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Irreducibility of the punctual quotient scheme of a surface

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Abstract. It is shown that the punctual quotient scheme Q_l^r parametrizing all zero-dimensional quotients $\mathcal{O}_{\mathbf{A}^2}^{\oplus r} \to T$ of length l and supported at some fixed point $0 \in \mathbf{A}^2$ in the plane is irreducible.

Let X be a smooth projective surface, E a locally free sheaf of rank $r \geq 1$ on X, and let $l \geq 1$ be an integer. Let $\mathrm{Quot}(E, l)$ denote Grothendieck's quotient scheme [8] that parametrizes all quotients $E \to T$, where T is a zero-dimensional sheaf of length l. Sending a quotient $E \to T$ to the point $\sum_{x \in X} l(T_x)x$ in the symmetric product $S^l(X)$ defines a morphism $\pi: \mathrm{Quot}(E, l) \to S^l(X)$ [8]. It is the purpose of this note to prove the following theorem.

Theorem 1. *The scheme* $\mathrm{Quot}(E, l)$ *is an irreducible scheme of dimension* $l(r+1)$. *The fibre of the morphism* $\pi: \mathrm{Quot}(E, l) \to S^l(X)$ *over a point* $\sum_x l_x x$ *is irreducible of dimension* $\sum_x (rl_x - 1)$.

Using the irreducibility result, one can check that a generic point in the fibre over $l_x \in S^l(X)$ represents a quotient $E \to T$, where $T \cong \mathcal{O}_{X,x}/(s, t^l)$ and s and t are appropriately chosen local parameters in $\mathcal{O}_{X,x}$, i.e. T is the structure sheaf of a curvilinear subscheme in X.

If $r=1$, i.e. if E is a line bundle, then $\mathrm{Quot}(E, l)$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^l(X)$. In this case, the first assertion of the theorem is due to Fogarty [6], whereas the second assertion was proved by Briançon [2]. For general $r \geq 2$, the first assertion of the theorem is a result due to J. Li and D. Gieseker [9], [7]. We give a different proof with a more geometric flavour, generalizing a technique from Ellingsrud and Strømme [5]. The second assertion is a new result for $r \geq 2$. After finishing this paper we learned about a different approach by Baranovsky [1].

The natural generalizations of the theorem to higher dimensional or singular varieties are false, as is already apparent in the $r=1$ case of the Hilbert schemes [3].

1. Elementary modifications

Let X be a smooth projective surface and $x \in X$. If N is a coherent $\mathcal{O}_X$-sheaf, then $e(N_x) = \hom_X(N, k(x))$ denotes the dimension of the fibre $N(x)$, which by Nakayama's lemma is the same as the minimal number of generators of the stalk N_x. If T is a coherent sheaf with zero-dimensional support, we denote by $i(T_x) = \hom_X(k(x), T)$ the dimension of the socle of T_x, i.e. the submodule $\mathrm{Soc}(T_x) \subset T_x$ of all elements that are annihilated by the maximal ideal in $\mathcal{O}_{X,x}$.

Lemma 2. *Let $[q: E \to T] \in \mathrm{Quot}(E, l)$ be a closed point and let N be the kernel of q. Then the socle dimension of T and the number of generators of N at x are related as*

$$e(N_x) = i(T_x) + r.$$

Proof. Write $e(N_x) = r + i$ for some integer $i \geq 0$. Then there is a minimal free resolution $0 \longrightarrow \mathcal{O}_{X,x}^i \xrightarrow{\alpha} \mathcal{O}_{X,x}^{r+i} \longrightarrow N_x \longrightarrow 0$, where all coefficients of the homomorphism α are contained in the maximal ideal of $\mathcal{O}_{X,x}$. We have $\mathrm{Hom}(k(x), T_x) \cong \mathrm{Ext}^1_X(k(x), N_x)$ and applying the functor $\mathrm{Hom}(k(x), \cdot\,)$ one finds an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_X(k(x), N_x) \longrightarrow \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \xrightarrow{\alpha'} \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^{r+i}).$$

But as α has coefficients in the maximal ideal, the homomorphism α' is zero. Thus $\mathrm{Hom}(k(x), T) \cong \mathrm{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \cong k(x)^i$. □

The main technique for proving the theorem will be induction on the length of T. Let N be the kernel of a surjection $E \to T$, let $x \in X$ be a closed point, and let $\lambda: N \to k(x)$ be any surjection. Define a quotient $E \to T'$ by means of the push-out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & k(x) & \xrightarrow{\mu} & T' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \lambda\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & N' & = & N' & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

In this way every element $\langle\lambda\rangle\in\mathbf{P}(N(x))$ determines a quotient $E\to T'$ together with an element $\langle\mu\rangle\in\mathbf{P}(\mathrm{Soc}(T'_x)^\vee)$. (Here $W^\vee:=\mathrm{Hom}_k(W,k)$ denotes the vector space dual to W.) Conversely, if $E\to T'$ is given, any such $\langle\mu\rangle$ determines $E\to T$ and a point $\langle\lambda\rangle$. We will refer to this situation by saying that T' is obtained from T by an elementary modification.

We need to compare the invariants for T and T'. Obviously, $l(T')=l(T)+1$. Applying the functor $\mathrm{Hom}(k(x),\cdot\,)$ to the upper row in the diagram we get an exact sequence

$$0\longrightarrow k(x)\longrightarrow \mathrm{Soc}(T'_x)\longrightarrow \mathrm{Soc}(T_x)\longrightarrow \mathrm{Ext}^1_X(k(x),k(x))\cong k(x)^2,$$

and therefore $|i(T_x)-i(T'_x)|\le 1$. Moreover, we have $0\le e(T'_x)-e(T_x)\le 1$. Two cases deserve closer inspection. Firstly, if e increases, then T' splits.

Lemma 3. *Consider the natural homomorphisms $f\colon \mathrm{Soc}(T'_x)\to T'_x\to T'(x)$ and $g\colon N(x)\to E(x)$. The following assertions are equivalent:*
(1) $e(T'_x)=e(T_x)+1$,
(2) $\langle\mu\rangle\notin\mathbf{P}(\ker(f)^\vee)$,
(3) $\langle\lambda\rangle\in\mathbf{P}(\mathrm{im}(g))$.
Moreover, if these conditions are satisfied, then $T'\cong T\oplus k(x)$ and $i(T'_x)=i(T_x)+1$.

Proof. Clearly, $e(T'_x)=e(T_x)+1$ if and only if $\mu(1)$ represents a non-trivial element in $T'(x)$ if and only if μ has a left inverse if and only if λ factors through E. □

Secondly, if i increases for all modifications λ from T to any T', then the same phenomenon occurs for all 'backwards' modifications μ' from T to any T^-.

Lemma 4. *Still keeping the notation above, let $E\to T'_\lambda$ be the modification of $E\to T$ determined by the point $\langle\lambda\rangle\in\mathbf{P}(N(x))$. Similarly, for $\langle\mu'\rangle\in\mathbf{P}(\mathrm{Soc}(T_x)^\vee)$ let $T^-_{\mu'}=T/\mu'(k(x))$. If $i(T'_{\lambda,x})=i(T_x)+1$ for all $\langle\lambda\rangle\in\mathbf{P}(N(x))$, then $i(T_x)=i(T^-_{\mu',x})-1$ for all $\langle\mu'\rangle\in\mathbf{P}(\mathrm{Soc}(T_x)^\vee)$ as well.*

Proof. Let

$$\Phi\colon \mathrm{Hom}_X(N,k(x))\longrightarrow \mathrm{Hom}_k(\mathrm{Ext}^1_X(k(x),N),\mathrm{Ext}^1_X(k(x),k(x)))$$

be the homomorphism which is adjoint to the natural pairing

$$\mathrm{Hom}_X(N,k(x))\otimes\mathrm{Ext}^1_X(k(x),N)\longrightarrow \mathrm{Ext}^1_X(k(x),k(x)).$$

By identifying $\mathrm{Soc}(T_x)\cong\mathrm{Ext}^1_X(k(x),N)$, we see that $i(T'_{\lambda,x})=1+i(T_x)-\mathrm{rank}(\Phi(\lambda))$. The action of $\Phi(\lambda)$ on a socle element $\mu'\colon k(x)\to T$ can be described by the following diagram of pull-backs and push-forwards

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle \mu'} & & \\
0 & \longrightarrow & N & \longrightarrow & N^-_{\mu'} & \longrightarrow & k(x) & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & k(x) & \longrightarrow & \xi & \longrightarrow & k(x) & \longrightarrow & 0.
\end{array}$$

The assumption that $i(T'_{\lambda,x})=1+i(T_x)$ for all λ, is equivalent to $\Phi=0$. This implies that for every μ' and every λ the extension in the third row splits, which in turn means that every λ factors through $N^-_{\mu'}$, i.e. that $N(x)$ embeds into $N^-_{\mu'}(x)$. Hence, for $T^-_{\mu'}=E/N^-_{\mu'}=\operatorname{coker}(\mu)$ we get $i(T^-_{\mu',x})=e(N^-_{\mu',x})-r=e(N_x)+1-r=i(T_x)+1$. $\square$

2. The global case

Let $Y_l=\operatorname{Quot}(E,l)\times X$, and consider the universal exact sequence of sheaves on Y_l,

$$0\longrightarrow \mathcal{N}\longrightarrow \mathcal{O}_{\mathrm{Quot}}\otimes E\longrightarrow \mathcal{T}\longrightarrow 0.$$

The function $y=(s,x)\mapsto i(\mathcal{T}_{s,x})$ is upper semi-continuous. Let $Y_{l,i}$ denote the locally closed subset $\{y=(s,x)\in Y_l \mid i(\mathcal{T}_{s,x})=i\}$ with the reduced subscheme structure.

Proposition 5. *The scheme Y_l is irreducible of dimension $(r+1)l+2$. For each $i\geq 0$ one has* $\operatorname{codim}(Y_{l,i},Y_l)\geq 2i$.

Clearly, the first assertion of the theorem follows from this.

Proof. The proposition will be proved by induction on l, the case $l=1$ being trivial: $Y_1=\mathbf{P}(E)\times X$, the stratum $Y_{1,1}$ is the graph of the projection $\mathbf{P}(E)\to X$ and $Y_{1,i}=\emptyset$ for $i\geq 2$. Hence suppose the proposition has been proved for some $l\geq 1$.

We describe the 'global' version of the elementary modification discussed above. Let $Z=\mathbf{P}(\mathcal{N})$ be the projectivization of the family $\mathcal{N}$ and let $\varphi=(\varphi_1,\varphi_2)\colon Z\to Y_l=\operatorname{Quot}(E,l)\times X$ denote the natural projection morphism. On $Z\times X$ there is a canonical epimorphism

$$\Lambda\colon(\varphi_1\times \mathrm{id}_X)^*\mathcal{N}\longrightarrow(\mathrm{id}_Z,\varphi_2)_*\varphi^*\mathcal{N}\longrightarrow(\mathrm{id}_Z,\varphi_2)_*\mathcal{O}_Z(1)=:\mathcal{K}.$$

As before we define a family $\mathcal{T}'$ of quotients of length $l+1$ by means of Λ,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{T}' & \longrightarrow & (\varphi_1,\mathrm{id}_X)^*\mathcal{T} & \longrightarrow & 0 \\
 & & {\scriptstyle \Lambda}\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & (\varphi_1,\mathrm{id}_X)^*\mathcal{N} & \longrightarrow & \mathcal{O}_Z\otimes E & \longrightarrow & (\varphi_1,\mathrm{id}_X)^*\mathcal{T} & \longrightarrow & 0.
\end{array}$$

Let $\psi_1: Z \to \mathrm{Quot}(E, l+1)$ be the classifying morphism for the family $\mathcal{T}'$, and define $\psi := (\psi_1, \psi_2 := \varphi_2): Z \to Y_{l+1}$. The scheme Z together with the morphisms $\varphi: Z \to Y_l$ and $\psi: Z \to Y_{l+1}$ allows us to relate the strata $Y_{l,i}$ and $Y_{l+1,j}$. Note that $\psi(Z) = \bigcup_{j \geq 1} Y_{l+1,j}$.

The fibre of φ over a point $(s,x) \in Y_{l,i}$ is given by $\mathbf{P}(\mathcal{N}_s(x)) \cong \mathbf{P}^{r-1+i}$, since $\dim(\mathcal{N}_s(x)) = r + i(T_{s,x}) = r+i$ by Lemma 2. Similarly, the fibre of ψ over a point $(s', x) \in Y_{l+1,j}$ is given by $\mathbf{P}(\mathrm{Soc}(\mathcal{T}'_{s',x})^\vee) \cong \mathbf{P}^{j-1}$. If T' is obtained from T by an elementary modification, then $|i(T') - i(T)| \leq 1$ as shown above. This can be stated in terms of φ and ψ as follows: For each $j \geq 1$ one has

$$\psi^{-1}(Y_{l+1,j}) \subset \bigcup_{|i-j| \leq 1} \varphi^{-1}(Y_{l,i}).$$

Using the induction hypothesis on the dimension of $Y_{l,i}$ and the computation of the fibre dimension of φ and ψ, we get

$$\dim(Y_{l+1,j}) + (j-1) \leq \max_{|i-j| \leq 1} \{(r+1)l + 2 - 2i + (r-1+i)\}$$

and

$$\dim(Y_{l+1,j}) \leq (r+1)(l+1) + 2 - 2j - \min_{|i-j| \leq 1} \{i - j + 1\}.$$

As $\min_{|i-j| \leq 1} \{i-j+1\} \geq 0$, this proves the dimension estimates of the proposition.

It suffices to show that Z is irreducible. Then $\mathrm{Quot}(E, l+1) = \psi_1(Z)$ and Y_{l+1} are irreducible as well.

Since X is a smooth surface, the epimorphism $\mathcal{O}_{\mathrm{Quot}} \otimes E \to \mathcal{T}$ can be completed to a finite resolution

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{O}_{\mathrm{Quot}} \otimes E \longrightarrow \mathcal{T} \longrightarrow 0$$

with locally free sheaves $\mathcal{A}$ and $\mathcal{B}$ on Y_l of rank n and $n+r$, respectively, for some positive integer n. It follows that $Z = \mathbf{P}(\mathcal{N}) \subset \mathbf{P}(\mathcal{B})$ is the vanishing locus of the composite homomorphism $\varphi^* \mathcal{A} \to \varphi^* B \to \mathcal{O}_{\mathbf{P}(\mathcal{B})}(1)$. In particular, assuming by induction that Y_l is irreducible, Z is locally cut out from an irreducible variety of dimension $(r+1)l + 2 + (r+n-1)$ by n equations. Hence every irreducible component of Z has dimension at least $(r+1)(l+1)$. But the dimension estimates for the stratum $Y_{l,i}$ and the fibres of φ over it yield

$$\dim(\varphi^{-1}(Y_{l,i})) \leq (r+1)l + 2 - 2i + (r+i-1) = (r+1)(l+1) - i,$$

which is strictly less than the dimension of any possible component of Z, if $i \geq 1$. This implies that the irreducible variety $\varphi^{-1}(Y_{l,0})$ is dense in Z. Moreover, since the fibre of ψ over $Y_{l+1,1}$ is zero-dimensional, $\dim(Y_{l+1}) = \dim(Y_{l+1,1}) + 2 = \dim(Z) + 2$ has the predicted value. $\square$

3. The local case

We now concentrate on quotients $E\to T$, where T has support in a single fixed closed point $x\in X$. For those quotients the structure of E is of no importance, and we may assume that $E\cong\mathcal{O}_X^r$. Let Q_l^r denote the closed subset

$$\{[\mathcal{O}_X^r\to T]\in\operatorname{Quot}(\mathcal{O}_X^r,l)\,|\,\operatorname{Supp}(T)=\{x\}\}$$

with the reduced subscheme structure. We may consider Q_l^r as a subscheme of $Y_{l,1}$ by sending $[q]$ to $([q],x)$. Then it is easy to see that $\varphi^{-1}(Q_l^r)=\psi^{-1}(Q_{l+1}^r)$. Let this scheme be denoted by Z'.

We will use a stratification of Q_l^r both by the socle dimension i and the number of generators e of T and denote the corresponding locally closed subset by $Q_{l,i}^{r,e}$. Moreover, let $Q_{l,i}^r=\bigcup_e Q_{l,i}^{r,e}$ and define $Q_l^{r,e}$ similarly. Of course, $Q_{l,i}^{r,e}$ is empty unless $1\leq i\leq l$ and $1\leq e\leq\min\{r,l\}$.

To prove the second half of the theorem it suffices to show the following.

Proposition 6. *The scheme Q_l^r is an irreducible variety of dimension $rl-1$.*

Lemma 7. *We have* $\dim(Q_{l,i}^{r,e})\leq(rl-1)-\big(2(i-1)+\binom{e}{2}\big)$.

Proof. The proof is done by induction on l. If $l=1$, then $Q_1^r\cong\mathbf{P}^{r-1}$, and $Q_{1,i}^{r,e}=\emptyset$ if $e\geq2$ or $i\geq2$. Assume that the lemma has been proved for some $l\geq1$.

Let $[q':\mathcal{O}_X^r\to T']\in Q_{l+1,j}^{r,e}$ be a closed point. Suppose that the map $\mu: k(x)\to T'(x)$ represents a point in $\psi^{-1}([q'])=\mathbf{P}(\operatorname{Soc}(T'_x)^\vee)$ and that $T_\mu=\operatorname{coker}(\mu)$ is the corresponding modification. If $i=i(T_{\mu,x})$ and $\varepsilon=e(T_{\mu,x})$, then, according to Section 1, the pair (i,ε) can take the following values

$$(i,\varepsilon)=(j-1,e-1),\ (j-1,e),\ (j,e)\text{ or }(j+1,e),\tag{1}$$

in other words

$$\psi^{-1}(Q_{l+1,j}^{r,e})\subset\varphi^{-1}(Q_{l,j-1}^{r,e-1})\cup\bigcup_{|i-j|\leq1}\varphi^{-1}(Q_{l,i}^{r,e}).$$

Subdivide $A=Q_{l,j}^{r,e}$ into four locally closed subsets $A_{i,\varepsilon}$ according to the generic value of (i,ε) on the fibres of ψ. Then

$$\dim(A_{i,\varepsilon})+(j-1)\leq\dim(Q_{l,i}^{r,\varepsilon})+d_{i,\varepsilon},$$

where $d_{i,\varepsilon}$ is the fibre dimension of the morphism

$$\varphi:\psi^{-1}(A_{i,\varepsilon})\cap\varphi^{-1}(Q_{l,i}^{r,\varepsilon})\longrightarrow Q_{l,i}^{r,\varepsilon}.$$

By the induction hypothesis we have bounds for $\dim(Q^{r,\varepsilon}_{l,i})$, and we can bound $d_{i,\varepsilon}$ in the four cases (1) as follows.

(A) Let $[q:\mathcal{O}^r_X\to T]\in Q^{r,e-1}_{l,j-1}$ be a closed point with $N=\ker(q)$. As we are looking for modifications T' with $e(T'_x)=e$, we are in the situation of Lemma 3 and may conclude

$$\begin{aligned}\varphi^{-1}([q])\cap\psi^{-1}(A_{e-1,j-1}) &\cong \mathbf{P}(\operatorname{im}(g: N(x)\to k(x)^r))\\ &\cong \mathbf{P}(\ker(k(x)^r\to T(x)))\cong \mathbf{P}^{r-e},\end{aligned}$$

since $\operatorname{im}(k(x)^r\to T(x))\cong k^{e-1}$. Hence $d_{j-1,e-1}=r-e$ and

$$\begin{aligned}\dim(A_{j-1,e-1}) &\le \dim Q^{r,e-1}_{l,j-1}+(r-e)-(j-1)\\ &\le \left\{(rl-1)-2(j-2)-\binom{e-1}{2}\right\}+(r-e)-(j-1)\\ &= \left\{(r(l+1)-1)-2(j-1)-\binom{e}{2}\right\}-(j-2).\end{aligned}$$

Note that this case only occurs for $j\ge 2$, so that $(j-2)$ is always nonnegative.

(B) In the three remaining cases

$$\varepsilon=e \quad\text{and}\quad i=j-1,\ j,\ \text{or}\ j+1,$$

we begin with the rough estimate $d_{i,e}\le r+i-1$ as in Section 2. This yields

$$\begin{aligned}\dim(A_{i,e}) &\le \left\{(rl-1)-2(i-1)-\binom{e}{2}\right\}+(r+i-1)-(j-1)\\ &= \left\{(r(l+1)-1)-2(j-1)-\binom{e}{2}\right\}-(i-j).\end{aligned}\tag{2}$$

Thus, if $i=j$ we get exactly the estimate asserted in the lemma, if $i=j+1$ the estimate is better than what we need by 1, but if $i=j-1$, the estimate is not good enough and fails by 1. It is this latter case that we must further study. Let $[q:\mathcal{O}^r_X\to T]$ be a point in $Q^{r,e}_{l,j-1}$ with $N=\ker(q)$. There are two alternatives.

(i) *Either* the fibre $\varphi^{-1}([q])\cap\psi^{-1}(A_{j-1,e})$ is a *proper* closed subset of $\mathbf{P}(N(x))$ which improves the estimate for the dimension of the fibre $\varphi^{-1}([q])$ by 1;

(ii) *or* this fibre *equals* with $\mathbf{P}(N(x))$, which means that the socle dimension increases for all modifications of T. In this case we conclude from Lemma 4 that also $i(T^-)=i(T)+1$ for every modification $T^-=\operatorname{coker}(\mu^-: k(x)\to T)$. But, as we just saw, calculation (2), applied to the contribution of $Q^{r,e}_{l-1,j}$ to $Q^{r,e}_{l,j-1}$, shows that the dimension estimate for the locus of such points $[q]$ in $Q^{r,e}_{l,j-1}$ can be improved by 1 compared to the dimension estimate for $Q^{r,e}_{l,j-1}$ as stated in the lemma.

Hence in either case we can improve estimate (3) by 1 and get

$$\dim(A_{j-1,e}) \leq (r(l+1)-1)-2(j-1)-\binom{e}{2}$$

as required. Thus, the lemma holds for $l+1$. $\square$

Lemma 8. *We have* $\psi(\varphi^{-1}(Q_l^{r,e}))\subset\overline{Q_{l+1}^{r,e}}$.

Proof. Let $[q\colon \mathcal{O}_X^r\to T]\in Q_{l,i}^{r,e}$ be a closed point with $N=\ker(q)$. Then $\varphi^{-1}([q])=\mathbf{P}(N(x))\cong\mathbf{P}^{r+i-1}$ and $\varphi^{-1}([q])\cap\psi^{-1}(Q_{l+1}^{r,e+1})\cong\mathbf{P}(\mathrm{im}(G))\cong\mathbf{P}^{r-e-1}$. Since we always have $e\geq 1$, $i\geq 1$, a dense open part of $\varphi^{-1}([q])$ is mapped to $Q_{l+1}^{r,e}$. $\square$

Lemma 9. *If* $r\geq 2$ *and if* Q_l^{r-1} *is irreducible of dimension* $(r-1)l-1$, *then* $Q_l^{r,<r}:=\bigcup_{e<r} Q_l^{r,e}$ *is an irreducible open subset of* Q_l^r *of dimension* $rl-1$.

Proof. Let M be the variety of all $r\times(r-1)$ matrices over k of maximal rank, and let $0\to\mathcal{O}_M^{r-1}\to\mathcal{O}_M^r\to\mathcal{L}\to 0$ be the corresponding tautological sequence of locally free sheaves on M. Consider the open subset $U\subset M\times Q_l^r$ of points $(A,[\mathcal{O}^r\to T])$ such that the composite homomorphism

$$\mathcal{O}^{r-1}\xrightarrow{A}\mathcal{O}^r\longrightarrow T$$

is surjective. Clearly, the image of U under the projection to Q_l^r is $Q_l^{r,<r}$. On the other hand, the tautological epimorphism

$$\mathcal{O}_{U\times X}^{r-1}\longrightarrow\mathcal{O}_{U\times X}^r\longrightarrow(\mathcal{O}_M\otimes T)|_{U\times X}$$

induces a classifying morphism $g'\colon U\to Q_l^{r-1}$. The morphism

$$g=(pr_1,g')\colon U\longrightarrow M\times Q_l^{r-1}$$

is surjective. In fact, it is an affine fibre bundle with fibre

$$g^{-1}(g(A,[\mathcal{O}^{r-1}\to T]))\cong\mathrm{Hom}_k(\mathcal{L}(A),T)\cong\mathbf{A}_k^l.$$

Since Q_l^{r-1} is irreducible of dimension $(r-1)l-1$ by assumption, U is irreducible of dimension $rl-1+\dim(M)$, and $Q_l^{r,<r}$ is irreducible of dimension $rl-1$. $\square$

Proof of Proposition 6. The irreducibility of Q_l^r will be proved by induction over r and l. The case $l=1$, r arbitrary is trivial; whereas the case l arbitrary, $r=1$ is the case of the Hilbert scheme, for which there exist several proofs ([2], [5]).

Assume therefore that $r \geq 2$ and that the proposition holds for (l, r) and $(l+1, r-1)$. We will show that it holds for $(l+1, r)$ as well.

Recall that $Z' := \varphi^{-1}(Q_l^r) = Q_l^r \times_{Y_l} Z$. Every irreducible component of Z' has dimension greater than or equal to $\dim(Q_l^r) + r - 1 = r(l+1) - 2$ (cf. Section 2). On the other hand, $\dim(\varphi^{-1}(Q_{l,i}^r)) \leq rl - 1 - 2(i-1) + (r+i-1) = r(l+1) - i$. Thus an irreducible component of Z' is either the closure of $\varphi^{-1}(Q_{l,1}^r)$ (of dimension $r(l+1)-1$) or the closure of $\varphi^{-1}(W)$ for an irreducible component $W \subset Q_{l,2}^r$ of maximal possible dimension $rl-3$. But according to Lemma 8 the image of $\varphi^{-1}(W)$ under ψ will be contained in the closure of $Q_{l+1}^{r,<r}$, unless W is contained in $Q_{l,2}^{r,r}$. But Lemma 7 says that $Q_{l,2}^{r,r}$ has codimension $\geq 2 + \binom{r}{2} \geq 3$ if $r \geq 2$, and hence cannot contain W for dimensional reasons. Hence any irreducible component of Z' is mapped by ψ into the closure of $Q_{l+1}^{r,<r}$, which is irreducible by Lemma 9 and the induction hypothesis. This finishes the proof of the proposition. □

References

1. BARANOVSKY, V., On punctual quot schemes for algebraic surfaces, *E-print,* math. AG/9703038.
2. BRIANÇON, J., Description de $\mathrm{Hilb}^n \mathbf{C}\{x,y\}$, *Invent. Math.* **41** (1977), 45–89.
3. IARROBINO, A., Reducibility of the families of 0-dimensional schemes on a variety, *Invent. Math.* **15** (1972), 72–77.
4. IARROBINO, A., Punctual Hilbert schemes, *Mem. Amer. Math. Soc.* **10**:188 (1977).
5. ELLINGSRUD, G. and STRØMME, S. A., An intersection number for the punctual Hilbert scheme, *Trans. Amer. Math. Soc.* **350** (1998), 2547–2552.
6. FOGARTY, J., Algebraic families on an algebraic surface, *Amer. J. Math.* **90** (1968), 511–521.
7. GIESEKER, D. and LI, J., Moduli of high rank vector bundles over surfaces, *J. Amer. Math. Soc.* **9** (1996), 107–151.
8. GROTHENDIECK, A., Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert, in *Séminaire Bourbaki Volume 1960/1961, Exposé* **221**, pp. 249–276, Benjamin, New York–Amsterdam, 1966.
9. LI, J., Algebraic geometric interpretation of Donaldson's polynomial invariants, *J. Differential Geom.* **37** (1993), 417–466.

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