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Small angle scattering and X-ray transform in classical mechanics

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Abstract. We consider the Newton equation

$$(*)\qquad \ddot{x}=F(x),\quad F(x)=-\nabla v(x),\quad x\in\mathbf{R}^d,$$
$$\text{where } v\in C^2(\mathbf{R}^d,\mathbf{R}),\ |\partial_x^j v(x)|\le c_{|j|}(1+|x|)^{-(\alpha+|j|)}$$

for $|j|\le 2$ and some $\alpha>1$.

We give estimates and asymptotics for scattering solutions and scattering data for the equation $(*)$ for the case of small angle scattering. We show that scattering data at high energies uniquely determine the X-ray transforms PF and Pv. Applying results on inversion of the X-ray transform P we obtain that for $d\ge 2$ scattering data at high energies uniquely determine F and v. For the case of potentials with compact support we give a connection between boundary value data and scattering data and for $d\ge 2$ we obtain, using known results, a uniqueness theorem in the inverse scattering problem at fixed energy.

1. Introduction

Consider the Newton equation

$$\ddot{x}=F(x),\quad F(x)=-\nabla v(x),\quad x\in\mathbf{R}^d, \tag{1.1}$$

$$\text{where } v\in C^2(\mathbf{R}^d,\mathbf{R}),\ |\partial_x^j v(x)|\le c_{|j|}(1+|x|)^{-(\alpha+|j|)} \tag{1.2}$$

for $|j|\le 2$ and some $\alpha>1$ (here j is the multiindex $j\in(\mathbf{N}\cup\{0\})^d$, $|j|=\sum_{n=1}^d j_n$).

For the equation (1.1) the energy

$$E=\tfrac{1}{2}\dot{x}(t)^2+v(x(t))$$

is an integral of motion.

Under the conditions (1.2), the following is valid (see [RS]): for any $(p_-,x_-)\in\mathbf{R}^{2d}$, $p_-\ne 0$, the equation (1.1) has a unique solution $x\in C^2(\mathbf{R},\mathbf{R}^d)$ such that

$$x(t)=p_-t+x_-+y_-(t), \tag{1.3}$$

where $\dot{y}_-(t)\to 0$, $y_-(t)\to 0$, as $t\to -\infty$; in addition, for almost any $(p_-,x_-)\in\mathbf{R}^{2d}$, $p_-\neq 0$,

$$x(t)=p_+t+x_++y_+(t), \tag{1.4}$$

where $p_+\neq 0$, $p_+=a(p_-,x_-)$, $x_+=b(p_-,x_-)$, $\dot{y}_+(t)\to 0$, $y_+(t)\to 0$, as $t\to+\infty$.

The map $S\colon\mathbf{R}^{2d}\to\mathbf{R}^{2d}$ given by the formulas

$$p_+=a(p_-,x_-),\quad x_+=b(p_-,x_-) \tag{1.5}$$

is called the scattering map for the equation (1.1).

By $\mathcal{D}(S)$ we denote the domain of definition of S; by $\mathcal{R}(S)$ we denote the range of S. (By definition, if $(p_-,x_-)\in\mathcal{D}(S)$, then $p_-\neq 0$ and $a(p_-,x_-)\neq 0$.)

Under the conditions (1.2), the map S has the following simple properties (see [RS]): the sets $\mathcal{D}(S)$ and $\mathcal{R}(S)$ are open subsets of $\mathbf{R}^{2d}$; $\mathrm{Mes}(\mathbf{R}^{2d}\backslash\mathcal{D}(S))=0$, $\mathrm{Mes}(\mathbf{R}^{2d}\backslash\mathcal{R}(S))=0$; the map $S\colon\mathcal{D}(S)\to\mathcal{R}(S)$ is continuous and preserves the element of volume; if $(p_-,x_-)\in\mathcal{D}(S)$, then $(-p_-,x_-)\in\mathcal{R}(S)$ and if $(p_+,x_+)\in\mathcal{R}(S)$, then $(-p_+,x_+)\in\mathcal{D}(S)$; $a(p_-,x_-)^2=p_-^2$.

If $v(x)\equiv 0$, then $a(p_-,x_-)=p_-$, $b(p_-,x_-)=x_-$, $(p_-,x_-)\in\mathbf{R}^d$, $p_-\neq 0$. Therefore for $a(p_-,x_-)$, $b(p_-,x_-)$ we will use the following representation

$$\begin{aligned} a(p_-,x_-)&=p_-+a_{\mathrm{sc}}(p_-,x_-),\\ b(p_-,x_-)&=x_-+b_{\mathrm{sc}}(p_-,x_-),\end{aligned}\quad (p_-,x_-)\in\mathcal{D}(S). \tag{1.6}$$

The map S restricted to

$$\Sigma_E(S)=\mathcal{D}(S)\cap\Sigma_E,\quad \text{where } \Sigma_E=\big\{(p_-,x_-)\in\mathbf{R}^{2d}\mid \tfrac{1}{2}p_-^2=E\big\},\ E>0,$$

is called the scattering map at fixed energy E.

We will use the fact that, under the conditions (1.2), the map S is uniquely determined by its restriction to $\mathcal{M}(S)=\mathcal{D}(S)\cap\mathcal{M}$, where

$$\mathcal{M}=\{(p_-,x_-)\in\mathbf{R}^{2d}\mid p_-\neq 0,\ p_-x_-=0\}.$$

In particular, the map S at fixed energy E is uniquely determined by its restriction to $\mathcal{M}_E(S)=\mathcal{D}(S)\cap\mathcal{M}_E$, where $\mathcal{M}_E=\Sigma_E\cap\mathcal{M}$.

In Section 3 we give estimates and asymptotics for the deflection $y_-(t)$ from (1.3) and for the scattering data $a_{\mathrm{sc}}(p_-,x_-)$, $b_{\mathrm{sc}}(p_-,x_-)$ from (1.6) (Theorem 3.1). These estimates and asymptotics are of interest, in particular, when the parameters c, α, d, $\hat{p}_-$, x_- are fixed and $|p_-|$ increases or c, α, d, p_-, $\hat{x}_-$ are fixed and $|x_-|$ increases or, e.g., α, d, p_-, x_- are fixed and c decreases (where $c_{|j|}$, α, d are constants from (1.2), $c=\max(c_1,c_2)$; $\hat{p}_-=p_-/|p_-|$, $\hat{x}_-=x_-/|x_-|$). In these cases $\sup_{t\in\mathbf{R}}|\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t)=p_-+\dot{y}_-(t)$ and p_-, and we deal with small angle scattering. Note that already, under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbf{R}}|\theta(t)|<\frac{1}{4}\pi$ and we deal with a rather small angle scattering. The term "small angle scattering" is adopted by us from Section 20 of [LL]. Note, however, that in [LL] the small angle scattering is considered only for a large impact parameter $|x_-|$ for the spherically symmetric case.

Consider

$$TS^{d-1}=\{(\theta,x)\,|\,\theta\in S^{d-1},\ x\in\mathbf{R}^d,\ \theta x=0\},$$

where S^{d-1} is the unit sphere in $\mathbf{R}^d$. Note that $TS^{d-1}\approx\mathcal{M}_E$, $E>0$.

Consider the X-ray transform P which maps each function f with the properties

$$f\in C(\mathbf{R}^d,\mathbf{R}^m),\quad |f(x)|=O(|x|^{-\beta}),\ \text{as } |x|\to\infty,\ \text{for some } \beta>1$$

into a function $Pf\in C(TS^{d-1},\mathbf{R}^m)$, where Pf is defined by

$$Pf(\theta,x)=\int_{-\infty}^{+\infty}f(t\theta+x)\,dt,\quad (\theta,x)\in TS^{d-1}.$$

In Theorem 4.1 (Section 4) we give, in particular, the asymptotic formulas

$$PF(\theta,x)=\lim_{s\to+\infty}sa_{\mathrm{sc}}(s\theta,x),$$
$$Pv(\theta,x)=\lim_{s\to+\infty}s^2\theta b_{\mathrm{sc}}(s\theta,x)$$

with explicit upper bounds for the difference with the limit for s large. (These results follow directly from Theorem 3.1.)

For $d\geq2$ Theorem 4.1 and methods for the reconstruction of f from Pf (see [GGG], [Na], [FN] and Section 4 of the present paper) permit to reconstruct F and v from scattering data at high energies.

In the present article we consider also the time delay

$$\tau(p_-,x_-)=\frac{p_-x_--a(p_-,x_-)b(p_-,x_-)}{|p_-|^2}.$$

In Proposition 3.1 we give asymptotics for $\tau(p_-,x_-)$ for small angle scattering; in Theorem 4.1 we give, in particular, formulas for the reconstruction of Pv from $\tau(p_-,x_-)$ at high energies.

To our knowledge the multidimensional inverse scattering problem (without assumptions of spherical symmetry) for the Newton equation (1.1) was not considered before. However, for the equation (1.1) in a bounded open strictly convex (in the strong sense) domain $D\subset\mathbf{R}^d$, $d\geq 2$, with smooth boundary ∂D the inverse boundary value problem at high energies and at fixed energy was considered in [GN]. In [GN] results are obtained using results of [BG2]. The work [BG2] is a detailed version of [BG1]. The work [BG1] generalizes, in particular, [B]. Results similar to results of [BG1] were given independently in [MR].

In Section 5 we obtain the following results: we give a connection between the boundary value data from [GN] and some other boundary value data (Lemma 5.1); for the case

$$v\in C^2(\mathbf{R}^d,\mathbf{R}),\quad \operatorname{supp} v\subset D$$

(where D has the properties mentioned above), we give a connection between boundary value data and the scattering data (Theorem 5.1), and for $d\geq 2$ we obtain (using results of [GN]) that the scattering data and D uniquely determine v (Theorem 5.2) at fixed, sufficiently large energy $E>E(v,D)$.

Let $\Omega(D)\subset TS^{d-1}$ denote the set of all rays which do not intersect D (with the properties mentioned above). Let $C_0^2(\mathbf{R}^d,\mathbf{R})=\{f\in C^2(\mathbf{R}^d,\mathbf{R})\,|\operatorname{supp} f \text{ is compact}\}$.

Conjecture A. *If $v\in C_0^2(\mathbf{R}^d,\mathbf{R})$, $d\geq 2$, and at fixed $E>0$ the identities $a(\sqrt{2E}\,\theta,x)\equiv\sqrt{2E}\,\theta$, $b(\sqrt{2E}\,\theta,x)\equiv x$ for $(\theta,x)\in\Omega(D)$ hold, then $\operatorname{supp} v\subseteq\bar{D}$.*

Conjecture A is a generalization of the Cormack–Helgason support theorem from the theory of the X-ray transform (see [Na]). (We have a proof of Conjecture A for the case of the Born approximation and for the case $v(x)\geq 0$.)

Conjecture B. *Under the conditions (1.2), $d\geq 2$, at fixed sufficiently large energy $E>E(v)$, the scattering data S uniquely determine v.*

Concerning the works on the inverse problem for the equation (1.1) in dimension 1 we can mention [A], [K], [AFC]. Concerning the works on the inverse scattering problem for the multidimensional equation (1.1) with spherically symmetric potential we can mention [Fi], [KKS], where this problem was considered in dimension 3 at fixed energy for the case of monotonous decreasing potential in $|x|$.

As related preceding works we would like to mention also the works that deal with asymptotics of scattering data at high energies for the Schrödinger equation and with the inverse scattering problem for this equation, see, e.g., [F], [R], [EW], [N1], [N2], [N3] and references given there.

2. A contraction map

If x satisfies the differential equation (1.1) and the initial condition (1.3), then

x satisfies the integral equation

$$x(t)=p_-t+x_-+\int_{-\infty}^{t}\int_{-\infty}^{\tau}F(x(s))\,ds\,d\tau,\quad \text{where } F(x)=-\nabla v(x),\ p_-\neq 0. \tag{2.1}$$

For $y_-(t)$ this equation takes the form

$$\begin{gathered} y_-(t)=A_{p_-,x_-}(y_-)(t),\\ \text{where } A_{p_-,x_-}(f)(t)=\int_{-\infty}^{t}\int_{-\infty}^{\tau}F(p_-s+x_-+f(s))\,ds\,d\tau,\ p_-\neq 0. \end{gathered} \tag{2.2}$$

From (2.2), (1.2) and from $y_-\in C(\mathbf{R},\mathbf{R}^d)$, $y_-(t)\to 0$, as $t\to-\infty$, it follows, in particular, that

$$y_-(t)\in C^1(\mathbf{R},\mathbf{R}^d)\quad\text{and}\quad |\dot y_-(t)|=O(|t|^{-\alpha}),\ |y_-(t)|=O(|t|^{-(\alpha-1)}),\ \text{as } t\to-\infty, \tag{2.3}$$

where $p_-\neq 0$ and x_- are fixed.

Consider the complete metric space

$$\begin{gathered} M_{T,r}=\{f\in C^1(]-\infty,T],\mathbf{R}^d)\mid \|f\|_T\le r\},\\ \text{where } \|f\|_T=\max\Big(\sup_{t\in]-\infty,T]}|\dot f(t)|,\ \sup_{t\in]-\infty,T]}|f(t)-t\dot f(t)|\Big) \end{gathered} \tag{2.4}$$

(where for $T=+\infty$ we understand $]-\infty,T[$ as $]-\infty,+\infty[$). From (2.3) it follows that, at fixed $T<+\infty$,

$$y_-(t)\in M_{T,r}\quad\text{for some } r \text{ depending on } y_-(t) \text{ and } T. \tag{2.5}$$

Lemma 2.1. *Under the conditions* (1.2), *the following is valid*: *if* $f\in M_{T,r}$, $0\le r\le 1$, $p_-x_-=0$ *and* $|p_-|>\sqrt{2}\,r$, *then*

$$\begin{aligned} \|A_{p_-,x_-}(f)\|_T &\le \varrho_T(d,c_1,\alpha,|p_-|,|x_-|,r)\\ &=\frac{dc_1 2^{\alpha+1}\big(|p_-|/\sqrt{2}+1-r\big)}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)T\big)^{\alpha-1}} \end{aligned} \tag{2.6a}$$

for $T\le 0$,

$$\begin{aligned} \|A_{p_-,x_-}(f)\|_T &\le \varrho(d,c_1,\alpha,|p_-|,|x_-|,r)\\ &=\frac{dc_1 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}} \end{aligned} \tag{2.6b}$$

for $T\leq+\infty$;

if $f_1,f_2\in M_{T,r}$, $0\leq r\leq 1$, *and* $|p_-|>\sqrt{2}\,r$, *then*

$$\|A_{p_-,x_-}(f_2)-A_{p_-,x_-}(f_1)\|_T\leq\lambda_T(d,c_2,\alpha,|p_-|,|x_-|,r)\|f_2-f_1\|_T, \tag{2.7a}$$

$$\lambda_T(d,c_2,\alpha,|p_-|,|x_-|,r)=\frac{d^2c_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)^2}{(\alpha-1)(|p_-|/\sqrt{2}-r)^3\big(1+|x_-|/\sqrt{2}-(|p_-|/\sqrt{2}-r)T\big)^{\alpha-1}}$$

for $T\leq 0$,

$$\|A_{p_-,x_-}(f_2)-A_{p_-,x_-}(f_1)\|_T\leq\lambda(d,c_2,\alpha,|p_-|,|x_-|,r)\|f_2-f_1\|_T, \tag{2.7b}$$

$$\lambda(d,c_2,\alpha,|p_-|,|x_-|,r)=\frac{d^2c_2 2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)^2}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}$$

for $T\leq+\infty$.

Note that

$$\begin{aligned}&\max\Big(\frac{\varrho_T(d,c_1,\alpha,|p_-|,|x_-|,r)}{r},\lambda_T(d,c_2,\alpha,|p_-|,|x_-|,r)\Big)\\ &\leq\mu_T(d,c,\alpha,|p_-|,|x_-|,r)\\ &=\frac{d^2c2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)^2}{r(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)T\big)^{\alpha-1}}\end{aligned} \tag{2.8a}$$

for $T\leq 0$,

$$\begin{aligned}&\max\Big(\frac{\varrho(d,c_1,\alpha,|p_-|,|x_-|,r)}{r},\lambda(d,c_2,\alpha,|p_-|,|x_-|,r)\Big)\\ &\leq\mu(d,c,\alpha,|p_-|,|x_-|,r)\\ &=\frac{d^2c2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)^2}{r(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}\end{aligned} \tag{2.8b}$$

for $T\leq+\infty$, where $c=\max(c_1,c_2)$, $0<r\leq 1$, $|p_-|>\sqrt{2}\,r$.

From Lemma 2.1 and the estimates (2.8) we obtain the following result.

Corollary 2.1. *Under the conditions* (1.2), $0<r\leq 1$, $p_-x_-=0$, $|p_-|>\sqrt{2}\,r$, *the following result is valid*:

if $\mu_T(d,c,\alpha,|p_-|,|x_-|,r)<1$, *then* A_{p_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T\leq 0$;

if $\mu(d,c,\alpha,|p_-|,|x_-|,r)<1$, *then* A_{p_-,x_-} *is a contraction map in* $M_{T,r}$ *for* $T\leq +\infty$.

Taking into account (2.5) and using Lemma 2.1, Corollary 2.1 and the lemma about contraction maps we will study the solution $y_-(t)$ of the equation (2.2) in $M_{T,r}$.

We will use also the following results.

Lemma 2.2. *Under the conditions* (1.2), $f\in M_{T,r}$, $0\leq r\leq 1$, $p_-x_-=0$, $|p_-|>\sqrt{2}\,r$, *the following is valid*:

$$|\dot{A}_{p_-,x_-}(f)(t)|\leq \zeta_-(d,c_1,\alpha,|p_-|,|x_-|,r,t) = \frac{dc_1 2^{\alpha+1}}{\alpha\big(|p_-|/\sqrt{2}-r\big)\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha}}, \tag{2.9}$$

$$|A_{p_-,x_-}(f)(t)|\leq \xi_-(d,c_1,\alpha,|p_-|,|x_-|,r,t) = \frac{dc_1 2^{\alpha+1}}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}, \tag{2.10}$$

for $t\leq T$, $t\leq 0$;

$$A_{p_-,x_-}(f)(t)=k_{p_-,x_-}(f)t+l_{p_-,x_-}(f)+H_{p_-,x_-}(f)(t), \tag{2.11}$$

where

$$k_{p_-,x_-}(f)=\int_{-\infty}^{+\infty}F(p_-s+x_-+f(s))\,ds, \tag{2.12a}$$

$$l_{p_-,x_-}(f)=\int_{-\infty}^{0}\int_{-\infty}^{\tau}F(p_-s+x_-+f(s))\,ds\,d\tau - \int_{0}^{+\infty}\int_{\tau}^{+\infty}F(p_-s+x_-+f(s))\,ds\,d\tau, \tag{2.12b}$$

$$|k_{p_-,x_-}(f)|\leq 2\zeta_-(d,c_1,\alpha,|p_-|,|x_-|,r,0), \tag{2.13a}$$

$$|l_{p_-,x_-}(f)|\leq 2\xi_-(d,c_1,\alpha,|p_-|,|x_-|,r,0), \tag{2.13b}$$

$$|\dot{H}_{p_-,x_-}(f)(t)|\leq \zeta_+(d,c_1,\alpha,|p_-|,|x_-|,r,t) = \frac{dc_1 2^{\alpha+1}}{\alpha\big(|p_-|/\sqrt{2}-r\big)\big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha}}, \tag{2.14}$$

$$|H_{p_-,x_-}(f)(t)|\leq \xi_+(d,c_1,\alpha,|p_-|,|x_-|,r,t) = \frac{dc_1 2^{\alpha+1}}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}} \tag{2.15}$$

for $T=+\infty$, $t\geq 0$.

Lemma 2.3. *Let the conditions* (1.2) *be valid, $y_-\in M_{T,r}$ be a solution of* (2.2), *$T=+\infty$, $0\le r\le 1$, $p_-x_-=0$, $|p_-|>\sqrt{2}\,r$, then*

$$|k_{p_-,x_-}(y_-)-k_{p_-,x_-}(0)|\le \varepsilon_a(d,c,\alpha,|p_-|,|x_-|,r)$$

$$(2.16\text{a})\qquad =\frac{d^2c2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)\varrho(d,c,\alpha,|p_-|,|x_-|,r)}{\alpha\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}\big)^\alpha},$$

$$|l_{p_-,x_-}(y_-)-l_{p_-,x_-}(0)|\le \varepsilon_b(d,c,\alpha,|p_-|,|x_-|,r)$$

$$(2.16\text{b})\qquad =\frac{d^2c2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)\varrho(d,c,\alpha,|p_-|,|x_-|,r)}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}},$$

where $c=\max(c_1,c_2)$.

Proofs of Lemmas 2.1, 2.2, 2.3 are given in Section 6.

3. Small-angle scattering

Under the conditions (1.2), for any $(p_-,x_-)\in\mathbf{R}^{2d}$, $p_-\neq 0$, the equation (1.1) has a unique solution $x\in C^2(\mathbf{R},\mathbf{R}^d)$ with the initial conditions (1.3). Consider the function $y_-(t)$ from (1.3). This function describes deflection from free motion.

Using Corollary 2.1 the lemma about contraction maps, and Lemmas 2.2 and 2.3 we obtain the following result.

Theorem 3.1. *Let the conditions* (1.2) *be satisfied, $\mu(d,c,\alpha,|p_-|,|x_-|,r)<1$, $(p_-,x_-)\in\mathbf{R}^{2d}$, $c=\max(c_1,c_2)$, $0<r\le 1$, $p_-x_-=0$, $|p_-|>\sqrt{2}\,r$. Then the deflection $y_-(t)$ has the properties*

$$(3.1)\qquad y_-\in M_{T,r},\quad T=+\infty;$$

$$(3.2)\qquad |\dot y_-(t)|\le \zeta_-(d,c_1,\alpha,|p_-|,|x_-|,r,t),$$

$$(3.3)\qquad |y_-(t)|\le \xi_-(d,c_1,\alpha,|p_-|,|x_-|,r,t)\qquad \textit{for } t\le 0;$$

$$(3.4)\qquad y_-(t)=a_{\mathrm{sc}}(p_-,x_-)t+b_{\mathrm{sc}}(p_-,x_-)+h(p_-,x_-,t),$$

where, with $\hat p_-=p_-/|p_-|$,

$$(3.5\text{a})\qquad \left|a_{\mathrm{sc}}(p_-,x_-)-\frac{1}{|p_-|}\int_{-\infty}^{+\infty}F(\hat p_-s+x_-)\,ds\right|\le \varepsilon_a(d,c,\alpha,|p_-|,|x_-|,r),$$

(3.5b)

$$\left|b_{\mathrm{sc}}(p_-,x_-)-\frac{1}{|p_-|^2}\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau}F(\hat p_-s+x_-)\,ds\,d\tau-\int_{0}^{+\infty}\int_{\tau}^{+\infty}F(\hat p_-s+x_-)\,ds\,d\tau\right)\right|$$
$$\leq \varepsilon_b(d,c,\alpha,|p_-|,|x_-|,r),$$

$$|a_{\mathrm{sc}}(p_-,x_-)|\leq 2\zeta_-(d,c_1,\alpha,|p_-|,|x_-|,r,0), \tag{3.6a}$$
$$|b_{\mathrm{sc}}(p_-,x_-)|\leq 2\xi_-(d,c_1,\alpha,|p_-|,|x_-|,r,0), \tag{3.6b}$$
$$|\dot h(p_-,x_-,t)|\leq \zeta_+(d,c_1,\alpha,|p_-|,|x_-|,r,t), \tag{3.7}$$
$$|h(p_-,x_-,t)|\leq \xi_+(d,c_1,\alpha,|p_-|,|x_-|,r,t) \tag{3.8}$$

for $t\geq 0$.

We remind that

$$\mu(d,c,\alpha,|p_-|,|x_-|,r)=\frac{d^2c2^{\alpha+3}}{r(\alpha-1)\big(1+|x_-|/\sqrt2\big)^{\alpha-1}}\frac{\big(|p_-|/\sqrt2-r+1\big)^2}{\big(|p_-|/\sqrt2-r\big)^3}, \tag{3.9}$$

$$\varepsilon_a(d,c,\alpha,|p_-|,|x_-|,r)=\frac{d^3c^22^{2\alpha+5}}{\alpha(\alpha-1)\big(1+|x_-|/\sqrt2\big)^{2\alpha-1}}\frac{\big(|p_-|/\sqrt2-r+1\big)^2}{\big(|p_-|/\sqrt2-r\big)^4}, \tag{3.10a}$$

$$\varepsilon_b(d,c,\alpha,|p_-|,|x_-|,r)=\frac{d^3c^22^{2\alpha+5}}{\alpha(\alpha-1)^2\big(1+|x_-|/\sqrt2\big)^{2\alpha-2}}\frac{\big(|p_-|/\sqrt2-r+1\big)^2}{\big(|p_-|/\sqrt2-r\big)^5}, \tag{3.10b}$$

$$2\zeta_-(d,c,\alpha,|p_-|,|x_-|,r,0)=\frac{dc2^{\alpha+2}}{\alpha\big(1+|x_-|/\sqrt2\big)^{\alpha}}\frac{1}{|p_-|/\sqrt2-r}, \tag{3.11a}$$

$$2\xi_-(d,c,\alpha,|p_-|,|x_-|,r,0)=\frac{dc2^{\alpha+2}}{\alpha(\alpha-1)\big(1+|x_-|/\sqrt2\big)^{\alpha-1}}\frac{1}{\big(|p_-|/\sqrt2-r\big)^2}, \tag{3.11b}$$

$$\zeta_\mp(d,c_1,\alpha,|p_-|,|x_-|,r,t)=\frac{dc_12^{\alpha+1}}{\alpha\big(1+|x_-|/\sqrt2\mp\big(|p_-|/\sqrt2-r\big)t\big)^{\alpha}}\frac{1}{|p_-|/\sqrt2-r}, \tag{3.12}$$

$$\xi_\mp(d,c_1,\alpha,|p_-|,|x_-|,r,t)=\frac{dc_12^{\alpha+1}}{\alpha(\alpha-1)\big(1+|x_-|/\sqrt2\mp\big(|p_-|/\sqrt2-r\big)t\big)^{\alpha-1}}\times\frac{1}{\big(|p_-|/\sqrt2-r\big)^2}. \tag{3.13}$$

We will use the following observations.

(1) Let $n\in\mathbf{N}$, $m,l\in\mathbf{N}\cup\{0\}$, $l+m<n$, then

$$\frac{s_1^l\big(s_1/\sqrt2-r+1\big)^m}{\big(s_1/\sqrt2-r\big)^n}>\frac{s_2^l\big(s_2/\sqrt2-r+1\big)^m}{\big(s_2/\sqrt2-r\big)^n} \tag{3.14}$$

for $\sqrt{2}\,r<s_1<s_2$, $0<r\le 1$;

$$\frac{s^l\left(s/\sqrt{2}-r_1+1\right)^m}{\left(s/\sqrt{2}-r_1\right)^n}<\frac{s^l\left(s/\sqrt{2}-r_2+1\right)^m}{\left(s/\sqrt{2}-r_2\right)^n} \tag{3.15}$$

for $\sqrt{2}\,r_1<\sqrt{2}\,r_2<s$, $0<r_1\le 1$, $0<r_2\le 1$.

(2) Let $z=z(d,c,\alpha,|x_-|,r)$ be the root of the equation

$$\mu(d,c,\alpha,z,|x_-|,r)=1, \quad z\in]\sqrt{2}\,r,+\infty[\tag{3.16}$$

(where the assumptions about d, c, α, $|x_-|$, r are the same as in Theorem 3.1). Then

$$\mu(d,c,\alpha,s,|x_-|,r)<1,\ \ s\in]\sqrt{2}r,+\infty[\quad\Longleftrightarrow\quad s>z(d,c,\alpha,|x_-|,r). \tag{3.17}$$

Theorem 3.1 gives, in particular, estimates and asymptotics for the scattering process when the parameters c, α, d, $\hat{p}_-$, x_- are fixed and $|p_-|$ increases or, e.g., c, α, d, p_-, $\hat{x}_-$ are fixed and $|x_-|$ increases. In these cases $\sup_{t\in\mathbf{R}}|\theta(t)|$ decreases, where $\theta(t)$ denotes the angle between the vectors $\dot{x}(t)=p_-+\dot{y}_-(t)$ and p_-, and we deal with small angle scattering. Note that already under the conditions of Theorem 3.1, without additional assumptions, there is the estimate $\sup_{t\in\mathbf{R}}|\theta(t)|<\frac{1}{4}\pi$ and we deal with a rather small angle scattering.

Using Theorem 3.1 we can obtain asymptotics and estimates for small angle scattering for functions which are expressed through $a(p_-,x_-)$ and $b(p_-,x_-)$. Consider, e.g., the time delay

$$\begin{aligned}\tau(p_-,x_-)&=\frac{p_-x_- - a(p_-,x_-)b(p_-,x_-)}{|p_-|^2}\\ &=\frac{(-p_-b_{\mathrm{sc}}(p_-,x_-)-x_-a_{\mathrm{sc}}(p_-,x_-)-a_{\mathrm{sc}}(p_-,x_-)b_{\mathrm{sc}}(p_-,x_-))}{|p_-|^2}\end{aligned} \tag{3.18}$$

and the function

$$\frac{-p_-b_{\mathrm{sc}}(p_-,x_-)}{|p_-|^2} \tag{3.19}$$

for $(p_-,x_-)\in\mathcal{D}(S)$.

Remark. To recall the physical sense of $\tau(p_-,x_-)$ note that

$$\tau(p_-,x_-)=\lim_{R\to+\infty}\left(T(p_-,x_-,R)-\frac{2R}{|p_-|}\right), \quad (p_-,x_-)\in\mathcal{D}(S), \tag{3.20}$$

where $T(p_-,x_-,R)$ is the total time during which the solution $x(t)$ of (1.1) and (1.3) satisfies $|x(t)|\le R$. Note that if $v(x)\equiv 0$, then $\tau(p_-,x_-)\equiv 0$ for $(p_-,x_-)\in\mathbf{R}^{2d}$, $p_-\ne 0$.

Proposition 3.1. *Under the conditions of Theorem* 3.1, *the following formulas are valid:*

$$\left|\frac{-p_- b_{\rm sc}(p_-,x_-)}{|p_-|^2} - \frac{1}{|p_-|^3}\int_{-\infty}^{+\infty} v(\hat{p}_-\tau + x_-)\,d\tau\right| \le \frac{\varepsilon_b(d,c,\alpha,|p_-|,|x_-|,r)}{|p_-|}, \tag{3.21}$$

$$\begin{aligned}&\left|\tau(p_-,x_-) - \frac{1}{|p_-|^3}(1 + x_-\nabla_{x_-})\int_{-\infty}^{+\infty} v(\hat{p}_-\tau + x_-)\,d\tau\right| \\ &\qquad \le \sigma(d,c,\alpha,|p_-|,|x_-|,r) \\ &\qquad = \frac{\varepsilon_b(d,c,\alpha,|p_-|,|x_-|,r)}{|p_-|} + \frac{|x_-|\varepsilon_a(d,c,\alpha,|p_-|,|x_-|,r)}{|p_-|^2} \\ &\qquad + \frac{4\zeta_-(d,c,\alpha,|p_-|,|x_-|,r,0)\xi_-(d,c,\alpha,|p_-|,|x_-|,r,0)}{|p_-|^2},\end{aligned} \tag{3.22}$$

where if, in addition, $|p_-| \ge 2\sqrt{2}$, *then*

$$\sigma(d,c,\alpha,|p_-|,|x_-|,r) \le \frac{\mathrm{const}\, d^3 c^2 2^{2\alpha}}{(\alpha-1)^2\left(1+|x_-|/\sqrt{2}\right)^{2\alpha-2}|p_-|^4}. \tag{3.23}$$

Proof. Under the conditions (1.2), for $(p_-,x_-) \in \mathbf{R}^{2d}$, $p_- \ne 0$, the following formulas are valid:

$$\begin{aligned}-\hat{p}_-\int_{-\infty}^{\tau} F(\hat{p}_- s + x_-)\,ds &= v(\hat{p}_-\tau + x_-),\\ \hat{p}_-\int_{\tau}^{+\infty} F(\hat{p}_- s + x_-)\,ds &= v(\hat{p}_-\tau + x_-),\end{aligned}$$

$$\begin{aligned}-\hat{p}_-\left(\int_{-\infty}^{0}\int_{-\infty}^{\tau} F(\hat{p}_- s + x_-)\,ds\,d\tau - \int_0^{+\infty}\int_{\tau}^{+\infty} F(\hat{p}_- s + x_-)\,ds\,d\tau\right) \\ = \int_{-\infty}^{+\infty} v(\hat{p}_-\tau + x_-)\,d\tau,\end{aligned} \tag{3.24}$$

$$\begin{aligned}-\int_{-\infty}^{+\infty} F(\hat{p}_- s + x_-)\,ds &= \nabla_{x_-}\int_{-\infty}^{\infty} v(\hat{p}_- s + x_-)\,ds,\\ \hat{p}_-\left(\nabla_{x_-}\int_{-\infty}^{+\infty} v(\hat{p}_- s + x_-)\,ds\right) &= 0.\end{aligned} \tag{3.25}$$

Under the conditions of Theorem 3.1, from (3.5b), (3.24) we obtain (3.21) and the formula

(3.26)
$$\left|\frac{-x_- a_{\rm sc}(p_-,x_-)}{|p_-|^2} - \frac{1}{|p_-|^3} x_-\nabla_{x_-}\int_{-\infty}^{+\infty} v(\hat p_- s + x_-)\,ds\right| \le \frac{|x_-|\varepsilon_a(d,c,\alpha,|p_-|,|x_-|,r)}{|p_-|^2}.$$

Under the conditions of Theorem 3.1, from (3.18), (3.21), (3.26) and (3.6) we obtain (3.22).

Under the additional condition that $|p_-|\ge 2\sqrt{2}$, the estimate (3.23) follows from the formulas for ε_a, ε_b, ζ_-, ξ_-.

4. Inverse scattering at high energies

Consider
$$TS^{d-1} = \{(\theta,x)\,|\,\theta\in S^{d-1},\ x\in\mathbf{R}^d,\ \theta x = 0\},$$
where S^{d-1} is the unit sphere in $\mathbf{R}^d$. We interpret TS^{d-1} as the set of all rays in $\mathbf{R}^d$. As a ray l we understand a straight line with fixed orientation. If $l=(\theta,x)\in TS^{d-1}$, then $l=\{y\in\mathbf{R}^d\,|\,y=t\theta+x,\ t\in\mathbf{R}\}$ (up to orientation) and θ gives the orientation of l.

Consider the X-ray transform P which maps each function f with the properties

$$f\in C(\mathbf{R}^d,\mathbf{R}^m),\quad |f(x)| = O(|x|^{-\beta}),\ \text{as } |x|\to\infty,\ \text{for some } \beta>1 \tag{4.1}$$

into a function $Pf\in C(TS^{d-1},\mathbf{R}^m)$, where Pf is defined by

$$Pf(\theta,x) = \int_{-\infty}^{\infty} f(t\theta+x)\,dt,\quad (\theta,x)\in TS^{d-1}. \tag{4.2}$$

Properties of the X-ray transform P and, in particular, the problem of reconstruction of f from Pf were being investigated in many works (see, e.g., [Na], [GGG], [FN]).

In this section, when considering Pf we always assume that f satisfies, at least, (4.1), although one can extend P to less regular functions.

The simplest property of P is

$$Pf(\theta,x) = Pf(-\theta,x),\quad (\theta,x)\in TS^{d-1}. \tag{4.3}$$

Some other simple properties of P are given in the following lemma.

Lemma 4.1. *Let*

$$f \in C^n(\mathbf{R}^d, \mathbf{R}), \tag{4.4a}$$

$$|f(x)| \le c(f)(1+|x|)^{-\beta(0)}, \tag{4.4b}$$

$$\partial_x^j f(x) = O(|x|^{-\beta(|j|)}), \qquad \text{as } |x| \to \infty, \text{ for } |j| \le n, \tag{4.4c}$$

$$\beta(s) > 1, \qquad s = 0, \dots, n. \tag{4.4d}$$

Then

$$Pf \in C^n(TS^{d-1}, \mathbf{R}) \tag{4.5a}$$

and, in particular,

$$|Pf(\theta, x)| \le \frac{2\sqrt{2}\, c(f)}{(\beta(0)-1)\big(1+|x|/\sqrt{2}\,\big)^{\beta(0)-1}} \tag{4.5b}$$

for $(\theta, x) \in TS^{d-1}$,

$$\partial_y^j f(\theta, A_\theta y) = O(|y|^{1-\beta(|j|)}), \quad \text{as } |y| \to \infty, \tag{4.5c}$$

for $|j| \le n$, *for* $\theta \in S^{d-1}$, $y \in \mathbf{R}^{d-1}$, *where* A_θ *is a linear isometric map of* $\mathbf{R}^{d-1}$ *on* $X_\theta = \{x \in \mathbf{R}^d \,|\, \theta x = 0\}$ *(as on a subspace of* $\mathbf{R}^d$*).*

To prove (4.5b) and (4.5c) we use the formulas

$$|t\theta + x| \ge (|t| + |x|)/\sqrt{2} \qquad \text{for } x \in X_\theta,$$

$$\int_{-\infty}^{+\infty} \frac{dt}{\big(1+(|s|+|x|)/\sqrt{2}\,\big)^{\sigma}} = \frac{2\sqrt{2}}{(\sigma-1)\big(1+|x|/\sqrt{2}\,\big)^{\sigma-1}}, \quad \sigma > 1.$$

For reconstruction of f from Pf for $d \ge 3$ there is, in particular, the following well-known scheme based on the methods of reconstruction of f from Pf for $d=2$. To reconstruct f at a point $x' \in \mathbf{R}^d$ we consider in $\mathbf{R}^d$ a two-dimensional plane Y containing x'. We consider in TS^{d-1} the subset $TS^1(Y)$ which is the set of all rays lying in Y. We restrict Pf on $TS^1(Y)$ and reconstruct $f(x')$ from these data using methods of reconstruction of f from Pf for $d=2$.

Remark. As coordinates on Y we take the Euclidean coordinates (with respect to the structure induced from $\mathbf{R}^d$) with centre at the point which is the nearest to 0 in $\mathbf{R}^d$. In such coordinates y on Y the following is valid. If f satisfies the conditions

(4.4), then the restriction $f|_Y$ satisfies these conditions in y with the same constant in (4.4b).

For reconstruction of f from Pf for $d=2$ there are, in particular, the formulas

$$f(x) = -\frac{\partial}{\partial x_1} I_2(x) + \frac{\partial}{\partial x_2} I_1(x), \tag{4.6a}$$

$$I_j(x) = \left(\frac{1}{2\pi}\right)^2 \int_{S^1} \theta_j \left(\text{p.v.} \int_{-\infty}^{+\infty} \frac{g(\theta,q)}{x\theta^\perp - q}\, dq\right) d\theta, \quad j = 1,2, \tag{4.6b}$$

$$g(\theta,q) = Pf(\theta, q\theta^\perp), \tag{4.6c}$$

where $\theta=(\theta_1,\theta_2)$, $\theta^\perp=(-\theta_2,\theta_1)$ and $d\theta$ denotes the standard Euclidean measure on S^1.

In addition,

$$\begin{aligned} I_1(x) &= \operatorname{Im}\left(\frac{1}{\pi}\iint_{\mathbf{R}^2} \frac{f(y)}{y_1+iy_2-(x_1+ix_2)}\, dy_1\, dy_2\right), \\ I_2(x) &= \operatorname{Re}\left(\frac{1}{\pi}\iint_{\mathbf{R}^2} \frac{f(y)}{y_1+iy_2-(x_1+ix_2)}\, dy_1\, dy_2\right). \end{aligned} \tag{4.7}$$

Using Lemma 4.1 and some properties of the Hilbert transform H,

$$Hg(s) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{+\infty} \frac{g(q)}{s-q}\, dq,$$

we can show, in particular, the following:

(1) under the conditions (4.4) with $n=0$, $d=2$, Pf determines $I_j(x)$ by the formulas (4.6b) and (4.6c) as a function from $L^p_{\text{loc}}(\mathbf{R}^2)$ for any $p\geq 2$;

(2) under the conditions (4.4) with $n=1$, $d=2$, Pf determines $I_j(x)$ by the formulas (4.6b) and (4.6c) as a function from $C(\mathbf{R}^2)$.

The system of formulas (4.6), (4.7) differs somewhat from well-known formulas. In fact, the system of formulas (4.6), (4.7) is similar to the formula (1.12) from [FN].

Consider PF and Pv. We assume that the conditions (1.2) are valid. Some simple properties of PF and Pv follow from (4.3) and Lemma 4.1. The functions PF and Pv arise in leading terms of asymptotics of scattering data for small angle scattering (see Theorem 3.1 and Proposition 3.1). In addition, scattering data at high energies determine PF and Pv uniquely.

Theorem 4.1. *Let the conditions* (1.2) *be valid,* $c=\max(c_1,c_2)$, $(\theta,x)\in TS^{d-1}$, $r\in]0,1]$.

Then

$$PF(\theta,x) = \lim_{s\to+\infty} s a_{\text{sc}}(s\theta, x) \tag{4.8a}$$

and, in addition,

$$\text{(4.8b)}\qquad |PF(\theta,x)-sa_{\rm sc}(s\theta,x)|\le \frac{d^3c^2 2^{2\alpha+5}}{\alpha(\alpha-1)\big(1+|x|/\sqrt{2}\big)^{2\alpha-1}}\,\frac{s\big(s/\sqrt{2}-r+1\big)^2}{\big(s/\sqrt{2}-r\big)^4}$$

for $s>z(d,c,\alpha,|x|,r)$;

$$\text{(4.9a)}\qquad Pv(\theta,x)=\lim_{s\to+\infty} s^2\theta b_{\rm sc}(s\theta,x)$$

and, in addition,

$$\text{(4.9b)}\qquad |Pv(\theta,x)-s^2\theta b_{\rm sc}(s\theta,x)|\le \frac{d^3c^2 2^{2\alpha+5}}{\alpha(\alpha-1)^2\big(1+|x|/\sqrt{2}\big)^{2\alpha-2}}\,\frac{s^2\big(s/\sqrt{2}-r+1\big)^2}{\big(s/\sqrt{2}-r\big)^5}$$

for $s>z(d,c,\alpha,|x|,r)$;

$$\text{(4.10a)}\qquad Pv(\theta,x)=-\frac{1}{|x|}\int_{|x|}^{+\infty}\Big(1+q\frac{d}{dq}\Big)Pv\Big(\theta,q\frac{x}{|x|}\Big)\,dq,\quad |x|\neq 0,$$

$$\text{(4.10b)}\qquad \Big(1+q\frac{d}{dq}\Big)Pv\Big(\theta,q\frac{x}{|x|}\Big)=\lim_{s\to+\infty} s^3\tau\Big(s\theta,q\frac{x}{|x|}\Big)$$

and, in addition,

$$\text{(4.10c)}\qquad \Big|\Big(1+q\frac{d}{dq}\Big)Pv\Big(\theta,q\frac{x}{|x|}\Big)-s^3\tau\Big(s\theta,q\frac{x}{|x|}\Big)\Big|\le\frac{\text{const } d^3c^2 2^{2\alpha}}{(\alpha-1)^2\big(1+q/\sqrt{2}\big)^{2\alpha-2}s}$$

for $s>\max\big(z(d,c,\alpha,q,r),2\sqrt{2}\big)$, *where* $q\ge|x|$.

The function $z(d,c,\alpha,|x|,r)$ is defined by (3.16). Some properties of the functions

$$\frac{s\big(s/\sqrt{2}-r+1\big)^2}{\big(s/\sqrt{2}-r\big)^4}\quad\text{and}\quad\frac{s^2\big(s/\sqrt{2}-r+1\big)^2}{\big(s/\sqrt{2}-r\big)^5}$$

from the right-hand sides of (4.8b) and (4.9b) are given by (3.14) and (3.15).

Theorem 4.1 follows from Theorem 3.1 and Proposition 3.1.

For $d\ge2$ Theorem 4.1 and the methods for the reconstruction of f from Pf permit the reconstruction of F and v from scattering data at high energies.

5. The inverse scattering and inverse boundary value problems

Let $D\subset\mathbf{R}^d$ be a bounded, open, strictly convex (in the strong sense) domain with smooth (say, infinitely smooth) boundary ∂D (without singular points). Let $\bar D=D\cup\partial D$. Consider the equation (1.1) in D, where

$$v\in C^n(\bar D,\mathbf{R}),\quad n=3, \tag{5.1a}$$

or

$$v\in C_0^n(\bar D,\mathbf{R}),\quad n=2, \tag{5.1b}$$

($C_0^n(\bar D,\mathbf{R})=\{v\in C^n(\bar D,\mathbf{R})\,|\,\operatorname{supp} v\subset D\}$).

Under the conditions (5.1a), at fixed sufficiently large E (i.e. $E>E(v,D)>\sup_{x\in\bar D} v(x)$) solutions $x(t)$ of the equation (1.1) in D have the following properties (see [GN]):

$$\begin{aligned}&\text{for each solution } x(t) \text{ there are } t_1,t_2\in\mathbf{R},\ t_1<t_2, \text{ such that}\\ &x\in C^{n+1}([t_1,t_2],\mathbf{R}^d),\ x(t_1),x(t_2)\in\partial D,\ x(t)\in D \text{ for } t\in]t_1,t_2[,\\ &x(s_1)\neq x(s_2) \text{ for } s_1,s_2\in[t_1,t_2],\ s_1\neq s_2;\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\text{for any two points } q_0,q\in\bar D,\ q_0\neq q, \text{ there is one and only one solution}\\ &x(t)=x(t,E,q_0,q) \text{ such that } x(0)=q_0,\ x(s)=q \text{ for some } s>0;\\ &\dot x(0,E,q_0,q)\in C^{n-1}((\bar D\times\bar D)\backslash\bar G,\mathbf{R}^d), \text{ where } \bar G \text{ is the diagonal in } \bar D\times\bar D.\end{aligned} \tag{5.3}$$

Remark 5.1. In this statement one can replace the conditions (5.1a) by the conditions (5.1b).

Let $E>E(v,D)$. Consider the solution $x(t,E,q_0,q)$ from (5.3) for $q_0,q\in\partial D$, $q_0\neq q$. Let $s=s(E,q_0,q)$ be defined as the root of the equation

$$x(s,E,q_0,q)=q,\quad s>0.$$

Let $k_0(E,q_0,q)=\dot x(0,E,q_0,q)$, $k(E,q_0,q)=\dot x(s(E,q_0,q),E,q_0,q)$.

The functions $s(E,q_0,q)$, $k(E,q_0,q)$ for $E>E(v,D)$, $(q_0,q)\in(\partial D\times\partial D)\backslash\partial G$, were taken as boundary value data in [GN].

For $d\geq 2$, under the conditions (5.1a), using the Maupertuis principle and results of [BG2] it was shown in [GN] that $s(E,q_0,q)$ on $\partial D\times\partial D$ at high energies E uniquely determines $v(x)$ in D and that $k(E,q_0,q)$ on $(\partial D\times\partial D)\backslash\partial G$ at fixed energy $E>E(v,D)$ uniquely determines $v(x)$ in D.

Remark 5.2. In these results from [GN] one can replace the conditions (5.1a) by the conditions (5.1b).

Note that

$$
\begin{aligned}
|k_0(E,q_0,q)| &= \sqrt{2(E-v(q_0))},\\
|k(E,q_0,q)| &= \sqrt{2(E-v(q))},\\
k_0(E,q_0,q) &= -k(E,q,q_0),
\end{aligned}
\tag{5.4}
$$

$E>E(v,D)$, $(q,q_0)\in(\partial D\times\partial D)\backslash\partial G$.

For $x\in\partial D$ we define

$$
\begin{aligned}
\Theta_x^- &= \{\theta\in S^{d-1}\mid x+t\theta\in D \text{ for } t\in]0,\varepsilon[\text{ for some } \varepsilon>0\},\\
\Theta_x^+ &= \{\theta\in S^{d-1}\mid x+t\theta\in \mathbf{R}^d\backslash\bar{D} \text{ for } t>0\}.
\end{aligned}
$$

Consider the functions $v(q_0)$, $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$, $s(E,q_0,\theta_0)$ for $E>E(v,D)$, $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$, where $q=q(E,q_0,\theta_0)$ is defined as the root of the equation

$$k_0(E,q_0,q)=\sqrt{E-v(q_0)}\,\theta_0,\quad q\in\partial D\backslash q_0,$$

and

$$
\begin{aligned}
\theta(E,q_0,\theta_0) &= k(E,q_0,q(E,q_0,\theta_0))/\sqrt{2(E-v(q(E,q_0,\theta_0)))},\\
s(E,q_0,\theta_0) &= s(E,q_0,q(E,q_0,\theta_0)).
\end{aligned}
$$

One can take the functions $v(q_0)$, $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$, $s(E,q_0,\theta_0)$, $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$, as boundary value data instead of the functions $s(E,q_0,q)$, $k(E,q_0,q)$, $(q_0,q)\in(\partial D\times\partial D)\backslash\partial G$, $E>E(v,D)$.

Lemma 5.1. *Under the conditions* (5.1a), (*or* (5.1b)), *at fixed* $E>E(v,D)$ *the functions* $s(E,q_0,q)$, $k(E,q_0,q)$, $(q_0,q)\in(\partial D\times\partial D)\backslash\partial G$, *uniquely determine the functions* $v(q_0)$, $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$, $s(E,q_0,\theta_0)$, $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$ *and vice versa.*

The direct statement of Lemma 5.1 follows from (5.4) and the definition of $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$ and $s(E,q_0,\theta_0)$.

To determine the functions in the converse statement we proceed in the following way: we determine $\theta_0=\theta_0(E,q_0,q)$ as the root of the equation

$$q(E,q_0,\theta_0)=q,\quad \theta_0\in\Theta_{q_0}^-$$

and we use the formulas

$$k(E,q_0,q)=\sqrt{2(E-v(q))}\,\theta(E,q_0,\theta_0(E,q_0,q)),$$
$$s(E,q_0,q)=s(E,q_0,\theta_0(E,q_0,q)).$$

Let us consider $C_0^2(\bar{D},\mathbf{R})$ as a subspace of $C_0^2(\mathbf{R}^2,\mathbf{R})$ (extending each function f from $C_0^2(\bar{D},\mathbf{R})$ by zero outside of $\bar{D}$).

Let $v\in C_0^2(\bar{D},\mathbf{R})\subset C_0^2(\mathbf{R}^d,\mathbf{R})$. Consider for the function v the boundary value data $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$, $s(E,q_0,\theta_0)$, $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$ (note that in this case $v(q_0)\equiv 0$, $q_0\in\partial D$); consider for v also the scattering data (as defined in Section 1) $a(p_-,x_-)$, $b(p_-,x_-)$, $(p_-,x_-)\in\mathcal{M}_E$ where $E>E(v,D)$.

Theorem 5.1. *Let $v\in C_0^2(\bar{D},\mathbf{R})\subset C_0^2(\mathbf{R}^d,\mathbf{R})$. Then at fixed $E>E(v,D)$ the scattering data $a(p_-,x_-)$, $b(p_-,x_-)$, $(p_-,x_-)\in\mathcal{M}_E$, uniquely determine the boundary value data $q(E,q_0,\theta_0)$, $\theta(E,q_0,\theta_0)$, $s(E,q_0,\theta_0)$, $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$, and vice versa.*

Proof. I. The direct statement.

Consider the system of equations

$$p_-t+x_-=q_0,\quad p_-/\sqrt{2E}=\theta_0 \tag{5.5}$$

for determination of $(p_-,x_-)\in\mathcal{M}_E$ and $t\in\mathbf{R}$ through $q_0\in\partial D$, $\theta_0\in\Theta_{q_0}^-$. One can solve this system by the formulas

$$p_-=\sqrt{2E}\,\theta_0,\quad x_-=q_0-(q_0\theta_0)\theta_0,\quad t=t_-(E,q_0,\theta_0)=(q_0\theta_0)/\sqrt{2E}\,. \tag{5.6}$$

Consider the equation

$$at+b\in\partial D \tag{5.7}$$

for determination of $t\in\mathbf{R}$ through

$$(a,b)\in\Sigma_E=\left\{(a,b)\in\mathbf{R}^{2d}\mid \tfrac{1}{2}a^2=E\right\}\quad\text{and}\quad\partial D.$$

There are functions $\chi(a,b)$, $\tau_\pm(a,b)$, $\gamma_\pm(a,b)$ depending on ∂D such that $\chi(a,b)=0$ if and only if the equation (5.7) has no solutions, $\chi(a,b)=1$ if and only if (5.7) has one and only one solution, $\chi(a,b)=2$ if and only if (5.7) has two different solutions; the functions $\tau_\pm(a,b)$ are defined if and only if $\chi(a,b)>0$, in addition, the function $\tau_-(a,b)$ denotes the minimal solution of (5.7) and the function $\tau_+(a,b)$ denotes the maximal solution of (5.7); $\gamma_\pm(a,b)=a\tau_\pm(a,b)+b$.

Let us observe that if $x(t)$ is the solution of (1.1) with initial data (1.3), where (p_-, x_-) is given by (5.6), then the equation

$$x(t) \in \partial D \quad \text{for } t \in \mathbf{R}$$

has two different solutions $t{=}t_-(E, q_0, \theta_0)$ and $t{=}\tau_+(a(p_-, x_-), b(p_-, x_-))$.

Using this observation and the formulas for solving the systems (5.5) and (5.7) we obtain the formulas

$$\begin{aligned} q(E, q_0, \theta_0) &= \gamma_+(a(p_-, x_-), b(p_-, x_-)), \\ \theta(E, q_0, \theta_0) &= a(p_-, x_-)/\sqrt{2E}\,, \\ s(E, q_0, \theta_0) &= \tau_+(a(p_-, x_-), b(p_-, x_-)) - t_-(E, q_0, \theta_0), \end{aligned}$$

where

$$p_- = \sqrt{2E}\,\theta_0, \quad x_- = q_0 - (q_0\theta_0)\theta_0.$$

Thus the direct statement is proved.

II. The converse statement.

For determination of the functions in the converse statement there are the formulas for $(p_-, x_-) \in \mathcal{M}_E$,

$$\begin{cases} a(p_-, x_-) = p_-, \\ b(p_-, x_-) = x_-, \end{cases}$$

if $\chi(p_-, x_-) \le 1$,

$$\begin{cases} a(p_-, x_-) = \sqrt{2E}\,\theta\big(E, \gamma_-(p_-, x_-), p_-/\sqrt{2E}\,\big), \\ b(p_-, x_-) = q\big(E, \gamma_-(p_-, x_-), p_-/\sqrt{2E}\,\big) \\ \qquad\qquad - \big(\tau_-(p_-, x_-) + s\big(E, \gamma_-(p_-, x_-), p_-/\sqrt{2E}\,\big)\big)a(p_-, x_-), \end{cases}$$

if $\chi(p_-, x_-) = 2$.

The proof of Theorem 5.1 is completed.

As a corollary of Remark 5.2, Lemma 5.1 and Theorem 5.1 we obtain the following result.

Theorem 5.2. *Let $v \in C_0^2(\bar{D}, \mathbf{R}) \subset C_0^2(\mathbf{R}^d, \mathbf{R})$, $d \ge 2$. Then at fixed $E > E(v, D)$ the scattering data $a(p_-, x_-)$, $b(p_-, x_-)$, $(p_-, x_-) \in \mathcal{M}_E$, and the domain D uniquely determine v.*

Remark 5.3. Suppose that v satisfies the conditions (5.1a) and $v|_{\partial D} \not\equiv 0$. Let us extend v by zero outside $\bar{D}$. Consider the scattering data for v (generalizing the definition from Section 1). In this case the formulas connecting the scattering data and the boundary value data are more complicated than in Theorem 5.1, one needs to take into account the boundary refraction.

6. Appendix: Proofs of Lemmas 2.1, 2.2 and 2.3

Proof of Lemma 2.1. The property

$$A(f)\in C^1(]-\infty,T],\mathbf{R}^d)\quad \text{for } f\in M_{T,r}\ (0\le r\le 1,\ r<|p_-|/\sqrt{2}) \tag{6.1}$$

follows from (2.2) and (1.2).

Consider

$$\begin{aligned} A_j(f)(t) &= \int_{-\infty}^{t}\int_{-\infty}^{\tau} F_j(x_-+p_-s+f(s))\,ds\,d\tau,\\ \frac{d}{dt}A_j(f)(t) &= \int_{-\infty}^{t} F_j(x_-+p_-s+f(s))\,ds \end{aligned}\qquad \text{for } f\in M_{T,r}. \tag{6.2}$$

We shall prove that

$$\left|\frac{d}{dt}A_j(f)(t)\right| \le \frac{c_1 2^{\alpha+1}}{\alpha\big(|p_-|/\sqrt{2}-r\big)\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha}} \tag{6.3a}$$

for $t\le T$, $t\le 0$;

$$\left|\frac{d}{dt}A_j(f)(t)\right| \le \frac{c_1 2^{\alpha+2}}{\alpha\big(|p_-|/\sqrt{2}-r\big)\big(1+|x_-|/\sqrt{2}\big)^{\alpha}} \tag{6.3b}$$

for $t\le T$ (without assuming that $t\le 0$).

From (6.2) and (1.2) it follows that

$$\left|\frac{d}{dt}A_j(f)(t)\right| \le c_1\int_{-\infty}^{t}(1+|x_-+p_-s+f(s)|)^{-(\alpha+1)}\,ds. \tag{6.4}$$

If $f\in M_{T,r}$, then

$$|f(s)|\le r|s|+r\quad \text{for } s\le T. \tag{6.5}$$

Let

$$f(s)=g_1(s)+g_2(s), \tag{6.6}$$

where

$$g_1(s)=(1+|s|)^{-1}f(s),\quad g_2(s)=|s|(1+|s|)^{-1}f(s).$$

From (6.5) and (6.6) it follows that

$$|g_1(s)|\le r,\ |g_2(s)|\le r|s|\quad \text{for } s\le T. \tag{6.7}$$

As $x_-p_-=0$,

$$|x_-+p_-s|\geq\big(1/\sqrt{2}\big)(|x_-|+|p_-|\,|s|). \tag{6.8}$$

From (6.6), (6.7) and (6.8) it follows that

$$\begin{aligned}2(1+|x_-+p_-s+f(s)|)&\geq 2+|x_-+p_-s+g_1(s)+g_2(s)|\\&\geq 2-r+|x_-+p_-s+g_2(s)|\\&\geq 1+|x_-|/\sqrt{2}+(|p_-|/\sqrt{2}-r)|s|\end{aligned} \tag{6.9}$$

for $s\leq T$ $(0\leq r\leq 1,\ r<|p_-|/\sqrt{2})$. From (6.4) and (6.9) it follows that

$$\left|\frac{d}{dt}A_j(f)(t)\right|\leq c_1 2^{\alpha+1}\int_{-\infty}^{t}\big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)|s|\big)^{-(\alpha+1)}\,ds \tag{6.10}$$

for $t\leq T$.

If $a>0$, $b>0$, $\beta>1$, then

$$\begin{aligned}&\int_{-\infty}^{t}(a+b|s|)^{-\beta}\,ds=\frac{1}{(\beta-1)b(a-bt)^{\beta-1}} && \text{for } t\leq 0;\\&\int_{-\infty}^{t}(a+b|s|)^{-\beta}\,ds=\frac{2}{(\beta-1)ba^{\beta-1}}-\frac{1}{(\beta-1)b(a+bt)^{\beta-1}}\\&\qquad\qquad\leq\frac{2}{(\beta-1)ba^{\beta-1}} && \text{for } 0\leq t.\end{aligned} \tag{6.11}$$

The formulas (6.3) follow from (6.10) and (6.11).

We shall prove that

(6.12a)

$$\left|A_j(f)(t)-t\frac{d}{dt}A_j(f)(t)\right|\leq\frac{c_1 2^{\alpha+1}}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}$$

for $t\leq T$, $t\leq 0$,

$$\left|A_j(f)(t)-t\frac{d}{dt}A_j(f)(t)\right|\leq\frac{c_1 2^{\alpha+2}}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}} \tag{6.12b}$$

for $t\leq T$.

For $t\leq 0$ we shall use that

$$\left|A_j(f)(t)-t\frac{d}{dt}A_j(f)(t)\right|\leq|A_j(f)(t)|+\left|t\frac{d}{dt}A_j(f)(t)\right|. \tag{6.13}$$

From (6.2) and (6.10) it follows that

$$(6.14)\quad |A_j(f)(t)| \le c_1 2^{\alpha+1} \int_{-\infty}^{t} \int_{-\infty}^{\tau} \bigl(1+|x_-|/\sqrt{2} + \bigl(|p_-|/\sqrt{2} - r\bigr)|s|\bigr)^{-(\alpha+1)}\, ds\, d\tau$$

for $t \le T$.

If $a>0$, $b>0$, $\beta>2$, then

$$(6.15)\quad \int_{-\infty}^{t} \int_{-\infty}^{\tau} (a+b|s|)^{-\beta}\, ds\, d\tau = \frac{1}{(\beta-1)(\beta-2)b^2(a-bt)^{\beta-2}}$$

for $t \le 0$.

From (6.14) and (6.15) it follows that

$$(6.16)\quad |A_j(f)(t)| \le \frac{c_1 2^{\alpha+1}}{\alpha(\alpha-1)\bigl(|p_-|/\sqrt{2} - r\bigr)^2 \bigl(1+|x_-|/\sqrt{2} - \bigl(|p_-|/\sqrt{2} - r\bigr)t\bigr)^{\alpha-1}}$$

for $t \le T$, $t \le 0$.

From (6.3a) it follows that

$$(6.17)\quad \left| t \frac{d}{dt} A_j(f)(t) \right| \le \frac{c_1 2^{\alpha+1}}{\alpha\bigl(|p_-|/\sqrt{2} - r\bigr)^2 \bigl(1+|x_-|/\sqrt{2} - \bigl(|p_-|/\sqrt{2} - r\bigr)t\bigr)^{\alpha-1}}$$

for $t \le T$, $t \le 0$.

The estimate (6.12a) follows from (6.13), (6.16) and (6.17). For $0 \le t \le T$ we shall use the formulas

(6.18)

$$\begin{aligned} A_j(f)(t) &= \int_{-\infty}^{0} \int_{-\infty}^{\tau} F_j(x_- + p_- s + f(s))\, ds\, d\tau + \int_0^t \int_{-\infty}^{\tau} F_j(x_- + p_- s + f(s))\, ds\, d\tau \\ &= A_j(f)(0) + \int_0^t \int_{-\infty}^{t} F_j(x_- + p_- s + f(s))\, ds\, d\tau \\ &\quad - \int_0^t \int_{\tau}^{t} F_j(x_- + p_- s + f(s))\, ds\, d\tau, \end{aligned}$$

$$(6.19)\quad A_j(f)(t) - t\frac{d}{dt} A_j(f)(t) = A_j(f)(0) - \int_0^t \int_{\tau}^{t} F_j(x_- + p_- s + f(s))\, ds\, d\tau.$$

For $A_j(f)(0)$ we use the estimate (6.16), i.e.

$$(6.20)\quad |A_j(f)(0)| \le \frac{c_1 2^{\alpha+1}}{\alpha(\alpha-1)\bigl(|p_-|/\sqrt{2} - r\bigr)^2 \bigl(1+|x_-|/\sqrt{2}\bigr)^{\alpha-1}}.$$

We estimate the second term on the right-hand side of (6.19) in the following way,

$$\begin{aligned}&\left|\int_0^t\int_\tau^t F_j(x_-+p_-s+f(s))\,ds\,d\tau\right| \\ &\qquad \le c_1 2^{\alpha+1}\int_0^t\int_\tau^t\left(1+|x_-|/\sqrt{2}+\left(|p_-|/\sqrt{2}-r\right)s\right)^{-(\alpha+1)}ds\,d\tau.\end{aligned}\tag{6.21}$$

If $a>0$, $b>0$, $\beta>2$, then

$$\begin{aligned}\int_0^t\int_\tau^t(a+bs)^{-\beta}\,ds\,d\tau &= \frac{1}{(\beta-1)(\beta-2)b^2a^{\beta-2}} \\ &\quad -\frac{1}{(\beta-1)(\beta-2)b^2(a+bt)^{\beta-2}}-\frac{t}{(\beta-1)b(a+bt)^{\beta-1}} \\ &\le \frac{1}{(\beta-1)(\beta-2)b^2a^{\beta-2}}\quad \text{for } t\ge 0.\end{aligned}\tag{6.22}$$

Thus,

$$\left|\int_0^t\int_\tau^t F_j(x_-+p_-s+f(s))\,ds\,d\tau\right| \le \frac{c_1 2^{\alpha+1}}{\alpha(\alpha-1)\left(|p_-|/\sqrt{2}-r\right)^2\left(1+|x_-|/\sqrt{2}\right)^{\alpha-1}}.\tag{6.23}$$

The estimate (6.12b) follows from (6.12a), (6.19), (6.20) and (6.23).

From (6.3) and (6.12) it follows that

$$\begin{aligned}&\max\left(\left|\frac{d}{dt}A(f)(t)\right|,\left|A(f)(t)-t\,\frac{d}{dt}A(f)(t)\right|\right) \\ &\qquad \le \frac{dc_1 2^{\alpha+1}\left(|p_-|/\sqrt{2}+1-r\right)}{(\alpha-1)\left(|p_-|/\sqrt{2}-r\right)^2\left(1+|x_-|/\sqrt{2}-\left(|p_-|/\sqrt{2}-r\right)t\right)^{\alpha-1}}\end{aligned}\tag{6.24a}$$

for $t\le T$, $t\le 0$;

$$\begin{aligned}&\max\left(\left|\frac{d}{dt}A(f)(t)\right|,\left|A(f)(t)-t\,\frac{d}{dt}A(f)(t)\right|\right) \\ &\qquad \le \frac{dc_1 2^{\alpha+1}\left(|p_-|/\sqrt{2}+1-r\right)}{(\alpha-1)\left(|p_-|/\sqrt{2}-r\right)^2\left(1+|x_-|/\sqrt{2}\right)^{\alpha-1}}\end{aligned}\tag{6.24b}$$

for $t\le T$.

The statements of (2.6) follow from (6.1) and (6.24).

Consider now $A_j(f_2)(t)-A_j(f_1)(t)$, $(d/dt)(A_j(f_2)(t)-A_j(f_1)(t))$ for $f_1,f_2\in M_{T,r}$ $(0\le r\le 1,\ r<|p_-|/\sqrt{2})$.

We shall prove that

(6.25a)
$$\left|\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right| \le \frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^\alpha}$$

for $t\le T$, $t\le 0$;

(6.25b)
$$\left|\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right| \le \frac{dc_2 2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha\big(|p_-|/\sqrt{2}-r\big)^2\big(1+|x_-|/\sqrt{2}\big)^\alpha}$$

for $t\le T$.

We shall start with the estimates

(6.26)
$$\left|\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right| \le \int_{-\infty}^{t} |F_j(x_-+p_-s+f_2(s))-F_j(x_-+p_-s+f_1(s))|\, ds;$$

(6.27)
$$\begin{aligned}&|F_j(x_-+p_-s+f_2(s))-F_j(x_-+p_-s+f_1(s))|\\ &\qquad\le \max_{\varepsilon\in[0,1]} |\nabla F_j(x_-+p_-s+(1-\varepsilon)f_1(s)+\varepsilon f_2(s))|\,|f_2(s)-f_1(s)|;\end{aligned}$$

(6.28)
$$\begin{aligned}&|\nabla F_j(x_-+p_-s+(1-\varepsilon)f_1(s)+\varepsilon f_2(s))|\\ &\qquad\overset{(1.2)}{\le} dc_2(1+|x_-+p_-s+(1-\varepsilon)f_1(s)+\varepsilon f_2(s)|)^{-(\alpha+2)}.\end{aligned}$$

Further, for $s\le T$, $\varepsilon\in[0,1]$ the following estimates are valid

(6.29)
$$|(1-\varepsilon)f_1(s)+\varepsilon f_2(s)|\le r(1+|s|),$$

(6.30)
$$2(1+|x_-+p_-s+(1-\varepsilon)f_1(s)+\varepsilon f_2(s)|)\ge 1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)|s|,$$

(6.31)
$$\begin{aligned}&|\nabla F_j(x_-+p_-s+(1-\varepsilon)f_1(s)+\varepsilon f_2(s))|\\ &\qquad\le dc_2 2^{\alpha+2}\big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)|s|\big)^{-(\alpha+2)},\end{aligned}$$

(6.32)
$$|f_2(s)-f_1(s)|\le \|f_2-f_1\|_T(1+|s|).$$

Using (6.26), (6.27), (6.31) and (6.32), we obtain

(6.33)
$$\begin{aligned}&\left|\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right|\\ &\le dc_2 2^{\alpha+2}\int_{-\infty}^{t}\big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)|s|\big)^{-(\alpha+2)}(1+|s|)\, ds\|f_2-f_1\|_T.\end{aligned}$$

If $a>0$, $b>0$, $\beta>2$, then

(6.34)
$$\int_{-\infty}^{t}(a+b|s|)^{-\beta}(1+|s|)\, ds\le \int_{-\infty}^{t}(a+b|s|)^{-\beta}\, ds+\int_{-\infty}^{t} b^{-1}(a+b|s|)^{-(\beta-1)}\, ds.$$

Using (6.34) we obtain for $a\geq 1$,

(6.35)
$$\int_{-\infty}^{t}(a+b|s|)^{-\beta}(1+|s|)\,ds\leq\frac{1}{(\beta-1)b(a-bt)^{\beta-1}}+\frac{1}{(\beta-2)b^2(a-bt)^{\beta-2}}$$
$$\leq\frac{b+1}{(\beta-2)b^2(a-bt)^{\beta-2}},\qquad t\leq 0,$$
$$\int_{-\infty}^{t}(a+b|s|)^{-\beta}(1+|s|)\,ds\leq\frac{2(b+1)}{(\beta-2)b^2a^{\beta-2}},\qquad t\geq 0.$$

The estimates (6.25) follow from (6.33) and (6.35).

We shall prove that

(6.36a)
$$\left|A_j(f_2)(t)-A_j(f_1)(t)-t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right|$$
$$\leq\frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}$$

for $t\leq T$, $t\leq 0$;

(6.36b)
$$\left|A_j(f_2)(t)-A_j(f_1)(t)-t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right|$$
$$\leq\frac{dc_2 2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}$$

for $t\leq T$.

For $t\leq 0$ we shall use that

(6.37)
$$\left|A_j(f_2)(t)-A_j(f_1)(t)-t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right|$$
$$\leq|A_j(f_2)(t)-A_j(f_1)(t)|+\left|t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right|.$$

From (6.2) and (6.25a) we obtain

(6.38)
$$|A_j(f_2)(t)-A_j(f_1)(t)|\leq\frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)}{\alpha\big(|p_-|/\sqrt{2}-r\big)^2}$$
$$\times\int_{-\infty}^{t}\frac{d\tau}{\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)\tau\big)^{\alpha}}\|f_2-f_1\|_T$$
$$\overset{(6.11)}{=}\frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}$$

for $t \le T$, $t \le 0$. From (6.25a) we obtain

(6.39)
$$\left| t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right| \le \frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}$$

for $t \le T$, $t \le 0$.

The estimate (6.36a) follows from (6.37), (6.38) and (6.39).

For $0 \le t \le T$ using (6.19) we obtain

$$\begin{aligned}\left|A_j(f_2)(t)-A_j(f_1)(t)-t\,\frac{d}{dt}(A_j(f_2)(t)-A_j(f_1)(t))\right| &\le |A_j(f_2)(0)-A_j(f_1)(0)| \\ &+\left|\int_0^t\int_\tau^t (F_j(x_-+p_-s+f_2(s))-F_j(x_-+p_-s+f_1(s)))\,ds\,d\tau\right|. \end{aligned} \tag{6.40}$$

Due to (6.38) we have

$$|A_j(f_2)(0)-A_j(f_1)(0)| \le \frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}. \tag{6.41}$$

Due to (6.27), (6.31) and (6.32) the second term on the right-hand side of (6.40) admits the estimate

$$\begin{aligned}&\left|\int_0^t\int_\tau^t (F_j(x_-+p_-s+f_2(s))-F_j(x_-+p_-s+f_1(s)))\,ds\,d\tau\right| \\ &\le dc_2 2^{\alpha+2}\int_0^t\int_\tau^t \big(1+|x_-|/\sqrt{2}+\big(|p_-|/\sqrt{2}-r\big)s\big)^{-(\alpha+2)}(1+s)\,ds\|f_2-f_1\|. \end{aligned} \tag{6.42}$$

If $a>0$, $b>0$, $\beta>3$, then

(6.43)
$$\int_0^t\int_\tau^t (a+bs)^{-\beta}(1+s)\,ds\,d\tau \le \int_0^t\int_\tau^t (a+bs)^{-\beta}\,ds\,d\tau + \int_0^t\int_\tau^t b^{-1}(a+bs)^{-(\beta-1)}\,ds\,d\tau.$$

Using (6.43) we obtain

$$\begin{aligned}\int_0^t\int_\tau^t (a+bs)^{-\beta}(1+s)\,ds\,d\tau &\le \frac{1}{(\beta-1)(\beta-2)b^2a^{\beta-2}}+\frac{1}{(\beta-2)(\beta-3)b^3a^{\beta-3}} \\ &\le \frac{b+1}{(\beta-2)(\beta-3)b^3a^{\beta-3}} \end{aligned} \tag{6.44}$$

for $t \ge 0$, $a \ge 1$. Thus,

$$\begin{aligned}&\left|\int_0^t\int_\tau^t (F_j(x_-+p_-s+f_2(s))-F_j(x_-+p_-s+f_1(s)))\,ds\,d\tau\right| \\ &\qquad\le \frac{dc_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)\|f_2-f_1\|_T}{\alpha(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}. \end{aligned} \tag{6.45}$$

The estimate (6.36b) follows from (6.36a), (6.40), (6.41) and (6.45).

From (6.25) and (6.36) it follows that

(6.46a)
$$\max\left(\left|\frac{d}{dt}(A(f_2)(t)-A(f_1)(t))\right|,\left|A(f_2)(t)-A(f_1)(t)-t\,\frac{d}{dt}(A(f_2)(t)-A(f_1)(t))\right|\right)$$
$$\leq\frac{d^2c_2 2^{\alpha+2}\big(|p_-|/\sqrt{2}+1-r\big)^2\|f_2-f_1\|_T}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}-\big(|p_-|/\sqrt{2}-r\big)t\big)^{\alpha-1}}$$

for $t\leq T$, $t\leq 0$;

(6.46b)
$$\max\left(\left|\frac{d}{dt}(A(f_2)(t)-A(f_1)(t))\right|,\left|A(f_2)(t)-A(f_1)(t)-t\,\frac{d}{dt}(A(f_2)(t)-A(f_1)(t))\right|\right)$$
$$\leq\frac{d^2c_2 2^{\alpha+3}\big(|p_-|/\sqrt{2}+1-r\big)^2\|f_2-f_1\|_T}{(\alpha-1)\big(|p_-|/\sqrt{2}-r\big)^3\big(1+|x_-|/\sqrt{2}\big)^{\alpha-1}}$$

for $t\leq T$.

The statements of (2.7) follow from (6.46). Lemma 2.1 is proved.

Proof of Lemma 2.2. The estimates (2.9) and (2.10) follow immediately from (6.3a) and (6.16). Further, we have

(6.47)
$$\begin{aligned}A_j(f)(t)&=\int_{-\infty}^{0}\int_{-\infty}^{\tau}F_j(x_-+p_-s+f(s))\,ds\,d\tau+\int_0^t\int_{-\infty}^{\tau}F_j(x_-+p_-s+f(s))\,ds\,d\tau\\&=\int_{-\infty}^{0}\int_{-\infty}^{\tau}F_j(x_-+p_-s+f(s))\,ds\,d\tau+\int_0^t\int_{-\infty}^{+\infty}F_j(x_-+p_-s+f(s))\,ds\,d\tau\\&\quad-\int_0^t\int_{\tau}^{+\infty}F_j(x_-+p_-s+f(s))\,ds\,d\tau\\&=t\int_{-\infty}^{+\infty}F_j(x_-+p_-s+f(s))\,ds+\int_{-\infty}^{0}\int_{-\infty}^{\tau}F_j(x_-+p_-s+f(s))\,ds\,d\tau\\&\quad-\int_0^{+\infty}\int_{\tau}^{+\infty}F_j(x_-+p_-s+f(s))\,ds\,d\tau\\&\quad+\int_t^{+\infty}\int_{\tau}^{+\infty}F_j(x_-+p_-s+f(s))\,ds\,d\tau.\end{aligned}$$

Thus, we obtain (2.11) and (2.12), where

$$H(f)(t)=\int_t^{+\infty}\int_{\tau}^{+\infty}F(x_-+p_-s+f(s))\,ds\,d\tau. \tag{6.48}$$

Using (2.12), (6.48), (6.2), (6.3a) and (6.16) and the observations

$$\int_{\tau}^{+\infty} \varphi(s)\, ds = \int_{-\infty}^{-\tau} \varphi(-s)\, ds, \quad \tau \geq 0, \tag{6.49}$$
$$f \in M_{T,r},\ T=+\infty,\ g(s)=f(-s) \quad \Longrightarrow \quad g \in M_{T,r}$$

we obtain the estimates (2.13), (2.14) and (2.15).

Lemma 2.2 is proved.

Proof of Lemma 2.3. Using (2.2) and (2.6b) we obtain

$$\|y_- - 0\|_T = \|y_-\|_T \leq \varrho(d,c,\alpha,|p_-|,|x_-|,r), \quad T=+\infty. \tag{6.50}$$

Using (2.12), (6.2), (6.25a) and (6.38) for $t=0$, (6.49), (6.50) we obtain the estimates (2.16).

Lemma 2.3 is proved.

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