Ark. Mat., **37** (1999), 211–220

On Rebelo's theorem on singularities of holomorphic flows

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Introduction

A holomorphic vector field Z on a complex manifold M is said to be $\mathbf{R}^+$ complete, resp. $\mathbf{R}$ complete or $\mathbf{C}$ complete if the initial value problem

$$\phi(0)=p, \quad \phi'(t)=Z(\phi(t))$$

can be solved in forward time, $t>0$, resp. in real time, $-\infty<t<+\infty$, or in complex time, $t\in\mathbf{C}$. Of course, complete in complex time implies complete in real time implies complete in positive time. On any Stein manifold that does not support any bounded, non-constant, plurisubharmonic function, complete in positive time implies complete in complex time ([1], generalizing [9]). In some sense, fields complete in positive time are much more abundant than those complete in real time. For example, in the unit disk, among non-constant fields vanishing at the origin, only the rotation fields are complete in real time but any small perturbation of the field $Z(\zeta)=-\zeta$ is complete in positive time. Rebelo's theorem says the following.

Theorem. (Rebelo [13].) *If a $\mathbf{C}$ complete holomorphic vector field on a two dimensional complex manifold has an isolated zero at some point p, then at this point the two jet of the field is not zero.*

Our goal is to show that there are several ways to easily strengthen this result. We will use the following notation: if Z is a holomorphic vector field defined near a point p in some complex manifold, $J_k(Z,p)$ will denote the k jet of Z at p.

Proposition 1. *Let M be a complex manifold of dimension two. Let Z be an $\mathbf{R}^+$ complete holomorphic vector field on M. Assume that Z has an isolated zero at p. Then $J_2(Z,p)\neq 0$. If $J_1(Z,p)=0$ then there is an embedded Riemann sphere Σ in M such that $p\in\Sigma$ and Z is tangent to Σ.*

Proposition 2 deals with vector fields which may not be complete, but which are limits of complete ones (for more on this topic see [5] and [8]), and it is also a more local statement.

Proposition 2. *Let M be a two dimensional complex manifold and let $p\in M$. Assume that Z is a holomorphic vector field defined on some neighborhood V of $p\in M$. Assume that Z is the uniform limit on compact sets in V of a sequence of $\mathbf{R}^+$ complete holomorphic vector fields defined on M, and that Z has an isolated zero at p. Then $J_2(Z,p)\neq 0$. If $J_1(Z,p)=0$ then there is a (germ of a) smooth holomorphic curve C through p such that Z is tangent to C and every holomorphic function on M is constant on C.*

Even if $V=M$ we have not been able to show the existence of an embedded Riemann sphere as in Proposition 1.

Since writing this paper we became aware of the paper [11] by Ghys and Rebelo. In [11], the authors obtain deep results on "semi-complete" holomorphic vector fields (that we prefer to call monodromic vector fields). They have a complete classification of their germs, at a point where their first jet is trivial. Using the Enriques–Kodaira classification of compact surfaces they show that only Hirzebruch surfaces F_n can carry holomorphic vector fields with isolated zero of order 2. Our paper is much more elementary in nature. The proof of Proposition 1 is in fact extremely short (see also [11, top of p. 1172]). Our results are in a somewhat different direction.

Remarks and questions

The above results are somewhat in contrast with the fact that, from another point of view, there are many complete holomorphic vector fields on $\mathbf{C}^n$. Indeed every polynomial vector field on $\mathbf{C}^n$ is a finite sum of complete polynomial vector fields. This fact, implicit in the Andersen–Lempert theory [3] and [4], was made explicit in [10].

The hypothesis of isolated zero is of course crucial: the vector field $(0, z_1^N)$ is indeed a $\mathbf{C}$ complete field on $\mathbf{C}^2$ (for any $N\in\mathbf{N}$).

A natural question is, of course, whether a $\mathbf{C}$ complete holomorphic vector field on a two dimensional Stein manifold can have two distinct isolated zeros? (It cannot have one of order ≥ 2.) The answer is yes (see the example after the proof of Proposition 2). However we do not know if this can happen in $\mathbf{C}^2$. Perhaps it is worthwhile at this point to recall the work of M. Suzuki, [13], [15]. In these papers

Suzuki gives a characterization (up to conjugation) of all polynomial flows and all 'proper' flows on $\mathbf{C}^2$ (see [15] for a definition of proper). These two lists, taken together, seem to include all known flows on $\mathbf{C}^2$. None of them has more than one isolated fixed point.

The Camacho–Sad theorem

This is the theorem which, as Rebelo saw, allows one to reduce two dimensional problems to easy one dimensional ones.

Theorem. (Camacho–Sad [6].) *Let M be a two dimensional complex manifold and $p\in M$. Let Z be a holomorphic vector field defined near p. Then there exists an irreducible one dimensional analytic set A, defined in a neighborhood of p, such that $p\in A$ and Z is tangent to $A\setminus\{p\}$ (A may have a singularity at p).*

A Camacho–Sad manifold through p

Here we assume that M is a two dimensional complex manifold and that Z is a holomorphic vector field defined on M with an isolated zero at $p\in M$. Let A be an analytic set (defined near p) as given by the Camacho–Sad theorem. By shrinking A if necessary, we assume that p is the only stationary point of Z in A. Pick $q\in A\setminus\{p\}$. Let L_q be its complex orbit. Then L_q is a one dimensional complex manifold in M (possibly not a closed submanifold). Of course, L_q is independent of q. Set $L=L_q\cup\{p\}$, and extend the topology of L_q to a topology on L by considering A to be a neighborhood of p in L. (This may not be the topology induced from M.) Finally, L has a natural structure as a smooth holomorphic manifold. This is clear in case p is a regular point of A. If A has a singularity at p, the analytic structure near p is obtained by the Puiseux parameterization, see Section 6.1 or 9.5 in [7]. This is a local parameterization of A by a neighborhood of 0 in $\mathbf{C}$, $\zeta\mapsto(h_1(\zeta),h_2(\zeta))$, where h_1 and h_2 are holomorphic, (h_1,h_2) is one to one, $(h_1(0),h_2(0))=p$, but possibly $h_1'(0)=h_2'(0)=0$. So L is a holomorphic manifold but the inclusion map $L\to M$ is not an immersion, and even when L is immersed, L may not be closed nor have its topology induced from the topology of M. This holomorphic manifold L is called *a Camacho–Sad manifold through p*.

Remark. The proofs of Propositions 1 and 2 are based on the existence of a Camacho–Sad manifold which exists, in general, only in dimension two ([12]).

The proofs are equally valid in higher dimensions if we *assume* the existence of an invariant one dimensional analytic set passing through p.

Definition. We say that a holomorphic vector field Z on a complex manifold M is *monodromic* if and only if the following holds: for every $q \in M$, whenever we have connected open sets Ω_j with $0 \in \Omega_j$, and $\phi_j: \Omega_j \to M$ satisfying $\phi_j' = Z(\phi_j)$, $j=1, 2$, such that $\phi_1(0)=\phi_2(0)=q$, then $\phi_1=\phi_2$ on $\Omega_1 \cap \Omega_2$.

Remark. We note here for further use that Z is monodromic if and only if for any $q \in M$ there is a connected open set $\Omega \subset \mathbf{C}$ with $0 \in \Omega$ and a mapping $\phi: \Omega \to M$ such that $\phi' = Z(\phi)$ and $\phi(0)=q$, and such that if $\zeta_n \in \Omega$ and $\zeta_n \to \partial\Omega$, then $\phi(\zeta_n)$ leaves every compact subset of M.

In the terminology of Rebelo, monodromic vector fields are called semi-complete. They are the topic of [13] and [11]. It is only for the convenience of the reader that we include a proof of the following lemma.

Lemma 1. *Suppose that Z is monodromic at p and that Γ is a curve in M such that $\Gamma(0)=p$, $\Gamma(1)=q$, Γ does not pass through any zeros of Z and*

$$\Gamma'(s) = \lambda(s) Z(\Gamma(s))$$

for some complex valued function λ. Then the curve $\gamma(t)=\int_0^t \lambda(\tau)\, d\tau$, $0 \le t \le 1$, lies in Ω and $\phi(\gamma(1))=q$. In particular if $\gamma(1)=0$ then $p=q$.

Proof. For small values of t, $\gamma(t)$ lies in Ω. For small values of t, $\phi(\gamma(t))$ satisfies the same differential equation as Γ with the same initial condition. So $\Gamma(t)=\phi(\gamma(t))$ for small t. If γ did not lie in Ω then there would be a sequence $t_n \to t_0$ with $\gamma(t_n) \in \Omega$ but tending to a point z_0 in the boundary of Ω. But then $\phi(\gamma(t_n)) = \Gamma(t_n) \to \Gamma(t_0) \in M$ contradicting the definition of monodromic.

Hence γ lies entirely in Ω and $\phi(\gamma(1))=\Gamma(1)=q$.

Monodromic vector fields in $(\mathbf{C}, 0)$

Lemma 2. *Let Y be a non-constant holomorphic vector field defined in some neighborhood V of 0 in $\mathbf{C}$ that vanishes at 0. If Y is monodromic then either $J_1(Y,0) \neq 0$ or Y is equivalent to $z^2 \partial/\partial z$ by a holomorphic change of variable in a (possibly smaller) neighborhood of 0.*

Proof. Let $Y=a(z)\partial/\partial z$ and assume that $a(z)$ vanishes only at the origin. Now consider a curve $\Gamma(t)$, $0 \le t \le 1$, in V that does not pass through the origin. We may write

$$\Gamma'(t) = \frac{\Gamma'(t)}{a(\Gamma(t))} a(\Gamma(t)).$$

So if we let $\lambda(t)=\Gamma'(t)/a(\Gamma(t))$ then Lemma 1 tells us that if $\int_0^1 \lambda(t)\,dt=0$ then Γ is a closed curve. But $\int_0^1 \lambda(t)\,dt=\int_\Gamma a(z)^{-1}\,dz$. In other words, if Y is monodromic and Γ is not a closed curve then $\int_\Gamma a(z)^{-1}\,dz\neq 0$. This allows us to eliminate most fields $a(z)\partial/\partial z$.

Now a non-zero holomorphic vector field can, via a local holomorphic change of variable near 0, be reduced to the normal form

$$Y = (z^p + \lambda z^{2p-1})\frac{\partial}{\partial z}.$$

This is well known (see [2, Proposition 3]). By making a complex linear change of time, which does not modify the question of monodromy, we can even reduce to either $Y=z^p\partial/\partial z$ ($\lambda=0$) or $Y=(z^p+z^{2p-1})\partial/\partial z$ ($\lambda\neq 0$). It is therefore enough to check that in no neighborhood of 0 are the vector fields $z^p\partial/\partial z$, $p\geq 3$, and $(z^p+z^{2p-1})\partial/\partial z$, $p\geq 2$, monodromic. We know from the above discussion that to check that the holomorphic vector field $Y=a(z)\partial/\partial z$ is not monodromic it is enough to find a *non-closed curve* Γ in $V\setminus\{0\}$ such that $\int_\Gamma a(z)^{-1}\,dz=0$.

Of course, the case $a(z)=z^p$, $p\geq 3$, is trivial. Next we look at the case $a(z)=z^2+z^3$. Fix $\varepsilon>0$ small enough. It is easy to show that there exists z_1 on the ray $\{z\in\mathbf{C}: z=\varepsilon+te^{-i\varepsilon},\ t<0\}$ close to 0 such that $\int_\varepsilon^{z_1}(z^2+z^3)^{-1}\,dz$ is real, simply by using $\int(z^2+z^3)^{-1}\,dz=-1/z-\log z+\log(z+1)$. Take $z_2=\bar{z}_1$. Then

$$\int_{z_1}^{\varepsilon}\frac{dz}{z^2+z^3}+\int_{\varepsilon}^{z_2}\frac{dz}{z^2+z^3}=0,$$

but the path made from the line segments $[z_1,\varepsilon]$ and $[\varepsilon,z_2]$ is not closed.

The non-monodromy of the field $(z^2+z^3)\partial/\partial z$ can also be seen from the dynamical point of view (see Figure 2 in [2]).

The case $a(z)=z^p+z^{2p-1}$ is proved in [13, Proposition 3.1]. This case can also be reduced (as in [2, p. 560]) to the case $a(z)=z^2+z^3$ by the singular change of variables $\zeta\mapsto z=\zeta^{p-1}$, since the pull-back of the differential form $(z^2+z^3)^{-1}\,dz$ is the form $(n-1)(\zeta^p+\zeta^{2p-1})^{-1}\,d\zeta$. Therefore by lifting by a determination of the $(n-1)^{\text{th}}$ root a non-closed curve along which the integral of $(z^2+z^3)^{-1}\,dz$ is zero, one gets a non-closed curve along which the integral of $(z^p+z^{2p-1})^{-1}\,dz$ is zero. For $p>2$ one could also reduce to the case $a(z)=z^p$, either by treating the case $a(z)=z^p+z^{2p-1}$ as a perturbation (as is done in [13]), or by conjugating the two cases, not on a full neighborhood of 0, but on a sector (of angle up to 2π).

The pull-back of a vector field via a Puiseux map

Let A be a germ of an irreducible analytic set at 0 in $\mathbf{C}^2$. Let h be a Puiseux parameterization of A, i.e. a map from $(\mathbf{C},0)$ into $(\mathbf{C}^2,0)$ which is injective and whose image is A. Assume that h vanishes to order m at 0 ($h(0)=h'(0)=...=h^{m-1}(0)=0$ but $h^m(0)\neq 0$). Let Z be a vector field on $A\setminus\{0\}$, tangential to A, vanishing to order $\geq k$ ($|(Z(q)|\leq C|q|^k$), with $k\geq 1$. The pull-back of Z is defined (a priori) on a neighborhood of 0 with 0 deleted, and is denoted by $h_*^{-1}(Z)$.

Lemma 3. (With the above notation.) *The vector field $h_*^{-1}(Z)$ extends holomorphically at 0 to a vector field vanishing to order at least $(k-1)m+1$, and therefore to order greater than 2 if $k>0$, $m>0$ unless $k=1$, or, $k=2$ and $m=1$, i.e. unless A vanishes to order one only or Z vanishes to order two only and A is non-singular.*

Proof. We have

$$|h_*^{-1}(Z)(\zeta)|=\frac{1}{|h'(\zeta)|}|Z(h(\zeta))|\leq C\left|\frac{\zeta^{km}}{\zeta^{m-1}}\right|\leq C(|\zeta|^{(k-1)m+1}).$$

Fields on Riemann surfaces

Lemma 4. *Let S be a connected Riemann surface. Let Z_1 be a non-zero $\mathbf{R}^+$ complete holomorphic field on S, vanishing at some point $p\in S$. Then either $S\approx\mathbf{C}$ or $S\approx U$, the unit disk in $\mathbf{C}$, in which case Z_1 vanishes to order one at p and has no other zero, or $S\approx P_1(\mathbf{C})$ in which case Z_1 may vanish to order two at p with no other zero or Z_1 vanishes to order one at p and has exactly one other zero, also a simple zero.*

Proof. The vector field Z_1 defines a semi-group (ϕ_t) of holomorphic injective maps from S into S. Since holomorphic injective maps from $\mathbf{C}$ to itself or from $P_1(\mathbf{C})$ to itself are bijective, the cases $S\approx\mathbf{C}$ or $S\approx P_1(\mathbf{C})$ are easily understood. Next, suppose that $S\approx U$. That is, we may assume that we have an $\mathbf{R}^+$ complete field X on U such that $X(0)=0$. First we note that if X were to vanish to order greater than one at 0, then we would have $\phi_t'(0)=1$, which would imply, by Schwarz's lemma, that $\phi_t(z)=z$ for all t and z. Hence X has a simple zero at 0. If $X(z_0)=0$ for some $z_0\neq 0$, then we would have $\phi_t(z_0)=z_0$ for all t, again by Schwarz's lemma.

For the general case: the vector field Z_1 lifts to an $\mathbf{R}^+$ complete field $\widehat{Z}_1$ on $\widehat{S}$, the universal cover of S. Unless $S\approx P_1(\mathbf{C})$, $\widehat{S}=\mathbf{C}$ or $\widehat{S}=U$ and since $\widehat{Z}_1$ is then allowed to have only one zero the cover is single sheeted, so $S\approx\mathbf{C}$ or $S\approx U$.

$\mathbf{R}^+$ complete fields

A connected subset Ω of the complex plane will be called an $\mathbf{R}^+$ domain if $z+t\in\Omega$ for all $z\in\Omega$ and all $t>0$. We note here some elementary facts about such domains.

Lemma 5. *If Ω is an $\mathbf{R}^+$ domain and $z,w\in\Omega$ with $\operatorname{Im} z>\operatorname{Im} w$ then there is $M>0$ such that $\{x+iy:\operatorname{Im} w\leq y\leq \operatorname{Im} z,\ x>M\}\subset\Omega$.*

Proof. If not, then for every $N>0$ there is a point x_N+iy_N in the complement of Ω with $x_N>N$ and $\operatorname{Im} w<y_N<\operatorname{Im} z$. Since Ω is an $\mathbf{R}^+$ domain the line segment $L_N=\{x+iy_N:x\leq x_N\}$ is contained in the complement of Ω. These line segments cluster on a full horizontal line L which lies in the complement of Ω and which separates z from w. This contradicts the connectedness of Ω.

From this lemma we easily obtain the following lemmas.

Lemma 6. *An $\mathbf{R}^+$ domain is simply connected.*

Lemma 7. *If Ω_1 and Ω_2 are $\mathbf{R}^+$ domains then so is $\Omega=\Omega_1\cap\Omega_2$.*

Proof. It is clear that Ω is invariant by translations to the right, so we need only check that Ω is connected, but this follows from Lemma 5.

Now suppose that M is a complex manifold. Suppose further that there is a holomorphic vector field Z defined on M that is $\mathbf{R}^+$ complete. Now fix $p\in M$, then we can find a solution ϕ_0 to the equation $\phi'=Z(\phi)$, $\phi(0)=p$ in a disk Δ centered at the origin. Because of the hypothesis, this solution extends to be a solution in $\Delta+\mathbf{R}^+$, an $\mathbf{R}^+$ domain. Now consider pairs (Ω,ϕ_Ω), where Ω is an $\mathbf{R}^+$ domain containing $\Delta+\mathbf{R}^+$ and ϕ_Ω is a (single valued!) solution of our differential equation that agrees with ϕ_0 in $\Delta+\mathbf{R}^+$. Suppose that $(\Omega_j,\phi_{\Omega_j})$, $j=1$, 2, are two such pairs. Note that since $\Omega_1\cap\Omega_2$ is connected and $\phi_{\Omega_1}=\phi_{\Omega_2}$ on a non-empty open subset of this intersection, we can find a single valued solution to our equation on the union of Ω_1 and Ω_2. Let $\widehat{\Omega}$ be the union of all such domains. By the above discussion we have a single valued solution $\hat{\phi}$ defined in $\widehat{\Omega}$ that agrees with ϕ_0 in $\Delta+\mathbf{R}^+$. Now suppose that $z\in\widehat{\Omega}$ and D is a disk with center at z such that D is not a subset of $\widehat{\Omega}$ and there is a solution f of the differential equation defined in D that agrees with $\hat{\phi}$ in a neighborhood of z. Then f extends to be a solution in $D+\mathbf{R}^+$, an $\mathbf{R}^+$ domain. By the proposition above $\widehat{\Omega}\cap(D+\mathbf{R}^+)$ is connected and $f=\hat{\phi}$ on a non-empty open subset of that intersection. This contradicts the maximality of the domain $\widehat{\Omega}$. It follows that no such analytic continuation f can exist. This implies the following.

Lemma 8. *If Z is an $\mathbf{R}^+$ complete field then Z is monodromic. Moreover, the open set Ω referred to in the definition of monodromic is simply connected.*

Proofs of Propositions 1 and 2

Although part of Proposition 1 is a special case of Proposition 2, for the sake of simplicity we start with the proof of Proposition 1, which requires fewer tools.

Proof of Proposition 1. Let L be a Camacho–Sad manifold through p. The vector field Z on M gives by restriction an $\mathbf{R}^+$ complete vector field Z_1 on the Riemann surface L. If Z vanishes to order k at p, then Z_1 vanishes to order greater than 2 at p in L, unless $k=1$ or $k=2$ and $m=1$ (in the notation of Lemma 3), i.e. $k=2$ and the inclusion map $L\to M$ is an immersion (as already noticed in [13]). If L is compact it is then of course an embedding. By applying Lemma 4, we then get either $k=1$ or $k=2$. If $k=2$, then L is an embedded Riemann sphere in M, going through p, and to which the vector field Z is tangent.

Proof of Proposition 2. Let A be an irreducible analytic set at p, as given by the Camacho–Sad theorem. Let $h: U\to A$ be a Puiseux map, $h(0)=p$, U being the unit disk in $\mathbf{C}$. Let $Y=a(z)\partial/\partial z=h_*^{-1}(Z)$. As we have seen from Lemma 3, $a(z)$ has a removable singularity at 0 and $a(0)=0$. Now we pick $z_0\in U$, $z_0\neq 0$, and consider the differential equation $\phi'=a(\phi)$, $\phi(0)=z_0$ in U. Let us be clear here. We consider the complex manifold U with the holomorphic vector field $a(z)\partial/\partial z$ and our first aim is to show that this field is monodromic in the sense of our definition. Suppose that we have connected open sets Ω_j and ϕ_j, $j=1$, 2, as in the remark following the definition of monodromic. Fix a point ζ_0 in $\Omega_1\cap\Omega_2$. Let γ_j be a curve that joins 0 to ζ_0 in Ω_j, $j=1$, 2. Let $K=h(\phi_1(\gamma_1))\cup h(\phi_2(\gamma_2))$. We may approximate Z as closely as we want near K by $\mathbf{R}^+$ complete fields. The solutions to these $\mathbf{R}^+$ complete fields will approximate $h(\phi_j)$ on $h(\gamma_j)$. Since these approximating fields are monodromic, it follows that $h(\phi_1(\zeta_0))=h(\phi_2(\zeta_0))$ and hence that $\phi_1(\zeta_0)=\phi_2(\zeta_0)$. That is, Y is monodromic. (The fact that limits of monodromic fields are monodromic has already been used by several authors ([5], [8], [11]).)

Now that we know that Y is monodromic we can apply Lemma 2 to conclude that $J_2(Y,0)\neq 0$, from which it follows that $J_2(Z,p)\neq 0$, by Lemma 3. Now, if $J_1(Z,p)=0$ then we would have $J_1(Y,0)=0$ and hence, by Lemma 2, Y would be (equivalent to) $z^2\partial/\partial z$.

Of course the solution to this equation is $\phi(\zeta)=z_0/(1-z_0\zeta)$ and its natural domain is $\Omega=\{\zeta:|\zeta-1/z_0|>1\}$. Let γ be a circle in Ω that is not homotopic to a point in Ω. Consider $K=h(\phi(\gamma))$. We approximate Z by $\mathbf{R}^+$ complete fields Z_n on K. The solutions ψ_n to the approximating fields will approximate $h(\phi)$ on γ. Notice that once the domain Ω_n of ψ_n contains γ it contains its interior as well, because Ω_n is an $\mathbf{R}^+$ domain. Now let f be a holomorphic function on M. Then $f(\psi_n)$ will converge to $f(h(\phi))$ on γ. This gives an analytic continuation of $f(h(\phi))$ to an entire function on $\mathbf{C}$. More precisely, we have an entire function g so that

$g(\zeta)=f(h(z_0/(1-z_0\zeta)))$ for $\zeta\in\Omega$. Letting $\zeta\to\infty$ we see that g is constant. From this it follows that f is constant on V.

An example. In this paragraph we give an example of a connected two dimensional Stein manifold M, and of a $\mathbf{C}$ complete holomorphic vector field Z on M with more than one isolated zero: Let $M=\{(z,w,\varrho)\in\mathbf{C}^3:\varrho^2=(1-zw)\}$ and $Z=z(\partial/\partial z)-w(\partial/\partial w)$. It is easily checked that M is a connected smooth manifold and that Z is tangent to M. In fact the flow corresponding to Z is $\phi_t(z,w,\varrho)=(e^t z, e^{-t}w,\varrho)$ which leaves M invariant in $\mathbf{C}^3$. The field Z has two isolated zeros on M, viz. $(0,0,\pm 1)$.

References

1. Ahern, P., Flores, M. and Rosay, J.-P., On $\mathbf{R}^+$ and $\mathbf{C}$ complete holomorphic vector fields, to appear in *Proc. Amer. Math. Soc.*
2. Ahern, P. and Rosay, J.-P., Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables, *Trans. Amer. Math. Soc.* **347** (1995), 543–572.
3. Andersen, E., Volume preserving automorphisms of $\mathbf{C}^n$, *Complex Variables Theory Appl.* **14** (1990), 223–225.
4. Andersen, E. and Lempert, L., On the group of holomorphic automorphisms of $\mathbf{C}^n$, *Invent. Math.* **110** (1992), 371–388.
5. Buzzard, G. and Fornæss, J. E., Complete holomorphic vector fields and time-1 maps, *Indiana Univ. Math. J.* **44** (1995), 1175–1182.
6. Camacho, C. and Sad, P., Invariant varieties through singularities of holomorphic vector fields, *Ann. of Math.* **115** (1982), 579–595.
7. Chirka, E., *Complex Analytic Sets*, Nauka, Moscow, 1985 (Russian). English transl.: Kluwer, Dordrecht, 1989.
8. Forstneric, F., Limits of complete holomorphic vector fields, *Math. Res. Lett.* **2** (1995), 401–414.
9. Forstneric, F., Actions of $(\mathbf{R},+)$ and $(\mathbf{C},+)$ on complex manifolds, *Math. Z.* **223** (1996), 123–153.
10. Forstneric, F. and Rosay, J.-P., Approximation of biholomorphic mappings by automorphisms of $\mathbf{C}^n$, *Invent. Math.* **112** (1993), 323–349.
11. Ghys, E. and Rebelo, J., Singularités des flots holomorphes II, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 1117–1174.
12. Gomez-Mont, X. and Luengo, I., Germs of holomorphic vector fields in $\mathbf{C}^3$ without a separatrix, *Invent. Math.* **109** (1992), 211–219.
13. Rebelo, J., Singularités des flots holomorphes, *Ann. Inst. Fourier (Grenoble)* **46** (1996), 411–428.
14. Suzuki, M., Propriétés topologiques des polynômes de deux variables complexes et automorphismes algébriques de l'espace $\mathbf{C}^2$, *J. Math. Soc. Japan* **26** (1973), 242–257.

15. SUZUKI, M., Sur les opérations holomorphes du groupe additif complexe sur l'espace de deux variables complexes I, *Ann. Sci. École Norm. Sup.* **10** (1977), 517–546.

Received October 28, 1997

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