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Multipliers from H^1 to L^p

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1. Introduction

We shall prove the following theorem combining the multiplier theorem of Hörmander–Mikhlin [5] for Hardy spaces with a result of Johnson, Corollary 2.1 in [6]. The original version of Hörmander–Mikhlin's multiplier theorem states that under the assumptions of Theorem 1 below with $q=1$ the operator is of weak-type $(1,1)$. The H^1 case was proved in [4] about a decade later by Fefferman and Stein. To simplify we will use the notation $\Delta_j=\{2^{j-1}\leq|\xi|\leq 2^{j+1}\}$.

Theorem 1. *Assume that* $m\in\mathcal{C}^k(\mathbf{R}^n\setminus\{0\})$ *and*

$$\int_{\Delta_j}\sum_{|\alpha|\leq k}|2^{j|\alpha|}D_\alpha m(\xi)|^2\,d\xi\leq 2^{nj(2-q)/q},\quad j\in\mathbf{Z},$$

where k *is the least integer* $>n(2-q)/2q$, *and* $1\leq q\leq 2$, *then the convolution by* $K=\widehat{m}$ *maps* H^1 *to* L^q.

As noted above the case $q=1$ is the theorem of Hörmander–Mikhlin, and the case $q=2$ gives us a weakened version of Johnson's result. The actual theorem of Johnson states that if

$$\int_{\Delta_j}|m(\xi)|^2\,d\xi\leq C,$$

then convolution with $\widehat{m}$ maps H^1 to L^2. However, one can easily see that our proof goes through in that case even if we drop the derivatives in the assumptions. Johnson also inverts the result, i.e. he proves that every convolution operator mapping H^1 to L^2 is of this type. One should also observe that if the kernel K has compact support then the operator maps H^1 to itself. Finally, if we take $m(\xi)=|\xi|^{-a}$ then we obtain the result of the Hardy–Littlewood–Sobolev theorem, i.e. the operator maps H^1 to L^q if $q=n/(n-a)$.

Related results could be found in Bagby [1] and de Michele–Inglis [3].

2. Proof

The proof interpolates the methods in Hörmander [5] and Björk [2].

Proof. It suffices to prove that the L^q norm is bounded for atoms (with a bound that is independent of the atom). Let a be an atom with support in a ball with radius r. This means that a also satisfies the following conditions

$$\int a(x)\,dx=0, \quad \|a\|_{L^\infty}\le r^{-n}.$$

As the operator is translation invariant we can assume that the atoms are centered at the origin. Since $2/(2-q)$ is the dual index to $2/q$ it is easy to see that

$$\|a*K\|_{L^q}\le\left(\int_{|x|>2r}|K*a(x)|^q\,dx\right)^{1/q}+r^{n(2-q)/2q}\|a*K\|_{L^2}. \tag{1}$$

We decompose $m=\sum m_j$ by setting $m_j(\xi)=m(\xi)\phi(2^{-j}\xi)$ where $\phi\in C_0^\infty$ has support in the set $\frac{1}{2}<|\xi|<2$ and for $\xi\neq 0$

$$\sum_{j=-\infty}^{\infty}\phi(2^{-j}\xi)=1.$$

By Leibniz' formula we obtain

$$\int\sum_{|\alpha|\le k}|2^{j|\alpha|}D_\alpha m_j|^2\,d\xi\le C2^{nj(2-q)/q}.$$

Using Parseval's formula and putting $K_j=\widehat{m}_j$ it follows that

$$\int(1+2^{2j}|x|^2)^k|K_j|^2\,dx\le C2^{nj(2-q)/q}. \tag{2}$$

Thus

$$\|K_j\|_{L^q}\le C2^{nj(2-q)/2q}\left(\int\frac{dx}{(1+2^{2j}|x|^2)^{k(q/(2-q))}}\right)^{(2-q)/2q}=C'. \tag{3}$$

In the same way (2) implies that

$$\left(\int_{|x|\ge t}|K_j|^q\,dx\right)^{1/q}\le C(2^jt)^{n(2-q)/2q-k},$$

which shows

$$\left(\int_{|x|\geq 2t} |K_j(x-y)-K_j(x)|^q\,dx\right)^{1/q} \leq C(2^j t)^{n(2-q)/2q-k},$$

under the assumption that $|y|\leq t$. Together with the following estimate, easily proved by S. Bernstein's inequality (which says in particular that the L^q-norm of the first derivative of a C^∞-function could be estimated by the radius of a ball, centered at the origin and containing the support of the Fourier transform of the function, times the L^q-norm of the function itself),

$$\left(\int |K_j(x-y)-K_j(x)|^q\,dx\right)^{1/q} \leq C2^{j+1}t \quad \text{for } |y|\leq t,$$

this proves that if $|y|\leq t$

$$\left(\int_{|x|\geq 2t} |K(x-y)-K(x)|^q\,dx\right)^{1/q} \leq C\sum_{j=-\infty}^{\infty} \min(2^j t, (2^j t)^{n(2-q)/2q-k}) \leq C.$$

Since a is an atom it is easy to see that this proves the required estimate for the first term in (1). It remains to show that

$$\|K*a\|_{L^2} \leq Cr^{-n(2-q)/2q}.$$

Since the m_j's have bounded overlap it follows that

$$\begin{aligned}
\|K*a\|_{L^2}^2 &\leq C\sum_{j=-\infty}^{\infty} \int |m_j(\xi)\hat{a}(\xi)|^2\,d\xi \\
&\leq C\sum_{j=-\infty}^{\infty} \|m_j\|_{L^{q'}}^2 \left(\int_{\Delta_j} |\hat{a}(\xi)|^{2q/(2-q)}\,d\xi\right)^{(2-q)/q} \\
&\leq \sum_{j=-\infty}^{\infty} C\left(\int_{\Delta_j} |\hat{a}(\xi)|^{2q/(2-q)}\,d\xi\right)^{(2-q)/q},
\end{aligned}$$

using (3) for the last inequality. Thus we have reduced to showing that

$$J := \sum_{j=-\infty}^{\infty} \left(\int_{\Delta_j} |\hat{a}(\xi)|^{2q/(2-q)}\,d\xi\right)^{(2-q)/q} \leq Cr^{-n(2-q)/q}.$$

We start with the case $q=1$. We get

$$J \leq C\|a\|_{L^2}^2 \leq Cr^{-n}.$$

In the case $q=2$ we want to estimate

$$\sum_{j=-\infty}^{\infty} \sup_{\Delta_j} |\hat{a}(\xi)|^2$$

by a constant. For this we divide the sum into two pieces

$$J_1 := \sum_{2^j \geq r^{-1}} \sup_{\Delta_j} |\hat{a}(\xi)|^2,$$
$$J_2 := \sum_{2^j < r^{-1}} \sup_{\Delta_j} |\hat{a}(\xi)|^2.$$

To estimate J_1 we introduce a function $\psi \in \mathcal{C}_0^\infty$ which is assumed to be 1 on the support of a and $\|\psi\|_{L^2} \leq Cr^{n/2}$. We may assume that $\psi(x) = \psi_0(x/r)$, where $\psi_0 \in \mathcal{C}_0^\infty$ is independent of r. We observe that

$$\hat{a}(\xi) = \hat{a} * \hat{\psi}(\xi).$$

Thus we have to show that

$$\sum_{2^j \geq r^{-1}} \sup_{\Delta_j} \left| \int \hat{a}(\eta)\hat{\psi}(\xi-\eta)\, d\eta \right|^2 \leq C. \tag{4}$$

Let $\xi \in \Delta_j$. If $\eta \notin \Delta_j^* = \Delta_{j-1} \cup \Delta_j \cup \Delta_{j+1}$ it follows that $|\xi-\eta| \geq 2^{j-2}$. Hence

$$\begin{aligned}\int_{\mathbf{R}^n \setminus \Delta_j^*} |\hat{\psi}(\xi-\eta)|^2\, d\eta &\leq \int_{|\eta|>2^{j-2}} |\hat{\psi}(\eta)|^2\, d\eta = r^{2n} \int_{|\eta|>2^{j-2}} |\hat{\psi}_0(r\eta)|^2\, d\eta \\ &\leq Cr^{-2} \int_{|\eta|>2^{j-2}} \eta^{-2n-2}\, d\eta \leq C' r^{-2} 2^{-j(n+2)}.\end{aligned}$$

By Schwarz' inequality the part of (4) where $\eta \notin \Delta_j^*$ is bounded, because $n+2>0$ and $\|a\|_{L^2} \leq r^{-n/2}$. We also have

$$\sum_{2^j \geq r^{-1}} \sup_{\Delta_j} \left| \int_{\Delta_j^*} \hat{a}(\eta)\hat{\psi}(\xi-\eta)\, d\eta \right|^2 \leq \sum_{2^j \geq r^{-1}} \int_{\Delta_j^*} |\hat{a}(\xi)|^2\, d\xi \|\psi\|_{L^2}^2 \leq C\|a\|_{L^2}^2 \|\psi\|_{L^2}^2,$$

since $\|a\|_{L^2} \le r^{-n/2}$ and $\|\psi\|_{L^2} \le r^{n/2}$ this finishes the estimate for J_1.

For J_2 we use the fact that $|\hat{a}(\xi)| \le r|\xi|$, from which it follows that

$$J_2 = C \sum_{2^j < r^{-1}} \sup_{\Delta_j} |\hat{a}(\xi)|^2 \le Cr^2 \sum_{2^j < r^{-1}} \sup_{\Delta_j} |\xi|^2 \le C.$$

The J_1 and J_2 estimates together provide the required estimate in the case $q=2$.

We shall now interpolate to get the general case. Let $p=2/(1-\theta)$ where $0 \le \theta \le 1$, let $c_j = \|\hat{a}\chi_j\|_{L^\infty}$ and $d_j = \|\hat{a}\chi_j\|_{L^2}$, where χ_j is the characteristic function for Δ_j. By interpolation we know that

$$\|\hat{a}\chi_j\|_{L^p} \le c_j^\theta d_j^{1-\theta}.$$

The $q=1$ estimate shows that $\|d_j\|_{l^2} \le Cr^{-n/2}$ and the $q=2$ estimate that $\|c_j\|_{l^2} \le C$. This implies that

$$\begin{aligned} \big\| \|\hat{a}\chi_j\|_{L^p} \big\|_{l^2} &\le \|c_j^\theta d_j^{1-\theta}\|_{l^2} \le \|c_j^{2\theta} d_j^{2-2\theta}\|_{l^1}^{1/2} \\ &\le \|c_j\|_{l^2}^\theta \|d_j\|_{l^2}^{1-\theta} \le C^\theta (Cr^{-n/2})^{1-\theta} = Cr^{-n/p}. \end{aligned}$$

So when we put $p=2q/(2-q)$ the proof is complete. $\square$

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