looking for phenomena of the opposite nature. If we understand "$\hat{\mu}$ is small" as "pointwise small" and "μ is not regular" as "μ and m are mutually singular", then we obtain the following classical problem.

Definition $\mathbf{T}$. A function $h\colon \mathbf{Z}_+\to\mathbf{R}_+$ is said to be $\mathbf{T}$-*admissible* if there exists a probability continuous *singular* measure $\mu\in M(\mathbf{T})$ such that $\hat{\mu}(k)=\mathcal{O}(h(|k|))$.

Problem. Characterize the admissible functions.

The famous Ivashëv-Musatov theorem shows that all of the standard test functions

$$h(k)=(k\log k\log\log k \dots \log_{(p)} k)^{-1/2}$$

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are **T**-admissible (see [I] for the precise conditions sufficient for admissibility, see also [K] for a simplified and somewhat different version). We refer the interested reader to the paper [BH] where other **T**-admissible functions are obtained and the history of the problem is discussed.

Often **T**-admissibility constructions can be carried out for all locally compact nondiscrete Abelian groups. In the present paper we consider a generalization of a different type. Namely, we are concerned with an analogue of the above problem on the complex unit sphere $S=S_n\subset\mathbf{C}^n$, $n\geq 2$.

Given a measure $\mu\in M(S)$, denote by μ_{pq}, $(p,q)\in\mathbf{Z}_+^2$, the projection of μ on $H(p,q)$, the space of the complex spherical harmonics (so μ_{pq} is a polynomial on the sphere; in dimension 1 we just have $\mu_{p0}(z)=\hat{\mu}(p)z^p$, $z\in\mathbf{T}$, $p\in\mathbf{Z}_+$).

Definition S. A function $h\colon\mathbf{Z}_+^2\to\mathbf{R}_+$ is said to be *S-admissible* if there exists a probability continuous *singular* measure $\mu\in M(S)$ such that $\|\mu_{pq}\|_2=\mathcal{O}(h(p,q))$.

If also $h(p,q)=0$ for all $(p,q)\in\mathbf{Z}_+^2$ such that $pq\neq 0$, then h is said to be *plh*-admissible (note that the corresponding μ is *pluriharmonic*, i.e. the Poisson integral of μ is a pluriharmonic function in the ball).

We show that the test function $h(k)=k^{-1/2}$ is *plh*-admissible (without loss of generality, we always put $h(0)=1$).

Theorem. *Put $h(p,0)=h(0,p)=p^{-1/2}$, $p\in\mathbf{N}$, and $h(p,q)=0$ if $pq\neq 0$. Then h in S-admissible.*

Remark 1. Every pluriharmonic measure $\mu\in M(S)$ is sufficiently regular. In particular, it is well known that μ has the full closed support and $|\mu|(E)=0$ if the (real) Hausdorff dimension of E does not exceed $2n-2$.

Remark 2. Obviously, if $h\in l^2$, then h is not admissible. Therefore, the theorem shows that there is no gap between necessary and sufficient conditions for S-admissibility in terms of the scale $\{k^\alpha\}_{k\in\mathbf{N}}$, $\alpha\in\mathbf{R}$.

Remark 3. The theorem has an **R**-interpretation since the measure μ is pluriharmonic. Namely, identify S_n and $S_{\mathbf{R}}^{2n-1}\subset\mathbf{R}^{2n}$, then $\|\mu_k\|_2=\mathcal{O}(1/\sqrt{k}\,)$ where μ_k is the projection of μ on $\mathcal{H}_k$, the space of the *real* spherical harmonics.

Notation. The notation of the paper is standard. In particular, σ is the normalized Lebesgue measure on S, $\sigma(S)=1$; the symbol $\|\cdot\|_p$ denotes the L^p-norm with respect to σ.

To finish the introduction, we give a simple and important example.

Example. There exists an $h\colon\mathbf{Z}_+^2\to\mathbf{R}_+$ such that $h\notin l^2$ and h is not S-admissible.

Proof. Put $h(0,0)=h(2^j,0)=h(0,2^j)=1$ for all $j\in\mathbf{Z}_+$ and put $h(p,q)=0$ otherwise. Suppose that $\mu\in M(S)$ and $\|\mu_{pq}\|_2=\mathcal{O}(h(p,q))$. We claim that $\mu\ll\sigma$.

This is well known and easy to see. Indeed, the Cauchy projection $C[\mu]$ is in the Hardy class $H^{1/2}(B)$. Therefore, $C[\mu]\in H^1(B)$ since $C[\mu]$ has a lacunary spectrum. Finally, we apply the F. and M. Riesz theorem (on the sphere) to the measure $\mu-C[\mu]\sigma$. $\square$

2. Auxiliary polynomials

Lemma. *Suppose that $K\in\mathbf{N}$. Then, for all $N\in\mathbf{N}$ large enough, there exist polynomials $W(N)=W(N,K)\in\sum_{l\geq 0}H(N+lK,0)$ such that*

$$\|W(N)\|_\infty\leq 1, \tag{1}$$

$$\|W(N)\|_2\geq \text{const}>0, \tag{2}$$

$$\|W_q(N)\|_\infty\leq 1, \tag{3}$$

$$\|W_q(N)\|_2\leq \text{const}\, q^{-1/2}, \tag{4}$$

where $W_q(N)$ is the $H(q,0)$-projection of $W(N)$.

Proof. For $\zeta,\eta\in S$, put $d^2(\zeta,\eta)=1-|\langle\zeta,\eta\rangle|^2$ and $E_\delta(\eta)=\{\zeta\in S:d(\zeta,\eta)<\delta\}$ for $0<\delta\leq 1$. Recall that d satisfies the triangle inequality and $\sigma(E_\delta)=\delta^{2n-2}$.

(1) *Construction for $K=1$ and $n=2$.* Put $\delta=N^{-1/2}$. Choose points $\{\eta_j\}_{j=1}^M\subset S$ such that $E_\delta(\eta_j)$ are mutually disjoint and $\bigcup_{j=1}^M E_{2\delta}(\eta_j)\supset S$. In particular, $N\asymp M\asymp\delta^{-2}$.

Define $g_j(z)=\langle z,\eta_j\rangle^{N+j}$, $1\leq j\leq M$, and $G(N)=\sum g_j$. We claim that the properties (1)–(4) hold for $G(N)$ (up to a multiplicative constant).

Indeed, we have $\|g_j\|_2^2=(N+j+1)^{-1}$ and $G_{N+j}(N)=g_j$. Therefore, (3) and (4) hold. The property (2) is also clear because $\|G(N)\|_2^2=\sum\|g_j\|_2^2\geq M/2N\geq\text{const}$.

Finally, we have to estimate $\|G(N)\|_\infty$. Fix a $\zeta\in S$. For $k\in\mathbf{Z}_+$, define $H_k=\{j:k\delta\leq d(\zeta,\eta_j)\leq(k+1)\delta\}$. First, the cardinality of H_k does not exceed $(k+2)^2$. Second, if $j\in H_k$, then $|g_j(\zeta)|\leq\exp(-k^2/2)$. Therefore

$$|G(N)(\zeta)|\leq\sum_{j=1}^M|g_j(\zeta)|\leq\sum_{k\geq 0}(k+2)^2\exp(-k^2/2):=\Sigma<\infty.$$

To finish the argument, put $W(N)=G(N)/\Sigma$.

(2) *$K=1$ and $n\geq 2$ is arbitrary.* As in the case $n=2$, put $\delta^2=N^{-1}$ and choose the points $\{\eta_j^l\}_{j=1}^N$, $1\leq l\leq M/N$ (it is useful to organize the sequence as a matrix). We have $M\asymp\delta^{-2n+2}$, so $M\asymp N^{n-1}$.

Define $f_j^{(l)}(z)=\langle z,\eta_j^l\rangle^{N+j}$, $1\le l\le M/N$. Now, make a randomization. Namely, let $r_l(t)$ be the Rademacher functions on $[0,1]$. Define

$$g_j(z,t)=\sum_l f_j^{(l)}(z)r_l(t) \quad \text{and} \quad h_j^L=\sum_{l=1}^L f_j^{(l)}(z)r_l(t).$$

Then there exists $\tau=\tau_j\in[0,1]$ such that $\|g_j(\,\cdot\,,\tau)\|_2^2\ge\sum_l\|f_j^{(l)}\|_2^2$. Note that $\|f_j^{(l)}\|_2^2\asymp N^{1-n}$, thus $\|g_j(\,\cdot\,,\tau)\|_2^2\ge C/N$. We fix such τ and C, and claim that there exists $L_0=L_0(j)\in[1,M/N]$ such that $\|h_j^{L_0}\|_2^2\asymp 1/N$. To show this, remark that $\|h_j^1\|_2^2\asymp N^{1-n}\le C/N$ (we assume that $n\ge3$). If $\|h_j^L\|_2^2\le C/N$ for all L, then we are done. Else choose such L that $\|h_j^L\|_2^2\le C/N$ but $\|h_j^{L+1}\|_2^2>C/N$. Then $\|h_j^L\|_2^2\asymp 1/N$ since $\|f_j^{(L+1)}\|_2^2\asymp N^{1-n}$.

Finally, put $g_j=h_j^{L_0}$ and $G(N)=\sum g_j$.

As above, the absolute value estimates provide (1) (up to a multiplicative constant). Since $G_{N+j}(N)=g_j$, (3) is clear also. By the definition of L_0, $\|g_j\|_2^2\le \text{const}\,/N$, so (4) holds. Since g_j are mutually orthogonal, we obtain $\|G(N)\|_2^2=\sum\|g_j\|_2^2\ge N\,\text{const}\,/N=\text{const}>0$. This yields (2).

(3) *K is arbitrary* (to simplify the notation, we assume that $n=2$). Take a sequence $0=a_0<a_1<\dots<a_K=1$ such that the sets $S_p=\{z\in S:a_p\le|z_1|\le a_{p+1}\}$, $p=0,1,\dots,K-1$, have equal areas (i.e. $\sigma(S_p)=1/K$).

For $p=0,1,\dots,K-1$, define $\delta_p^2=2^{-p}N^{-1}$. We take points $\{\eta_j^p\}_{j=1}^{M(p)}$ such that $E_{\delta_p}(\eta_j^p)\subset S_p$ are mutually disjoint and $E_{2\delta_p}(\eta_j^p)$ cover S_p (we can do this if N is sufficiently large). Note that $M(p)\asymp 2^pN/K$. It is convenient to assume that $M(p)<2^pN/K$ (we just forget other points).

Finally, we define $f_j^{(p)}(z)=\langle z,\eta_j^p\rangle^{2^pN+Kj}$, $j=1,2,\dots,M(p)$, $g^{(p)}=\sum_j f_j^{(p)}$ and $G(N)=\sum_p g^{(p)}$.

Fix $\zeta\in S_p$, then

$$|G(N)(\zeta)|\le \sum_{\substack{l=0 \text{ or}\\ |p-l|\le1}}|g^{(l)}(\zeta)|+\left|\sum_{\substack{l\ne0\\ |p-l|>1}} g^{(l)}(\zeta)\right|:=\Sigma_1+\Sigma_2.$$

First, the estimates given in the case $K=1$ provide $\Sigma_1\le4\Sigma$.

Second, put $\delta^2=\delta_0^2=N^{-1}$. Choose points $\{\eta_m\}_{m=1}^M\subset S$ such that $E_\delta(\eta_m)$ are mutually disjoint and $\bigcup_{m=1}^M E_{2\delta}(\eta_m)\supset S$. Now, fix $m\in\{1,\dots,M\}$ and $l\in\{1,\dots,K-1\}$ such that $|p-l|>1$. Consider the set $I(l,m)=\{j:\eta_j^l\in E_{2\delta}(\eta_m)\}$ (we suppose that $I(l,m)\ne\emptyset$). Clearly card $I(l,m)\le\text{const}\,2^l$. Since $|p-l|>1$, we have $|\langle\zeta,\eta_j^l\rangle|^{2^lN+Kj}\le$

const $|\langle\zeta,\eta_m\rangle|^{2^{l-1}N}$ if $j\in I(l,m)$ (we always assume that N is sufficiently large). Therefore

$$\sum_{j\in I(l,m)} |\langle\zeta,\eta_j^l\rangle|^{2^l N+Kj} \le \text{const}\,|\langle\zeta,\eta_m\rangle|^N$$

for all $N\in\mathbf{N}$ large enough.

Note that, for every m, $E_{2\delta}(\eta_m)$ has a non-empty intersection with at most two sets S_l. On the other hand, recall that $\bigcup_{m=1}^M E_{2\delta}(\eta_m)\supset S$. Hence

$$\Sigma_2 \le \text{const} \sum_{m=1}^{M} |\langle\zeta,\eta_m\rangle|^N \le \text{const}\,\Sigma.$$

Therefore, we have $\|G(N)\|_\infty \le \text{const}$.

Note that $f_j^{(p)}$ are mutually orthogonal and $\|f_j^{(p)}\|_2^2 \asymp (2^p N)^{-1}$, so $\|g^{(p)}\|_2^2 \ge$ const$/K$ and we obtain (2). The properties (3)–(4) are clear. □

3. The proof of the theorem

Let $Pr\colon L^2(S)\to\{f\in L^2(S): P[f]$ is a pluriharmonic function$\}$ be the orthogonal projection. Given a polynomial φ on S (a symbol), the corresponding operator of the Hankel type is defined by the equality $H_\varphi[f]=\varphi Pr[f]-Pr[\varphi f]$, $f\in L^2(S)$. Then $H_\varphi\colon C(S)\to C(S)$ is a compact operator. Therefore

$$\text{(5)} \qquad [\|f_j\|_{C(S)}\le 1 \text{ and } f_j\to 0 \text{ weakly in } L^2(S)] \quad\Rightarrow\quad \|H_\varphi f_j\|_{C(S)}\to 0.$$

The property (5) leads to the definition of the *pluriharmonic* Riesz product based on a sequence of Ryll–Wojtaszczyk polynomials (see [D]). In the present paper we use the polynomials $W(N)$ provided by the lemma. Since the spectrum of $W(N)$ is not the only point, our measure is the pluriharmonic version of the classical *generalized* Riesz product.

Generalized pluriharmonic Riesz product construction.

Step 1. Fix $N_1\in\mathbf{N}$ and put $\psi_1=1+W(N_1,1)/2$, $\varphi_1=\operatorname{Re}\psi_1>0$.

Step $k+1$. Assume, as induction hypothesis, that a holomorphic polynomial $\psi=\psi_k$ is constructed and $\varphi=\varphi_k=\operatorname{Re}\psi_k$, $\varphi>0$. Put $K=2\deg\varphi+2$. Suppose that $\beta=\{\beta(N)\}$ is a sign sequence (i.e. $\beta(N)\in\{\pm1\}$) and $U=\{U(N)\}$ is a sequence of unitary operators on $\mathbf{C}^n$. Then put $W(N)=\frac{1}{2}\beta(N)W(N,K)\circ U(N)$, $N\in\mathbf{N}$.

Let $\varphi=\sum_{l\in\mathbf{Z}} f_l$ and $W(N)=\sum_{j\in\mathbf{Z}_+} W_j(N)$ be the homogeneous decompositions (i.e. $f_j\in H(j,0)$ if $j\in\mathbf{Z}_+$ and $f_j\in H(0,j)$ if $-j\in\mathbf{Z}_+$). We assume, as induction hypothesis, that $\|f_l\|_{C(S)}\leq 1$ for all $l\in\mathbf{Z}$ (this property trivially holds in step 1). Now define

$$\Phi(N):=Pr(\varphi[1+\operatorname{Re}W(N)])=\varphi+(\varphi\operatorname{Re}W(N)-H_\varphi[\operatorname{Re}W(N)]),$$
$$g_{lj}(N)=f_lW_j(N)-H_{f_l}[W_j(N)].$$

Note that, for $N>\deg\varphi$, $\Phi(N)=\operatorname{Re}\Psi(N)$, where $\Psi(N)=\psi+Pr(\varphi W(N))$ is a holomorphic polynomial.

Claim. *There exist sequences β and U such that the relations*

$$\operatorname{spec}(\varphi)\cap\operatorname{spec}(\Phi(N)-\varphi)=\emptyset, \tag{6}$$
$$\Phi(N)>0, \tag{7}$$
$$\|g_{lj}(N)\|_2\leq 2\|W_j(N)\|_2\leq \operatorname{const} j^{-1/2}, \tag{8}$$
$$\|g_{lj}(N)\|_{C(S)}\leq\|f_l\|_{C(S)}\leq 1, \tag{9}$$
$$\|\Phi(N)\|_{1/2}\leq(1-\operatorname{const})\|\varphi\|_{1/2} \tag{10}$$
$$\|\Psi(N)-\psi\|_{1/2}\leq\|\varphi\|_{1/2} \tag{11}$$

obtain for all $N\in\mathbf{N}$ large enough (of course, β and U depend on k).

Proof of the claim. Using (1) and (2) we construct (and fix) the sequences β and U such that (10) holds (we proceed as in [D]).

To ensure (6), we just consider sufficiently large $N\in\mathbf{N}$. Remark that $W(N)\xrightarrow{wL^2}0$ and $\varphi>0$, therefore, (1) and (5) give (7). On the other hand, $\|f_l\|_{C(S)}\leq 1$, $\|W_j(N)\|_{C(S)}\leq\frac{1}{2}$, and $W_j(N)\xrightarrow{wL^2}0$ as $N\to\infty$. Thus, (1) and (3)–(5) provide (8) and (9). Finally, $\Psi(N)-\psi=\varphi W(N)-H_\varphi[W(N)]$ and $\|W(N)\|_{C(S)}\leq\frac{1}{2}$, so we use (5) and obtain (11) for all sufficiently large $N\in\mathbf{N}$. This finishes the proof of the claim.

Fix $N_{k+1}\in\mathbf{N}$ such that the above properties hold and define $\psi_{k+1}=\Psi(N_{k+1})$, respectively $\varphi_{k+1}=\Phi(N_{k+1})$. By the definition of K, the $H(p,q)$-projections of $\varphi_{k+1}-\varphi_k$ (the "new $H(p,q)$-projections") are $g_{lj}(N_{k+1})/2$ or $\overline{g_{lj}(N_{k+1})}/2$, therefore, (9) guarantees that the induction construction proceeds.

By (6) and (7), $\varphi_k\sigma\xrightarrow{w^*}\mu$ for a probability measure μ which is said to be the generalized Riesz product based on the polynomials $W(N)$.

By (10) and (11), $\|\psi_{k+1}-\psi_k\|_{1/2}\leq(1-\operatorname{const})^{k-1}\|\varphi_1\|_{1/2}$. Therefore, the sequence $\{\psi_k\}$ converges in $H^{1/2}(B)$ to a function f. Since $\varphi_k=\operatorname{Re}\psi_k$, (10) gives $\operatorname{Re}f^*=0$ a.e., where f^* stands for the boundary values of f. On the other hand, $\operatorname{Re}f$ is the Poisson integral of μ, so μ is a singular measure.

Finally, consider the new $H(p,q)$-projections $g_{lj}(N_{k+1})/2 \in H(j+l,0)$. Since $j+l \leq 3j/2$, the estimate (8) gives $\|\mu_{p0}\|_2 \leq \text{const } p^{-1/2}$. The proof of the theorem is complete. □

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Evgueni Doubtsov
Centre de Recerca Matemàtica
Institut d'Estudis Catalans
Apartat 50
E-08193 Bellaterra (Barcelona)
Spain
Current address:
St. Petersburg State University
Bibliotechnaya pl. 2
Staryi Petergof
198904 St. Petersburg
Russia
email: es@dub.pdmi.ras.ru