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Higher order commutators for a class of rough operators

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Abstract. In this paper we study the $(L^p(u^p), L^q(v^q))$ boundedness of the higher order commutators $T^m_{\Omega,\alpha,b}$ and $M^m_{\Omega,\alpha,b}$ formed by the fractional integral operator $T_{\Omega,\alpha}$, the fractional maximal operator $M_{\Omega,\alpha}$, and a function $b(x)$ in $\mathrm{BMO}(\nu)$, respectively.

Our results improve and extend the corresponding results obtained by Segovia and Torrea in 1993 [9].

1. Introduction

Suppose that $0<\alpha<n$, $\Omega(x)$ is homogeneous of degree zero on $\mathbf{R}^n$ and $\Omega(x')\in L^s(S^{n-1})$ $(s>1)$, where S^{n-1} denotes the unit sphere in $\mathbf{R}^n$. Then the fractional integral operator $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy,$$

and the fractional maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)|\,|f(y)|\,dy.$$

In 1971, Muckenhoupt and Wheeden [8] gave (L^p,L^q) boundedness with power weight for the rough fractional integral operator $T_{\Omega,\alpha}$. This is an extension of the Hardy–Littlewood–Sobolev theorem. For general $A(p,q)$ weights, we gave the weighted (L^p,L^q) boundedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ in [5]. In 1993, Chanillo, Watson and Wheeden [1] proved that when $s\geq n/(n-\alpha)$, the operator $T_{\Omega,\alpha}$ is of weak type $(1,n/(n-\alpha))$. Recently, weak type inequalities with power weights for $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ have been obtained by one of the authors [4].

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Before stating our results, let us give some definitions. For ν a nonnegative locally integrable function on $\mathbf{R}^n$, a function $b(x)$ is said to belong to $\mathrm{BMO}(\nu)$, if there is a constant $C>0$ such that for any cube Q in $\mathbf{R}^n$ with its sides parallel to the coordinate axes

$$\int_Q |b(x)-b_Q|\,dx \le C\int_Q \nu(x)\,dx,$$

where $b_Q=(1/|Q|)\int_Q b(x)\,dx$.

Let Ω be homogeneous of degree zero on $\mathbf{R}^n$ and satisfy $\int_{S^{n-1}}\Omega(x')\,dx'=0$. Then the integral modulus of continuity of order s ($s\ge 1$) of Ω is defined by

$$\omega(t)=\sup_{|\varrho|<t}\left(\int_{S^{n-1}}|\Omega(\varrho x')-\Omega(x')|^s\,dx'\right)^{1/s},$$

where ϱ is a rotation in $\mathbf{R}^n$ and $|\varrho|=\|\varrho-I\|$.

A nonnegative locally integrable function $u(x)$ on $\mathbf{R}^n$ is said to belong to $A(p,q)$ ($1<p,q<\infty$), if there is a constant $C>0$ such that

$$\sup_Q\left(\frac{1}{|Q|}\int_Q u(x)^q\,dx\right)^{1/q}\left(\frac{1}{|Q|}\int_Q u(x)^{-p'}\,dx\right)^{1/p'}\le C<\infty,$$

where $p'=p/(p-1)$.

In this paper we shall follow the idea developed in [5] to consider the weighted (L^p,L^q) boundedness for a class of higher order commutators formed by $T_{\Omega,\alpha}$, $M_{\Omega,\alpha}$ and $\mathrm{BMO}(\nu)$ function $b(x)$ which are defined as

$$T^m_{\Omega,\alpha,b}f(x)=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x)-b(y)]^m f(y)\,dy \tag{1.1}$$

and

$$M^m_{\Omega,\alpha,b}f(x)=\sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy. \tag{1.2}$$

In 1993, Segovia and Torrea [9] gave the weighted boundedness of higher order commutators for vector-valued integral operators with a pair of weights using the Rubio de Francia extrapolation idea for weighted norm inequalities. As an application of this result, they obtained $(L^p(u^p),L^q(v^q))$ boundedness of $T^m_{\Omega,\alpha,b}$ and $M^m_{1,\alpha,b}$, where Ω satisfies some smoothness condition.

Theorem A. ([9]) *Suppose that* $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, Ω *is a homogeneous function of degree zero defined on* $\mathbf{R}^n$, *and* $\int_{S^{n-1}}\Omega(x')\,dx'=0$. *For* $m\in\mathbf{Z}_+$, *if the integral modulus of continuity of order* s $(s>1)$ *of* Ω *satisfies*

$$\int_0^1 \log^m(1/t)\omega(t)\,\frac{dt}{t}<\infty, \tag{1.3}$$

then for $b\in\mathrm{BMO}(\nu)$, $u(x)^{s'},v(x)^{s'}\in A(p/s',q/s')$ *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $T^m_{\Omega,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}|T^m_{\Omega,\alpha,b}f(x)v(x)|^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

Theorem B. ([9]) *Suppose that* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$. *Then for* $b\in\mathrm{BMO}(\nu)$, $u(x),v(x)\in A(p,q)$ *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $M^m_{1,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}[M^m_{1,\alpha,b}f(x)v(x)]^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

In this paper we shall prove the following results.

Theorem 1. *Suppose that* $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, Ω *is homogeneous of degree zero defined on* $\mathbf{R}^n$ *and* $\Omega\in L^s(S^{n-1})$, *then for functions* $b\in\mathrm{BMO}(\nu)$, $u(x)^{s'},v(x)^{s'}\in A(p/s',q/s')$ *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $T^m_{\Omega,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}|T^m_{\Omega,\alpha,b}f(x)v(x)|^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

Theorem 2. *Suppose that* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $s>q$. *If* Ω *is homogeneous of degree zero defined on* $\mathbf{R}^n$ *and* $\Omega\in L^s(S^{n-1})$, *then for functions* $b\in\mathrm{BMO}(\nu)$, $u(x)^{-s'},v(x)^{-s'}\in A(q'/s',p'/s')$, *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $T^m_{\Omega,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}|T^m_{\Omega,\alpha,b}f(x)v(x)|^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

On the higher order commutator $M^m_{\Omega,\alpha,b}$ of the fractional maximal operator $M_{\Omega,\alpha}$ we have the following results.

Theorem 3. *Suppose that* $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$, Ω *is homogeneous of degree zero defined on* $\mathbf{R}^n$ *and* $\Omega\in L^s(S^{n-1})$, *then for functions* $b\in \mathrm{BMO}(\nu)$, $u(x)^{s'}, v(x)^{s'}\in A(p/s', q/s')$ *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $M^m_{\Omega,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}[M^m_{\Omega,\alpha,b}f(x)v(x)]^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

Theorem 4. *Suppose that* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, $s>q$. *If* Ω *is homogeneous of degree zero defined on* $\mathbf{R}^n$ *and* $\Omega\in L^s(S^{n-1})$, *then for functions* $b\in \mathrm{BMO}(\nu)$, $u(x)^{-s'}, v(x)^{-s'}\in A(q'/s', p'/s')$, *and* $u(x)v(x)^{-1}=\nu^m$, *there is a constant* C, *independent of* f, *such that* $M^m_{\Omega,\alpha,b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}[M^m_{\Omega,\alpha,b}f(x)v(x)]^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}.$$

Remark 1. By comparing the results in this paper with the results in [9], we see that the cancellation condition and smoothness condition (1.3) of Ω in Theorem A have been removed in Theorem 1. Moreover, the theorems in this paper are also extensions of Theorem A and B.

Remark 2. In [2] and [3], we gave the weighted boundedness of $T^m_{\Omega,\alpha,b}$ and $M^m_{\Omega,\alpha,b}$ for one weight function, respectively. The theorems in this paper are also extensions of results in [2] and [3].

2. Proof of the theorems

Let us recall the definitions of A_p $(1\le p<\infty)$ weights and some elementary properties of A_p weights and $A(p,q)$ weights. A nonnegative locally integrable function $w(x)$ on $\mathbf{R}^n$ is said to belong to A_p $(1<p<\infty)$, if there is a constant $C>0$ such that for any cube Q,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1}\le C<\infty.$$

Using the elementary properties of A_p weights [6], we can prove that if $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, then we have

$$\begin{aligned}(2.1)\qquad u(x)\in A(p,q) \iff u(x)^{-p'}\in A_{1+p'/q} &\iff u(x)^q\in A_{1+q/p'}\\ &\iff u(x)^q\in A_{q(n-\alpha)/n}.\end{aligned}$$

Lemma 1. *Suppose that* $0<\alpha<n$, $s'>1$, $1<p/s'<n/\alpha$, $1/(q/s')=1/(p/s')-\alpha/n$. *Then for* $b\in\mathrm{BMO}(\nu)$, $u(x)^{s'}, v(x)^{s'}\in A(p/s',q/s')$ *and* $u(x)v(x)^{-1}=\nu^m$, *there is a* C, *independent of* f, *such that the commutator* $N^{ms'}_{\alpha,s',b}$ *satisfies*

$$\left(\int_{\mathbf{R}^n}[N^{ms'}_{\alpha,s',b}f(x)v(x)]^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)u(x)|^p\,dx\right)^{1/p}, \tag{2.2}$$

where $N^{ms'}_{\alpha,s',b}$ *is the commutator for the fractional maximal operator of order* s' *defined by*

$$N^{ms'}_{\alpha,s',b}f(x)=\sup_{r>0}\left(\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|b(x)-b(y)|^{ms'}|f(y)|^{s'}\,dy\right)^{1/s'}.$$

Proof. Clearly, $N^{ms'}_{\alpha,s',b}f(x)=(M^{ms'}_{1,\alpha,b}(|f|^{s'})(x))^{1/s'}$, and we have

$$\begin{aligned}\left(\int_{\mathbf{R}^n}[N^{ms'}_{\alpha,s',b}f(x)v(x)]^q\,dx\right)^{1/q}&=\left(\int_{\mathbf{R}^n}[M^{ms'}_{1,\alpha,b}(|f|^{s'}(x))]^{q/s'}v(x)^q\,dx\right)^{1/q}\\&=\left[\left(\int_{\mathbf{R}^n}[M^{ms'}_{1,\alpha,b}(|f|^{s'}(x))v(x)^{s'}]^{q/s'}\,dx\right)^{s'/q}\right]^{1/s'}.\end{aligned}$$

Since $uv^{-1}=\nu^m$, we get $(u^{s'})(v^{s'})^{-1}=\nu^{ms'}$. By Theorem B,

$$\begin{aligned}\left(\int_{\mathbf{R}^n}[M^{ms'}_{1,\alpha,b}(|f|^{s'}(x))v(x)^{s'}]^{q/s'}\,dx\right)^{s'/q}&\le C\left(\int_{\mathbf{R}^n}[|f(x)|^{s'}u(x)^{s'}]^{p/s'}\,dx\right)^{s'/p}\\&=C\left(\int_{\mathbf{R}^n}|f(x)|^pu(x)^p\,dx\right)^{s'/p}.\end{aligned}$$

Thus,

$$\left(\int_{\mathbf{R}^n}[N^{ms'}_{\alpha,s',b}f(x)v(x)]^q\,dx\right)^{1/q}\le C\left(\int_{\mathbf{R}^n}|f(x)|^pu(x)^p\,dx\right)^{1/p}.$$

This is (2.2).

Let us first give the proof of Theorem 3. By the conditions of Theorem 3, we know that for $r>0$,

$$\left(\int_{|x-y|<r}|\Omega(x-y)|^s\,dy\right)^{1/s}\le Cr^{n/s}\|\Omega\|_{L^s(S^{n-1})}.$$

Hence

$$\begin{aligned} M^m_{\Omega,\alpha,b}f(x) &= \sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy \\ &\le \sup_{r>0}\frac{1}{r^{n-\alpha}}\left(\int_{|x-y|<r}|\Omega(x-y)|^s\,dy\right)^{1/s} \\ &\quad\times\left(\int_{|x-y|<r}|b(x)-b(y)|^{ms'}|f(y)|^{s'}\,dy\right)^{1/s'} \\ &\le C\sup_{r>0}\frac{1}{r^{n-\alpha}}r^{n/s}\left(\int_{|x-y|<r}|b(x)-b(y)|^{ms'}|f(y)|^{s'}\,dy\right)^{1/s'} \\ &= C\sup_{r>0}\left(\frac{1}{r^{n-\alpha s'}}\int_{|x-y|<r}|b(x)-b(y)|^{ms'}|f(y)|^{s'}\,dy\right)^{1/s'} \\ &= CN^{ms'}_{\alpha s',s',b}f(x). \end{aligned}$$

From $1<s'<p<n/\alpha$ and $1/q=1/p-\alpha/n$, we have $0<\alpha s'<n$, $1<p/s'<n/\alpha s'$ and $1/(q/s')=1/(p/s')-\alpha s'/n$. Thus, by Lemma 1 we get

$$\begin{aligned} \left(\int_{\mathbf{R}^n}[M^m_{\Omega,\alpha,b}f(x)v(x)]^q\,dx\right)^{1/q} &\le C\left(\int_{\mathbf{R}^n}[N^{ms'}_{\alpha s',s',b}f(x)v(x)]^q\,dx\right)^{1/q} \\ &\le C\left(\int_{\mathbf{R}^n}|f(x)|^p u(x)^p\,dx\right)^{1/p}. \end{aligned}$$

The result of Theorem 3 is proved.

The proof of Theorem 1 is based on the following lemmas. Let us first give a pointwise relation between $T^m_{\Omega,\alpha,b}$ and $M^m_{\Omega,\alpha,b}$.

Lemma 2. *For any $\varepsilon>0$ with $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, we have*

$$|T^m_{\Omega,\alpha,b}f(x)|\le C[M^m_{\Omega,\alpha+\varepsilon,b}f(x)]^{1/2}[M^m_{\Omega,\alpha-\varepsilon,b}f(x)]^{1/2},\quad x\in\mathbf{R}^n, \tag{2.3}$$

where C depends only on α, ε, n.

Proof. The idea of the proof will be taken from [10]. However, it is worth pointing out that the important technique used here was suggested first by Hedberg in [7]. For $x\in\mathbf{R}^n$ and $\varepsilon>0$ with $0<\alpha-\varepsilon<\alpha+\varepsilon<n$, we choose a $\delta>0$ such that

$$\delta^{2\varepsilon}=M^m_{\Omega,\alpha+\varepsilon,b}f(x)/M^m_{\Omega,\alpha-\varepsilon,b}f(x).$$

Write

$$
\begin{aligned}
T^m_{\Omega,\alpha,b}f(x) &= \int_{|x-y|<\delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x)-b(y)]^m f(y)\,dy \\
&\quad + \int_{|x-y|\geq\delta} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x)-b(y)]^m f(y)\,dy \\
&:= I_1 + I_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
|I_1| &\leq \sum_{j=0}^{\infty} \int_{2^{-j-1}\delta\leq|x-y|<2^{-j}\delta} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(x)-b(y)|^m|f(y)|\,dy \\
&\leq \sum_{j=0}^{\infty}(2^{-j-1}\delta)^{-(n-\alpha)} \int_{|x-y|<2^{-j}\delta} |\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy \\
&= 2^{n-\alpha}\sum_{j=0}^{\infty}(2^{-j}\delta)^{\varepsilon}\frac{1}{(2^{-j}\delta)^{n-\alpha+\varepsilon}} \int_{|x-y|<2^{-j}\delta} |\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy \\
&\leq C\delta^{\varepsilon} M^m_{\Omega,\alpha-\varepsilon,b}f(x).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\leq \sum_{j=1}^{\infty} \int_{2^{j-1}\delta\leq|x-y|<2^{j}\delta} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(x)-b(y)|^m|f(y)|\,dy \\
&\leq C\sum_{j=1}^{\infty}(2^{j}\delta)^{-\varepsilon}\frac{1}{(2^{j}\delta)^{n-\alpha-\varepsilon}} \int_{|x-y|<2^{j}\delta} |\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy \\
&\leq C\delta^{-\varepsilon} M^m_{\Omega,\alpha+\varepsilon,b}f(x).
\end{aligned}
$$

Thus, by the above selection of δ we get

$$
\begin{aligned}
|T^m_{\Omega,\alpha,b}f(x)| &\leq C[\delta^{\varepsilon} M^m_{\Omega,\alpha-\varepsilon,b}f(x) + \delta^{-\varepsilon} M^m_{\Omega,\alpha+\varepsilon,b}f(x)] \\
&= C[M^m_{\Omega,\alpha+\varepsilon,b}f(x)]^{1/2}[M^m_{\Omega,\alpha-\varepsilon,b}f(x)]^{1/2},
\end{aligned}
$$

and the proof of Lemma 2 is complete.

The following two lemmas characterize an important property of $A(p,q)$ weights and they are also the key for proving Theorem 1.

Lemma 3. *Suppose that* $0<\alpha<n$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$ *and* $u(x),v(x)\in A(p,q)$. *Then there is an* $\varepsilon>0$ *such that*

$$\text{(i)}\qquad \varepsilon<\alpha<\alpha+\varepsilon<n,$$

$$\text{(ii)}\qquad 1/p>(\alpha+\varepsilon)/n,\quad 1/q<(n-\varepsilon)/n,$$

$u(x),v(x)\in A(p,q_\varepsilon)$ *and* $u(x),v(x)\in A(p,\bar{q}_\varepsilon)$, *where* $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$ *and* $1/\bar{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$.

Proof. For $\alpha>0$, $1/q<1$,we can take $\delta_1>0$ such that $\delta_1<\alpha$ and $1/q+\delta_1/n<1$. Let $1/q_{\delta_1}=1/p-(\alpha-\delta_1)/n=1/q+\delta_1/n$, then $q>q_{\delta_1}>1$ and $1+p'/q<1+p'/q_{\delta_1}$. By (2.1) and the inclusion relation between A_p weight classes, we have $u^{-p'},v^{-p'}\in A_{1+p'/q}\subset A_{1+p'/q_{\delta_1}}$, which is equivalent to

$$\text{(2.4)}\qquad u(x),v(x)\in A(p,q_{\delta_1}),$$

by (2.1).

On the other hand, there is an η with $0<\eta<1/q$, such that $u^{-p'}\in A_{1+p'(1/q-\eta)}$, by the reverse Hölder's inequality or A_p weights. Hence we can choose $\delta_2>0$ small enough such that $\delta_2<\min\{\alpha,n-\alpha\}$, $1/p>(\alpha+\delta_2)/n$ and $\delta_2/n<\eta$ hold at the same time. Now let $1/q_{\delta_2}=1/p-(\alpha+\delta_2)/n$, then since $1/p>(\alpha+\delta_2)/n$ and $\delta_2/n<\eta$ we get $0<1/q_{\delta_2}<1$ and $1/q_{\delta_2}=1/q-\delta_2/n>1/q-\eta$. From this we have $u^{-p'}\in A_{1+p'(1/q-\eta)}\subset A_{1+p'/q_{\delta_2}}$. By (2.1), this is equivalent to $u(x)\in A(p,q_{\delta_2})$. Obviously, given the same discussion for $v(x)$, we can also get a $\sigma_2>0$ (corresponding to $v(x)$), which possesses the conditions satisfied by δ_2 (corresponding to $u(x)$). Hence we have also $v(x)\in A(p,q_{\sigma_2})$. Let $\varepsilon_1=\min\{\delta_2,\sigma_2\}$, then we have

$$\text{(2.5)}\qquad u(x),v(x)\in A(p,q_{\varepsilon_1}).$$

Finally, let $\varepsilon=\min\{\delta_1,\varepsilon_1\}$ and $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$, then by (2.4) and (2.5) we get $u(x),v(x)\in A(p,q_\varepsilon)$ and $u(x),v(x)\in A(p,\bar{q}_\varepsilon)$.

Lemma 4. *Suppose that* $0<\alpha<n$, $1\le s'<p<n/\alpha$, $1/q=1/p-\alpha/n$ *and that* $u(x)^{s'},v(x)^{s'}\in A(p/s',q/s')$. *Then there is an* $\varepsilon>0$ *such that*

$$\text{(iii)}\qquad \varepsilon<\alpha<\alpha+\varepsilon<n,$$

$$\text{(iv)}\qquad 1/p>(\alpha+\varepsilon)/n,\quad 1/q<(n-\varepsilon)/n,$$

and $u(x)^{s'},v(x)^{s'}\in A(p/s',q_\varepsilon/s')$, $u(x)^{s'},v(x)^{s'}\in A(p/s',\bar{q}_\varepsilon/s')$ *hold at the same time, where* $1/q_\varepsilon=1/p-(\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon=1/p-(\alpha-\varepsilon)/n$.

Proof. As $1/(q/s')=1/(p/s')-\alpha s'/n$, by Lemma 3 there is an $\eta>0$ such that $\eta<\alpha s'<\alpha s'+\eta<n$, $1/(p/s')>(\alpha s'+\eta)/n$, $1/(q/s')<(n-\eta)/n$ and that $u(x)^{s'},v(x)^{s'}\in$

$A(p/s', q_\eta)$, $u(x)^{s'}, v(x)^{s'} \in A(p/s', \bar{q}_\eta)$ hold at the same time, where $1/q_\eta = 1/(p/s') - (\alpha s' + \eta)/n$, $1/\bar{q}_\eta = 1/(p/s') - (\alpha s' - \eta)/n$.

Now let $\varepsilon = \eta/s'$, $q_\varepsilon = s' q_\eta$ and $\bar{q}_\varepsilon = s' \bar{q}_\eta$, then it is easy to see that ε satisfies (iii), (iv) and $u(x)^{s'}, v(x)^{s'} \in A(p/s', q_\varepsilon/s')$, $u(x)^{s'}, v(x)^{s'} \in A(p/s', \bar{q}_\varepsilon/s')$ hold at the same time, where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon = 1/p - (\alpha - \varepsilon)/n$. This completes the proof of Lemma 4.

The proof of Theorem 1. Under the conditions of Theorem 1, by Lemma 4, there is an $\varepsilon > 0$ such that (iii) and (iv) hold, and

$$u(x)^{s'}, v(x)^{s'} \in A(p/s', q_\varepsilon/s') \quad \text{and} \quad u(x)^{s'}, v(x)^{s'} \in A(p/s', \bar{q}_\varepsilon/s')$$

hold at the same time, where $1/q_\varepsilon = 1/p - (\alpha+\varepsilon)/n$, $1/\bar{q}_\varepsilon = 1/p - (\alpha-\varepsilon)/n$. Let $l_1 = 2q_\varepsilon/q$, $l_2 = 2\bar{q}_\varepsilon/q$, then $1/l_1 + 1/l_2 = 1$. For the above given $\varepsilon > 0$, using Lemma 2 and Hölder's inequality, we have

$$\begin{aligned}
\|T^m_{\Omega,\alpha,b} f\|_{q,v^q} &\le C\left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha+\varepsilon,b} f(x) v(x)]^{q/2} [M^m_{\Omega,\alpha-\varepsilon,b} f(x) v(x)]^{q/2}\, dx\right)^{1/q} \\
&\le C\left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha+\varepsilon,b} f(x) v(x)]^{q l_1/2}\, dx\right)^{1/q l_1} \\
&\quad \times \left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha-\varepsilon,b} f(x) v(x)]^{q l_2/2}\, dx\right)^{1/q l_2} \\
&= C\left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha+\varepsilon,b} f(x) v(x)]^{q_\varepsilon}\, dx\right)^{1/2q_\varepsilon} \\
&\quad \times \left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha-\varepsilon,b} f(x) v(x)]^{\bar{q}_\varepsilon}\, dx\right)^{1/2\bar{q}_\varepsilon}.
\end{aligned}$$

From Lemma 4 and Theorem 3, it follows that

$$\left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha+\varepsilon,b} f(x) v(x)]^{q_\varepsilon}\, dx\right)^{1/2q_\varepsilon} \le C\|f\|^{1/2}_{p,u^p},$$

and

$$\left(\int_{\mathbf{R}^n} [M^m_{\Omega,\alpha-\varepsilon,b} f(x) v(x)]^{\bar{q}_\varepsilon}\, dx\right)^{1/2\bar{q}_\varepsilon} \le C\|f\|^{1/2}_{p,u^p}.$$

Thus, we get

$$\|T^m_{\Omega,\alpha,b} f\|_{q,v^q} \le C\|f\|_{p,u^p}.$$

This is the conclusion of Theorem 1.

Remark 3. If we define the commutator $\tilde{T}^m_{\Omega,\alpha,b}$ by

$$\tilde{T}^m_{\Omega,\alpha,b}f(x)=\int_{\mathbf{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}|b(x)-b(y)|^m f(y)\,dy,$$

then from the proof of Lemma 2, we know that (2.3) still holds if one has $\tilde{T}^m_{\Omega,\alpha,b}$ instead of $T^m_{\Omega,\alpha,b}$. Thus, under the conditions of Theorem 1 we have also

$$\|\tilde{T}^m_{\Omega,\alpha,b}f\|_{q,v^q}\le C\|f\|_{p,u^p}.$$

The proof of Theorem 2. From the definition we know that the commutator $T^m_{\Omega,\alpha,b}$ is a linear operator. Then we have $(T^m_{\Omega,\alpha,b})^*=T^m_{\Omega^*,\alpha,b}$, where $\Omega^*(x)=(-1)^m\overline{\Omega(-x)}$. Clearly, Ω^* satisfies the same conditions as Ω. We have

$$\|T^m_{\Omega,\alpha,b}f\|_{q,v^q}=\sup_g\left|\int_{\mathbf{R}^n}T^m_{\Omega,\alpha,b}f(x)g(x)\,dx\right|,$$

where the supremum is taken over all g with $\|g\|_{q',v^{-q'}}\le 1$. Since $(T^m_{\Omega,\alpha,b})^*$ is the adjoint operator of $T^m_{\Omega,\alpha,b}$,

$$\int_{\mathbf{R}^n}T^m_{\Omega,\alpha,b}f(x)g(x)\,dx=\int_{\mathbf{R}^n}f(x)(T^m_{\Omega,\alpha,b})^*g(x)\,dx.$$

Thus,

$$\|T^m_{\Omega,\alpha,b}f\|_{q,v^q}=\sup_g\left|\int_{\mathbf{R}^n}T^m_{\Omega,\alpha,b}f(x)g(x)\,dx\right|\le\|f\|_{p,u^p}\sup_g\|(T^m_{\Omega,\alpha,b})^*g\|_{p',u^{-p'}}.$$

From the conditions in Theorem 2, we see that $1/p'=1/q'-\alpha/n$ and $s'<q'<n/\alpha$. Since $(u^{-1})^{s'},(v^{-1})^{s'}\in A(q'/s',p'/s')$, and noticing that $(v^{-1})(u^{-1})^{-1}=uv^{-1}=\nu^m$, using the conclusion of Theorem 1, we get

$$\|(T^m_{\Omega,\alpha,b})^*g\|_{p',u^{-p'}}\le C\|g\|_{q',v^{-q'}}.$$

Therefore,

$$\|T^m_{\Omega,\alpha,b}f\|_{q,v^q}\le\|f\|_{p,u^p}\sup_g\|(T^m_{\Omega,\alpha,b})^*g\|_{p',u^{-p'}}\le C\|f\|_{p,u^p}.$$

This is the conclusion of Theorem 2.

Remark 4. From the proof of Theorem 2 and Remark 3, we know that under the conditions of Theorem 2,

$$\|\tilde{T}^m_{\Omega,\alpha,b}f\|_{q,v^q}\le C\|f\|_{p,u^p}.$$

The proof of Theorem 4. The conclusion of Theorem 4 is a direct consequence of the following lemma and Remark 4.

Lemma 5. *Let* $0<\alpha<n$, $\Omega\in L^1(S^{n-1})$. *Then we have*

$$M^m_{\Omega,\alpha,b}f(x)\le \bar{T}^m_{|\Omega|,\alpha,b}(|f|)(x), \quad x\in\mathbf{R}^n.$$

In fact, fix $r>0$, we have

$$\begin{aligned}\bar{T}^m_{|\Omega|,\alpha,b}(|f|)(x) &\ge \int_{|x-y|<r}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|b(x)-b(y)|^m|f(y)|\,dy\\ &\ge \frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy.\end{aligned}$$

Taking the supremum for $r>0$ on both sides of the inequality above, we get

$$\bar{T}^m_{|\Omega|,\alpha,b}(|f|)(x)\ge \sup_{r>0}\frac{1}{r^{n-\alpha}}\int_{|x-y|<r}|\Omega(x-y)|\,|b(x)-b(y)|^m|f(y)|\,dy.$$

This is just our desired conclusion.

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