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Space-time scattering for the Schrödinger equation

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Abstract. Results are obtained on the scattering theory for the Schrödinger equation

$$i\partial_t u(t,x) = -\Delta_x u(t,x) + V(t,x)u(t,x) + F(u(t,x))$$

in spaces $L^r(\mathbf{R}; L^q(\mathbf{R}^d))$ for a certain range of r, q, the so-called space-time scattering. In the linear case (i.e. $F \equiv 0$) the relation with usual configuration space scattering is established.

1. Introduction

We discuss the space-time scattering for a class of Schrödinger operators with a multiplicative potential depending explicitly on time and with non-linear interactions. In the paper [8] T. Kato studied the scattering theory for the non-linear Schrödinger equation

$$i\partial_t u(t,x) = -\Delta_x u(t,x) + F(u(t,x))$$

in spaces $L^r(\mathbf{R}; L^q(\mathbf{R}^d))$ for a certain range of r, q. We call this type of results space-time scattering, the idea being to consider time and position on an equal footing.

In this paper we study the space-time scattering for the Schrödinger equation

$$i\partial_t u(t,x) = -\Delta_x u(t,x) + V(t,x)u(t,x) + F(u(t,x)) \tag{1.1}$$

for a class of explicitly time-dependent potentials V and the same class of non-linearities as in [8]. The goal is to obtain a better understanding of the relation between the space-time scattering and the usual scattering theory. Associated with the equation (1.1) with $F \equiv 0$ is a unitary propagator $U(t,s)$. The free Schrödinger group is denoted by $U_0(t)$. The usual wave operators are defined by

$$W_\pm(s) = \underset{t\to\pm\infty}{s\text{-}\lim}\, U(s,t)U_0(t-s).$$

They have been studied for a large class of potentials, see e.g. [3], [9], [10], [14], and references therein. Essential to our comparison with the space-time scattering is the result in [5] that under our assumptions (see Assumption 2.1) the wave operators extend to bounded invertible operators on $L^q(\mathbf{R}^d)$, $1\le q\le\infty$, with norm bounded uniformly in $s\in\mathbf{R}$.

The main results comparing the two approaches are stated in Theorems 4.9 and 4.10. In particular, the last result shows that the well-known perturbation expansion of the scattering operator for a time-dependent perturbation (see e.g. [11, p. 184]) can be given in closed form under our assumptions. Note that the potential is not assumed to be small in some norm.

The present approach has certain similarities with the Howland–Yajima approach [4], [15], but differs by considering a range of r, q, not just $r=q=2$, for the spaces $L^r(\mathbf{R};L^q(\mathbf{R}^d))$, and by not introducing an auxiliary time parameter.

Let us briefly describe the contents of the paper. In Section 2 we recall some results from [5]. In Section 3 we introduce various operators and their mapping properties. This section relies heavily on the results in [5], [8]. In Section 4 we establish the main results in space-time scattering in the linear case, and compare with the usual scattering theory. In Section 5 we extend the results to include a class of non-linearities, and establish the space-time scattering for small initial data. This section relies on ideas from [8]. In a subsequent paper the results obtained here will be used to discuss some perturbation results for the non-linear Schrödinger equation.

2. Preliminaries

We start by fixing our notation. After that we recall some results from [5].

Let $\mathcal{H}=L^2(\mathbf{R}^d)$ and let $H_0=-\Delta$ with domain $\mathcal{D}(H_0)=H^2(\mathbf{R}^d)$, the usual Sobolev space of order 2. The propagator for the free Schrödinger equation is denoted by $U_0(t)=\exp(-itH_0)$.

Let $V(t,x)$ be a real-valued function on $\mathbf{R}\times\mathbf{R}^d$. We denote by $V(t)$ the multiplication by $V(t,\cdot)$ on $\mathcal{H}$, and by $\widehat{V}(t,\xi)$ the Fourier transform with respect to the x-variable. The Banach space of finite regular complex measures on $\mathbf{R}^d$ is denoted by $\mathcal{M}(\mathbf{R}^d)$.

Assumption 2.1. *Let $V(t,x)$ be a real-valued function such that $\widehat{V}\in L^1(\mathbf{R};\mathcal{M}(\mathbf{R}^d))$.*

This assumption is imposed throughout the paper. A consequence of this assumption is that $V\in L^1(\mathbf{R};L^\infty(\mathbf{R}^d))$. Let $H(t)=H_0+V(t)$ for each $t\in\mathbf{R}$. Then

$H(t)$ is self-adjoint on $\mathcal{H}$ with domain $\mathcal{D}(H(t))=\mathcal{D}(H_0)$ for all $t\in\mathbf{R}$. Associated with the problem

$$i\frac{d}{dt}\psi(t)=H(t)\psi(t), \quad \psi(s)=\psi_0, \tag{2.1}$$

is the propagator $U(t,s)$ such that the mild solution to (2.1) is given by $\psi(t)=U(t,s)\psi_0$. The family $U(t,s)$ consists of unitary operators on $\mathcal{H}$ with the properties $U(t,t)=I$ (the identity operator on $\mathcal{H}$), and $U(t,s)U(s,r)=U(t,r)$ for all $t,s,r\in\mathbf{R}$.

It is well known that the condition $V\in L^1(\mathbf{R};L^\infty(\mathbf{R}^d))$ implies the existence of a unitary propagator $U(t,s)$ for the problem (2.1). The function $\psi(t)=U(t,s)\psi_0$ solves the integral equation

$$\psi(t)=U_0(t-s)\psi_0-i\int_s^t U_0(t-\tau)V(\tau)\psi(\tau)\,d\tau \tag{2.2}$$

and in that sense $\psi(t)$ solves (2.1). Additional conditions are needed on V in order to get a strongly differentiable solution $\psi(t)$. See [16] and references therein for some results in that direction. See also the discussion in [3].

The wave operators for the problem (2.1) and the free Schrödinger equation are given by

$$W_\pm(s)=\operatorname*{s\text{-}lim}_{t\to\pm\infty} U(s,t)U_0(t-s). \tag{2.3}$$

We recall the main result from [5].

Theorem 2.2. ([5]) *Let V satisfy Assumption 2.1. Then the following results hold:*

(i) *For each $s\in\mathbf{R}$ the limits*

$$W_\pm(s)=\lim_{t\to\pm\infty} U(s,t)U_0(t-s)$$

exist in operator norm in $\mathcal{B}(L^2(\mathbf{R}^d))$ and are unitary.

(ii) *The operators $W_\pm(s)$ extend to bounded operators on $L^p(\mathbf{R}^d)$, $1\le p\le\infty$. Furthermore, $W_\pm(s)$ are invertible in $\mathcal{B}(L^p(\mathbf{R}^d))$, and we have*

$$\sup_{s\in\mathbf{R}}\|W_\pm(s)\|_{\mathcal{B}(L^p)}<\infty$$

and

$$\sup_{s\in\mathbf{R}}\|W_\pm(s)^{-1}\|_{\mathcal{B}(L^p)}<\infty.$$

We recall the intertwining relation (in $\mathcal{B}(L^2(\mathbf{R}^d))$)

$$U(t,s)W_\pm(s)=W_\pm(t)U_0(t-s), \quad t,s\in\mathbf{R}. \tag{2.4}$$

3. Space-time estimates

It is well known that space-time estimates for the free propagator $U_0(t)$ play an important role for the study of both Schrödinger operators with time-dependent potentials [3], [16], [17] and for non-linear Schrödinger operators [1], [6], [7], [13].

Theorem 2.2 and the intertwining relation (2.4) allow us to transfer these estimates from $U_0(t-s)$ to $U(t,s)$. We need these results below, so we will state them in detail, referring to [7], [8] for the proofs for $U_0(t-s)$. See also the results in [2].

We define two operators, initially from $L^2(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}; L^2(\mathbf{R}^d))$, as follows. Let $\phi\in L^2(\mathbf{R}^d)$ and $s\in\mathbf{R}$,

$$\Gamma_0(s)\phi = U_0(t-s)\phi, \tag{3.1}$$

$$\Gamma(s)\phi = U(t,s)\phi. \tag{3.2}$$

For $f\in C_0(\mathbf{R}; L^2(\mathbf{R}^d))$ (the continuous functions with compact support) the adjoints are given by

$$\Gamma_0(s)^* f = \int_{-\infty}^{\infty} U_0(s-t)f(t)\,dt, \tag{3.3}$$

$$\Gamma(s)^* f = \int_{-\infty}^{\infty} U(s,t)f(t)\,dt. \tag{3.4}$$

For $f\in C_0(\mathbf{R}; L^2(\mathbf{R}^d))$ we define maps with values in $L^\infty(\mathbf{R}; L^2(\mathbf{R}^d))$ as

$$(G^0_\pm f)(t) = \int_{\pm\infty}^{t} U_0(t-s)f(s)\,ds, \tag{3.5}$$

$$(G_\pm f)(t) = \int_{\pm\infty}^{t} U(t,s)f(s)\,ds. \tag{3.6}$$

We have the following relations, valid for any $s\in\mathbf{R}$,

$$G^0_- - G^0_+ = \Gamma_0(s)\Gamma_0(s)^*, \tag{3.7}$$

$$G_- - G_+ = \Gamma(s)\Gamma(s)^*. \tag{3.8}$$

On $L^\infty(\mathbf{R}; L^2(\mathbf{R}^d))$ we define

$$(\mathsf{W}_\pm f)(t) = W_\pm(t)f(t) \tag{3.9}$$

and then we get from the intertwining relation the following results:

$$G_+ = \mathsf{W}_+ G^0_+ \mathsf{W}_+^{-1}, \tag{3.10}$$

$$G_- = \mathsf{W}_- G^0_- \mathsf{W}_-^{-1}, \tag{3.11}$$

$$\Gamma(s) = \mathsf{W}_+ \Gamma_0(s) W_+(s)^{-1}, \tag{3.12}$$

$$\Gamma(s) = \mathsf{W}_- \Gamma_0(s) W_-(s)^{-1}. \tag{3.13}$$

The operators $\mathsf{W}_\pm$ were also considered in [4].

Let us now recall the notation from [8]. We denote by $\square$ the closed unit square $[0,1]\times[0,1]$ in $\mathbf{R}^2$. We write $P\in\square$, $P=(1/q,1/r)$, $1\le q\le\infty$, $1\le r\le\infty$, with the usual convention $1/\infty=0$. We use the notation

$$L(P)=L^r(\mathbf{R};L^q(\mathbf{R}^d)),\quad P=(1/q,1/r)\in\square.$$

The norm on $L(P)$ is denoted by $\|\!|\cdot|\!\|_P$ or $\|\!|\cdot|\!\|_{L(P)}$. For $P\in\square$ we write $P=(x(P),y(P))$ for the coordinates, and define the function $\pi:\square\to\mathbf{R}$ by $\pi(P)=x(P)+2y(P)/d$.

The following notation for special points in $\square$ comes from [8].

For $d\ge2$,

$$\begin{aligned}
B&=\left(\tfrac{1}{2},0\right), & B'&=\left(\tfrac{1}{2},1\right),\\
C&=\left(\frac{1}{2}-\frac{1}{d},\frac{1}{2}\right), & C'&=\left(\frac{1}{2}+\frac{1}{d},\frac{1}{2}\right),\\
D&=\left(\frac{d-2}{2(d-1)},\frac{d}{2(d-1)}\right), & D'&=\left(\frac{d}{2(d-1)},\frac{d-2}{2(d-1)}\right),\\
E&=\left(\frac{1}{2}-\frac{1}{d},1\right), & E'&=\left(\frac{1}{2}+\frac{1}{d},0\right),\\
F&=\left(\frac{1}{2}-\frac{1}{d},0\right), & F'&=\left(\frac{1}{2}+\frac{1}{d},1\right),
\end{aligned}$$

and for $d=1$,

$$\begin{aligned}
C&=\left(0,\tfrac{1}{4}\right), & C'&=\left(1,\tfrac{3}{4}\right),\\
D&=\left(0,\tfrac{1}{2}\right), & D'&=\left(1,\tfrac{1}{2}\right),\\
E&=\left(0,\tfrac{1}{2}\right), & E'&=\left(1,\tfrac{1}{2}\right),\\
F&=(0,0), & F'&=(1,1).
\end{aligned}$$

We denote by T the triangle determined by B, E, and F, and by T' the triangle determined by B', E', and F'. The triangles are open except that $B\in T$ and $B'\in T'$. The triangle $\widehat{T}$ is determined by B, C, and D. We have $\widehat{T}\subset T$, and the side $]CD[$ is included in $\widehat{T}$. Similarly, $\widehat{T}'\subset T'$ is determined by B', C', and D'. See Figure 1.

Theorem 3.1. *Assume $P\in T$, $Q\in T'$, and $\pi(Q)-\pi(P)=2/d$. Then $G^0_\pm\in\mathcal{B}(L(Q),L(P))$ and $G_\pm\in\mathcal{B}(L(Q),L(P))$.*

Proof. For $G^0_\pm$ this is [8, Theorem 2.1], and for $G_\pm$ the result follows from Theorem 2.2, (2.4), and the result for $G^0_\pm$.

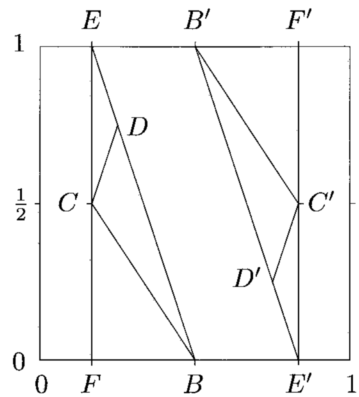

Figure 1. The unit square and some important points (for $d \geq 3$).

Theorem 3.2. *Assume* $\frac{1}{2} \leq 1/p < d/2(d-1)$ (*for* $d=1$ *we assume* $\frac{1}{2} \leq 1/p \leq 1$). *Let* $P \in \widehat{T} \cup [B, C[$ *with* $\pi(P) = 1/p$. *Then*

$$\Gamma_0(s), \Gamma(s) \in \mathcal{B}(L^p(\mathbf{R}^d), L(P)).$$

Let q *be conjugate to* p *and let* $Q \in \widehat{T}' \cup [B', C'[$ *with* $\pi(Q) = 1/q + 2/d$. *Then*

$$\Gamma_0(s)^*, \Gamma(s)^* \in \mathcal{B}(L(Q), L^q(\mathbf{R}^d)).$$

Proof. For $\Gamma_0(s)$ and $\Gamma_0(s)^*$ the results are in [8, Theorem 3.2], and for $\Gamma(s)$ and $\Gamma(s)^*$ they follow from these results, Theorem 2.2, and (2.4).

One more assumption on V will be needed.

Assumption 3.3. *Let* V *be a real-valued function such that* $V \in L(R)$ *for some* $R \in \square$ *satisfying* $y(R) > 0$ *and* $\pi(R) = 2/d$.

Definition 3.4. Let V satisfy Assumption 3.3 for some R. A pair $P, Q \in \square$ is called V-admissible, if $P \in T$, $Q \in T'$, and $Q = P + R$.

We sometimes shorten notation and call P a V-admissible point, if there exists Q such that the pair P, Q is V-admissible.

Lemma 3.5. *Let* V *satisfy Assumption* 3.3. *Then there exists at least one* V*-admissible pair* $P, Q \in \square$.

Proof. The result is verified by straightforward computations, which we omit.

Remark 3.6. If V satisfies Assumption 2.1 then the pair B, B' is V-admissible. If V furthermore satisfies Assumption 3.3 for some R_0, then by interpolation this assumption is satisfied by all R on the line segment $[(0,1), R_0]$.

Lemma 3.7. *Let V satisfy Assumptions* 2.1 *and* 3.3. *Let P, Q be a V-admissible pair. Then the following identities hold in $\mathcal{B}(L(Q), L(P))$*

$$G_-^0 - G_- = iG_-^0 V G_- = iG_- V G_-^0, \tag{3.14}$$

$$G_+^0 - G_+ = iG_+^0 V G_+ = iG_+ V G_+^0. \tag{3.15}$$

Proof. Note that the assumptions imply $\pi(Q)-\pi(P)=\pi(R)=2/d$. We only prove the first equality in (3.14). Let $f\in C_0(\mathbf{R}; L^2(\mathbf{R}^d))$. First change the order of integration, and then use (2.2) to get (recall that $V\in L^1(\mathbf{R}; L^\infty(\mathbf{R}^d))$)

$$\begin{aligned}(iG_-^0 V G_- f)(t) &= i\int_{-\infty}^t U_0(t-s)V(s)\int_{-\infty}^s U(s,\tau)f(\tau)\,d\tau\,ds\\ &= i\int_{-\infty}^t\int_\tau^t U_0(t-s)V(s)U(s,\tau)f(\tau)\,ds\,d\tau\\ &= \int_{-\infty}^t (U_0(t-\tau)-U(t,\tau))f(\tau)\,d\tau = (G_-^0 f)(t)-(G_- f)(t).\end{aligned}$$

The second equality in (3.14) is proved using

$$U(t,\tau)-U_0(t-\tau) = -i\int_\tau^t U(t,s)V(s)U_0(s-\tau)\,ds,$$

instead of (2.2). This proves (3.14) on a dense subset of $L(Q)$. The extension to all of $L(Q)$ follows, since all terms are bounded operators from $L(Q)$ to $L(P)$. Note that $V\in\mathcal{B}(L(P), L(Q))$ due to Assumption 3.3 and Hölder's inequality.

The results in (3.15) are proved analogously.

Lemma 3.8. *Let V satisfy Assumptions* 2.1 *and* 3.3, *and let P, Q be a V-admissible pair. Then $1+iG_-^0V$ is invertible in $\mathcal{B}(L(P))$ with inverse $1-iG_-V$. Similarly, $1+iG_+^0V$ is invertible with inverse $1-iG_+V$.*

Proof. The result is a consequence of (3.14) and (3.15), as is seen by a straightforward computation.

4. Space-time scattering in the linear case

We now establish the results on space-time scattering in the linear case. For that purpose we recall further definitions and results from [8].

Definition 4.1. A distribution $u\in\mathcal{S}'(\mathbf{R}\times\mathbf{R}^d)$ is called a *free wave*, if

$$(i\partial_t+\Delta_x)u=0 \quad \text{in } \mathcal{S}'(\mathbf{R}\times\mathbf{R}^d).$$

It is well known (see for example [12]) that a free wave u can be written as $u=\Gamma_0(0)\phi$ for some $\phi\in\mathcal{S}'(\mathbf{R}^d)$. Furthermore, $U_0(t)$ is a C^∞-group on $\mathcal{S}'(\mathbf{R}^d)$.

Definition 4.2. Let $P\in\square$. We define

$$\mathcal{L}_0(P)=\{u\in L(P)\,|\,u \text{ is a free wave}\}.$$

If P is on the line $]BE]$ or to the right of this line (see Figure 1), then $\mathcal{L}_0(P)=\{0\}$. Otherwise, $\mathcal{L}_0(P)$ is a closed subspace of $L(P)$ which is quite large, due to the result [8, Theorem 3.6].

We note the following easy result.

Lemma 4.3. $\mathcal{L}_0(B)=\{\Gamma_0(s)\psi\,|\,s\in\mathbf{R},\ \psi\in L^2(\mathbf{R}^d)\}$.

One can use a space smaller than $\mathcal{S}'(\mathbf{R}^d)$ in the description of properties of the free waves, and thus get stronger results concerning free waves. We recall that the weighted L^2-space is given by $L^{2,1}(\mathbf{R}^d)=\{\phi\in L^2(\mathbf{R}^d)\,|\,\langle x\rangle\phi\in L^2(\mathbf{R}^d)\}$. The Ginibre–Velo space is given by $\Sigma=H^1(\mathbf{R}^d)\cap L^{2,1}(\mathbf{R}^d)$. The dual Hilbert space is denoted by Σ^* and is obtained using the duality given by the inner product on $L^2(\mathbf{R}^d)$. Thus $\Sigma\hookrightarrow L^2(\mathbf{R}^d)\hookrightarrow\Sigma^*$. The group $U_0(t)$ is a continuous group on both Σ and Σ^*. However, it is not uniformly bounded. We denote by $C_\infty(\mathbf{R};\Sigma^*)$ the continuous functions $v\colon\mathbf{R}\to\Sigma^*$ such that $v(t)\to 0$ in Σ^* as $|t|\to\infty$. The bounded continuous functions are denoted by $BC(\mathbf{R};\Sigma^*)$.

Lemma 4.4. ([8]) *Let $P\in T$ and $u\in\mathcal{L}_0(P)$. Then $u\in C_\infty(\mathbf{R};\Sigma^*)$.*

Proof. See [8, Lemma 4.1].

Lemma 4.5. ([8]) *Assume $f\in L(Q)$, $Q\in T'$. Then we have*

(i) $G^0_\pm f\in C_\infty(\mathbf{R};\Sigma^*)$;

(ii) *let $h_\pm(t)=U_0(-t)(G^0_\pm f)(t)$, then $h_\pm\in BC(\mathbf{R};\Sigma^*)$ and $h_\pm(t)\to 0$ in Σ^*, as $t\to\pm\infty$.*

Proof. See [8, Lemmas 4.2 and 4.3].

Definition 4.6. Let $P\in T$ and let $u,v\in L(P)$. Then u is said to be asymptotic to v at $\pm\infty$, if

$$U_0(-t)(u(t)-v(t))\to 0 \quad \text{in } \Sigma^*, \text{ as } t\to\pm\infty.$$

In that case we write $u\sim v$ at $\pm\infty$.

One more piece of terminology is needed. We abuse notation and let $H(t)=-\Delta+V(t)$ denote the operator also acting on certain functions $u\in L(P)$.

Definition 4.7. Let $u\in L(P)$, $P\in T$. We say that $(i\partial_t-H(t))u=0$ holds *in the weak sense*, if

$$\langle (i\partial_t-H(t))\Psi, u\rangle=0$$

for all $\Psi\in\mathcal{S}(\mathbf{R}\times\mathbf{R}^d)$.

Note that this definition makes sense when V satisfies Assumptions 2.1 and 3.3. Here the duality $\langle\cdot,\cdot\rangle$ between $\mathcal{S}(\mathbf{R}\times\mathbf{R}^d)$ and $\mathcal{S}'(\mathbf{R}\times\mathbf{R}^d)$ is obtained from the inner product on $L^2(\mathbf{R}\times\mathbf{R}^d)$.

Given this framework we can present our results on space-time scattering in the linear case.

Theorem 4.8. *Let V satisfy Assumptions* 2.1 *and* 3.3. *Let P be V-admissible.*

(i) *Let $u\in L(P)$ satisfy $(i\partial_t-H(t))u=0$ in the weak sense. Then there exist unique free waves $u_\pm\in\mathcal{L}_0(P)$ such that*

$$u\sim u_\pm \quad \text{at } \pm\infty.$$

Furthermore, the map $u_-\mapsto u_+$ is given by

$$u_+=(1+iG_+^0V)(1+iG_-^0V)^{-1}u_-. \tag{4.1}$$

(ii) *Let $u_-\in\mathcal{L}_0(P)$. Then $u=(1-iG_-V)u_-\in L(P)$ solves $(i\partial_t-H(t))u=0$ in the weak sense, and $u\sim u_-$ at $-\infty$. An analogous result holds in the $+\infty$-case.*

Proof. Assume that $u\in L(P)$ and $(i\partial_t-H(t))u=0$ in the weak sense. Let $u_-=(1+iG_-^0V)u$. We have

$$(i\partial_t-H_0)u_-=(i\partial_t-H_0)u+(i\partial_t-H_0)(iG_-^0Vu)=(i\partial_t-H_0-V)u=0.$$

Since $u\in L(P)$, we have $u_-\in\mathcal{L}_0(P)$. Furthermore,

$$U_0(-t)(u_--u)=U_0(-t)iG_-^0Vu\to 0 \quad \text{in } \Sigma^*, \text{ as } t\to-\infty,$$

by Lemma 4.5. In the $+\infty$-case we take $u_+=(1+iG_+^0V)u$ and proceed in the same manner. Uniqueness of $u_\pm$ is trivial. This proves part (i).

To prove part (ii), let $u_-\in\mathcal{L}_0(P)$ and $u=(1-iG_-V)u_-\in L(P)$. By Lemma 3.8

$$1-iG_-V=(1+iG_-^0V)^{-1}=1-iG_-^0V(1+iG_-^0V)^{-1}. \tag{4.2}$$

Thus, using $(i\partial_t-H_0)u_-=0$ and $(i\partial_t-H_0)(-iG_-^0)=I$, we get

$$\begin{aligned}(i\partial_t-H(t))u&=(i\partial_t-H_0-V)(u_--iG_-^0V(1+iG_-^0V)^{-1}u_-)\\&=-Vu_-+V(1+iG_-^0V)^{-1}u_-+VG_-^0V(1+iG_-^0V)^{-1}u_-=0\end{aligned}$$

by (4.2).

We also have

$$U_0(-t)(u-u_-)=U_0(-t)(-iG^0_- V(1+iG^0_- V)^{-1})u_- \to 0 \quad \text{in } \Sigma^*, \text{ as } t\to -\infty,$$

by Lemma 4.5. The $+\infty$-case is handled analogously.

We introduce the notation

$$\mathsf{S}=(1+iG^0_+ V)(1+iG^0_- V)^{-1} \tag{4.3}$$

for the operator in (4.1), and call S the *space-time scattering operator* on $\mathcal{L}_0(P)$.

The following two theorems establish the relation between the usual configuration space scattering theory and the space-time approach.

Theorem 4.9. *Let V satisfy Assumptions* 2.1 *and* 3.3, *and let $P\in T$ be V-admissible. Then the following results hold on $\mathcal{L}_0(P)$:*

$$\mathsf{W}_\pm = 1-iG_\pm V, \tag{4.4}$$

$$\mathsf{S}=\mathsf{W}_+^{-1}\mathsf{W}_-. \tag{4.5}$$

Proof. We start by computing on $L^2(\mathbf{R}^d)$. Let $\phi\in L^2(\mathbf{R}^d)$. Then (2.2) is rewritten as

$$U_0(s-t)U(t,s)\phi=\phi-i\int_s^t U_0(s-\tau)V(\tau)U(\tau,s)\phi\, d\tau.$$

Hence

$$U_0(s-t)U(t,s)\phi-U_0(s-T)U(T,s)\phi=-i\int_T^t U_0(s-\tau)V(\tau)U(\tau,s)\phi\, d\tau.$$

Taking the limit $T\to-\infty$, which is justified since $V\in L^1(\mathbf{R};L^\infty(\mathbf{R}^d))$, we get

$$U_0(s-t)U(t,s)\phi=W_-(s)^{-1}\phi-i\int_{-\infty}^t U_0(s-\tau)V(\tau)U(\tau,s)\phi\, d\tau.$$

Take $\psi=W_-(s)^{-1}\phi$ in order to get

$$U(t,s)W_-(s)\psi=U_0(t-s)\psi-i\int_{-\infty}^t U_0(t-\tau)V(\tau)U(\tau,s)W_-(s)\psi\, d\tau.$$

Using the notation from Section 3, this equation can be written

$$\Gamma(s)W_-(s)\psi=\Gamma_0(s)\psi-iG^0_- V\Gamma(s)W_-(s)\psi$$

or, using Lemma 3.8 and (3.13),

$$\Gamma_0(s)\psi=(1+iG^0_- V)\Gamma(s)W_-(s)\psi=(1-iG_- V)^{-1}\mathsf{W}_-\Gamma_0(s)\psi.$$

The result in the case $P=B$ now follows from Lemma 4.3. The general case follows from a density argument. The result for S follows from this result and the definition.

The usual scattering operator then has the following representation formula.

Theorem 4.10. *Let V satisfy Assumptions* 2.1 *and* 3.3. *Let $s\in\mathbf{R}$. Then*

$$W_+(s)^{-1}W_-(s)=1-i\Gamma_0(s)^*V(1+iG_-^0V)^{-1}\Gamma_0(s) \tag{4.6}$$

on $L^2(\mathbf{R}^d)$.

Proof. Using Lemma 3.8 and (3.7) we get for any $s\in\mathbf{R}$,

$$\begin{aligned}(1+iG_+^0V)(1-iG_-V)&=(1+iG_-^0V+i(G_+^0-G_-^0)V)(1-iG_-V)\\&=1-i\Gamma_0(s)\Gamma_0(s)^*V(1-iG_-V).\end{aligned}$$

Thus if $u_\pm=\Gamma_0(s)\phi_\pm$, $u_+=\mathsf{S}u_-$ (cf. Lemma 4.3), for some fixed $s\in\mathbf{R}$, then

$$\begin{aligned}u_+=\Gamma_0(s)\phi_+&=(1-i\Gamma_0(s)\Gamma_0(s)^*V(1-iG_-V))\Gamma_0(s)\phi_-\\&=\Gamma_0(s)(1-i\Gamma_0(s)^*V(1-iG_-V)\Gamma_0(s))\phi_-.\end{aligned}$$

Since $\Gamma_0(s)$ is injective, we conclude the result (4.6).

Remark 4.11. The expression (4.6) is the closed form of the well-known perturbation expansion from physics textbooks, see e.g. [11, p. 184]. Explicitly, with the usual notation, we have a formal expansion

$$\begin{aligned}W_+(s)^{-1}W_-(s)=1-i&\int_{-\infty}^{\infty}e^{i(t-s)H_0}V(t)e^{-i(t-s)H_0}\,dt\\&+i^2\int_{-\infty}^{\infty}e^{i(t-s)H_0}V(t)\int_{-\infty}^{t}e^{-i(t-\tau)H_0}V(\tau)e^{-i(\tau-s)H_0}\,d\tau\,dt+\dots,\end{aligned}$$

which converges, if $\|V\|_{\mathcal{B}(L(P),L(Q))}$ is sufficiently small.

Remark 4.12. By Theorem 2.2 the operator $W_+(s)^{-1}W_-(s)$ is bounded on any $L^p(\mathbf{R}^d)$, whereas the mapping properties of the individual operators in the right-hand side of (4.6) established in Theorem 3.2 and in [8] are insufficient to draw the same conclusion for the composite operators. However, the expression in (4.6) is densely defined on the relevant $L^p(\mathbf{R}^d)$.

Remark 4.13. In some cases a stronger statement than the free wave asymptotic statement in Definition 4.6 holds. Let $\phi_-\in L^2(\mathbf{R}^d)\cap L^q(\mathbf{R}^d)$, $1/q=x(P)$, and let

$$\begin{aligned}u_-&=\Gamma_0(0)\phi_-,\\u&=\Gamma(0)W_-(0)\phi_-.\end{aligned}$$

Note that $u=\mathsf{W}_-u_-$. We have, using the various definitions from Section 3,

$$U_0(-t)(u_--u)=\phi_--U_0(-t)U(t,0)W_-(0)\phi_-.$$

Then

$$\|U_0(-t)(u_- - u)\|_{L^q} \le \|1 - U_0(-t)U(t,0)W_-(0)\|_{\mathcal{B}(L^q)} \|\phi_-\|_{L^q} \to 0, \quad \text{as } t \to -\infty,$$

by [5, Theorem 1.2]. The L^q-norm is stronger than the Σ^*-norm by the conditions on P and the Sobolev embedding theorem. Thus this result is stronger than the free wave asymptotic statement.

Let us briefly look at an application of the representation (4.6) for the scattering operator. To simplify the discussion we take $s=0$. We have

$$S(0) = 1 - i\Gamma_0(0)^* V \Gamma_0(0) - \Gamma_0(0)^* V G_-^0 V (1 + iG_-^0 V)^{-1} \Gamma_0(0).$$

The second term has in the Fourier space (momentum space) an integral kernel which can be written as

$$K(\xi,\eta) = i \int_{-\infty}^{\infty} e^{it(\xi^2 - \eta^2)} \widehat{V}(t, \xi - \eta)\, dt. \tag{4.7}$$

If we assume

$$\left(\frac{d}{dt}\right)^j \widehat{V}(t) \in L^1(\mathbf{R}; L^1(\mathbf{R}^d)), \quad j = 0, \dots, N,$$

integration by parts yields an estimate

$$\|(1 - \chi_{R+\varrho}) K \chi_R\| \le C_N \varrho^{-N}, \quad \varrho > 1.$$

Here χ_R denotes multiplication by the characteristic function for the ball $\{|\xi|^2 < R\}$ in Fourier space. Thus we have good momentum localization properties for the scattering operator $S(0)$ for this class of potentials. With additional effort explicit error estimates are obtainable. This result is to be compared with the case of time-independent V, where $S(0)$ commutes with H_0, and therefore is decomposable in the spectral representation of H_0 into a direct integral of scattering matrices. Note that the expression (4.7) formally becomes the usual Born term for a V independent of t.

5. Non-linear scattering for small data

In this section we briefly discuss some results where a nonlinear term is added. These results are minor extensions of those obtained in [8]. We consider nonlinearities given by a function $F: \mathbf{C} \to \mathbf{C}$. The following assumptions are needed.

Assumption 5.1. *The function F is continuously differentiable in the real sense and satisfies $F(0)=0$ and*

$$|F'(\zeta)|=|\partial_x F(\zeta)|+|\partial_y F(\zeta)|\le M|\zeta|^{k-1},\quad k>1.$$

Under this assumption we have

$$|F(\zeta)|\le M|\zeta|^k. \tag{5.1}$$

Assumption 5.2. *We assume that*

$$\frac{d+2+\sqrt{d^2+12d+4}}{2d}<k<1+\frac{4}{d-2}.$$

(For $d=1,2$ the right-hand side is $+\infty$).

To apply the mapping results we need to be able to pick $P\in T$ and $Q\in T'$ with $Q=kP$. The conditions in Assumption 5.2 imply that this is possible, see [8]. We also need the conditions in Assumption 3.3. Thus we introduce the following assumption.

Assumption 5.3. *Let $P\in T$, $Q\in T'$, and $R\in\square$, and let $k>1$, such that k satisfies Assumption 5.2. Furthermore, we assume $R=(k-1)P$, and V a real-valued function, $V\in L(R)$.*

Under these assumptions the map $u\mapsto F(u)$ is bounded from $L(P)$ to $L(Q)$. Thus $G_\pm F\in\mathcal{B}(L(P))$.

Lemma 5.4. *The maps $1+iG_\pm F$ are injective on $L(P)$.*

Proof. The argument in [8] can be used unchanged, due to Theorem 3.1.

Theorem 5.5. *Let V, F, and P satisfy Assumptions 5.2 and 5.3. Assume that $u\in L(P)$ solves*

$$i\partial_t u-H(t)u-F(u)=0 \tag{5.2}$$

in the weak sense. Then there exist $u_\pm\in\mathcal{L}_0(P)$ such that $u\sim u_\pm$ at $\pm\infty$. We have

$$u_\pm=(1+iG_\pm^0 V)(u+iG_\pm F(u)). \tag{5.3}$$

The maps $u\mapsto u_-$ and $u\mapsto u_+$ are injective and uniformly continuous on bounded sets of solutions.

Proof. These results follow easily from the results already proved. We note the following consequence of Lemma 3.7:

$$(1+iG_-^0 V)(1+iG_-F)=1+iG_-^0 V+i(iG_-^0 VG_-+G_-)F=1+iG_-^0(V+F). \tag{5.4}$$

This equation allows us to move freely between the solution sets.

Theorem 5.6. *There exist balls (all centered at zero) $B_\pm \subset \mathcal{L}_0(P)$, $B \subset L(P)$, such that the following results hold:*

(i) *given $u_- \in B_-$, then there exists a global solution (in the weak sense) $u \in B$ to (5.2), such that $u \sim u_-$ at $-\infty$; furthermore, there is $u_+ \in B_+$, such that $u \sim u_+$ at $+\infty$;*

(ii) *the map $\mathsf{S}: u_- \mapsto u_+$ is well-defined, continuous, and injective from B_- to $\mathcal{L}_0(P)$;*

(iii) $\mathsf{S}(B_-) \supseteq B_+$;

(iv) $u, u_\pm \in C_\infty(\mathbf{R}; \Sigma^*)$.

Remark 5.7. We can also find open sets $\mathcal{O}_\pm \subset \mathcal{L}_0(P)$, $0 \in \mathcal{O}_\pm$, such that S is an isomorphism of $\mathcal{O}_-$ onto $\mathcal{O}_+$.

Proof. Let $\Phi_{v_-}(u) = v_- - iG_- F(u)$, where $v_- \in L(P)$ is fixed. There exists a $\delta > 0$ such that for each $v_- \in \widetilde{B}_- = \{v_- \in L(P) \,|\, \|v_-\|_P \leq \frac{1}{2}\delta\}$ the equation $\Phi_{v_-}(u) = u$ has a unique solution $u \in B = \{u \in L(P) \,|\, \|u\|_P \leq \delta\}$. Furthermore, u depends continuously on $v_- \in \widetilde{B}_-$. By continuity, we can find a ball $B_- \subset \mathcal{L}_0(P)$ (centered at 0) such that

$$(1 + iG_-^0 V)^{-1} B_- \subseteq \widetilde{B}_-.$$

Let $u_- \in B_-$ and denote by u the fixed point above with the choice $v_- = (1 + iG_-^0 V)u_-$. Then clearly u solves (5.2) in the weak sense, and $u \sim u_-$ at $-\infty$, by the argument in the proof of Theorem 5.5. We find u_+ from Theorem 5.5. This argument can be reversed, and, after adjusting the radii of the balls, we find B_- and B_+. The remaining results follow easily.

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