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Boundary growth theorems for superharmonic functions

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Abstract. This paper examines the boundary behaviour of superharmonic functions on a half-space in terms of their behaviour along lines normal to the boundary. It is shown that, if the set of lines along which such functions grow quickly is (in a certain sense) metrically dense, then the set of lines along which they are bounded below is topologically small.

1. Introduction

Let f be a non-constant holomorphic function on the unit disc, and let E be the set of boundary points where f has radial limit 0. A classical theorem of Luzin and Privalov (see [14] or [4, p. 150, Corollary 3]) asserts that, if $E\cap J$ has positive outer measure for every subarc J of a given arc I, then $E\cap I$ is of first (Baire) category. A generalization of this result to superharmonic functions, due to Arsove [1], when reformulated for the half-plane and slightly refined, is as follows. Recall that a subset E of $\mathbf{R}$ is said to be *metrically dense* in an open interval I if $E\cap J$ has positive outer Lebesgue measure for every open subinterval J of I.

Theorem A. *Let u be a superharmonic function on $\mathbf{R}\times(0,+\infty)$ and I be an open interval. If the set $\{x\in\mathbf{R}:\limsup_{y\to 0+} u(x,y)=+\infty\}$ is metrically dense in I, then the set $\{x\in I:\liminf_{y\to 0+} u(x,y)>-\infty\}$ is of first category.*

Rippon [16, Theorem 6] showed that the natural analogue of Theorem A in higher dimensions is false: there exists a superharmonic function u on $\mathbf{R}^2\times(0,+\infty)$ such that

$$u(x,y,z)\to+\infty, \quad \text{as } z\to 0+,\ (x,y)\in\mathbf{R}^2\setminus E,$$

where E is a first category subset of $\mathbf{R}^2$ with zero area measure. However, the author [11] has recently shown that Theorem A can be extended to higher dimensions using the fine topology, that is, the coarsest topology which makes every superharmonic function continuous. (See Doob [5, 1.XI] for its basic properties and its relationship to the notion of thinness.) Let U be a non-empty finely open subset of $\mathbf{R}^n$. A set E is said to be *metrically fine dense* in U if, for every non-empty finely open subset V of U, the set $E \cap V$ has positive outer Lebesgue measure. Also, E is said to be of *first fine category* if it can be expressed as a countable union of sets E_k such that the fine closure of each E_k has empty fine interior. Relevant facts here are that every non-empty finely open set has positive measure, and that the fine topology has the Baire property. Also, the fine topology on $\mathbf{R}^n$ is strictly finer than the Euclidean topology when $n \geq 2$, but the topologies coincide when $n=1$. (The superharmonic functions on $\mathbf{R}$ are precisely the concave functions and so are already continuous.)

Points of $\mathbf{R}^n$, $n \geq 2$, will be denoted by X or (X', x) where $X' \in \mathbf{R}^{n-1}$, and the half-space $\mathbf{R}^{n-1} \times (0, +\infty)$ will be denoted by D. Theorem A has the following generalization to all dimensions (see [11, Theorem 1]).

Theorem B. *Let u be a superharmonic function on D and U' be a non-empty finely open subset of $\mathbf{R}^{n-1}$. If the set $\{X' \in \mathbf{R}^{n-1} : \limsup_{x \to 0+} u(X', x) = +\infty\}$ is metrically fine dense in U', then the set*

$$\left\{X' \in U' : \liminf_{x \to 0+} u(X', x) > -\infty\right\} \tag{1}$$

is of first fine category in $\mathbf{R}^{n-1}$.

Below we show that there is a family of results of this type dealing with various growth rates for superharmonic functions along lines normal to the boundary. Let $\alpha \geq 0$, let $E \subseteq \mathbf{R}^n$ and U be a finely open subset of $\mathbf{R}^n$. If, for every non-empty finely open subset V of U, the set $E \cap V$ has positive α-dimensional Hausdorff measure (resp. $E \cap V$ is non-polar), then we say that E is *α-metrically fine dense in U* (resp. *capacitarily fine dense in U*).

Theorem 1. *Let $n \geq 2$ and $n-2 < \alpha \leq n-1$, let u be a superharmonic function on D and let U' be a finely open subset of $\mathbf{R}^{n-1}$. If the set*

$$\left\{X' \in \mathbf{R}^{n-1} : \limsup_{x \to 0+} x^{n-1-\alpha} u(X', x) = +\infty\right\} \tag{2}$$

is α-metrically fine dense in U', then the set (1) *is of first fine category in $\mathbf{R}^{n-1}$.*

The special case of Theorem 1 where $\alpha = n-1$ is Theorem A above. Other values of α are much more difficult to treat, and new arguments are required. When $n=2$, the fine-topological concepts can be replaced by their Euclidean counterparts (and the proof is much simpler, as we indicate at the end of Section 3), but this is not the case when $n \geq 3$, see Example 1 in Section 6.

Theorem 1 fails when $\alpha \leq n-2$ (see Example 3(b) in Section 6), but a related result is obtained by strengthening (2).

Theorem 2. *Let $n \geq 3$ and $n-3 < \alpha \leq n-2$, let u be a superharmonic function on D and let U' be a finely open subset of $\mathbf{R}^{n-1}$. If the set*

$$\{X' \in \mathbf{R}^{n-1} : x^{n-1-\alpha} u(X', x) \to +\infty, \text{ as } x \to 0+\} \tag{3}$$

is α-metrically fine dense in U', then the set (1) *is of first fine category in $\mathbf{R}^{n-1}$.*

The sharpness of the growth rates in (2) and (3) will be demonstrated by Examples 2 and 3 in Section 6. If $n \geq 3$ and $0 \leq \alpha \leq n-3$, and if a set $E' \subseteq \mathbf{R}^{n-1}$ is capacitarily fine dense in a finely open set U', then E' is α-metrically fine dense in U' because of the well-known relationship between Hausdorff measure and capacity (see [3, IV]). However, the converse is also true since, if there is a non-empty finely open subset V' of U' such that $E' \cap V'$ is polar, then $V' \backslash E'$ is a non-empty finely open subset of U' which is disjoint from E', and so E' is not α-metrically fine dense in U'. Hence the case where $n \geq 3$ and $0 \leq \alpha \leq n-3$ is covered by the following result.

Theorem 3. *Let $n \geq 3$, let $\Psi_n : (0,1) \to \mathbf{R}$ be given by $\Psi_n(t) = t^2$, $n \geq 4$, and $\Psi_3(t) = t^2 \log(1/t)$, let u be a superharmonic function on D and U' be a finely open subset of $\mathbf{R}^{n-1}$. If the set*

$$\left\{X' \in \mathbf{R}^{n-1} : \liminf_{x \to 0+} \Psi_n(x) u(X', x) > 0\right\}$$

is capacitarily fine dense in U', then the set (1) *is of first fine category in $\mathbf{R}^{n-1}$.*

Following some preliminary lemmas in Section 2, Theorems 1–3 are proved in Sections 3–5 and several examples illustrating the sharpness of these results are provided in Section 6.

2. Preliminary lemmas

2.1. We refer to Doob [5, 1.XII] for the notion of minimal thinness.

Lemma 1. *Let $n \geq 3$ and $A' \subseteq \mathbf{R}^{n-1}$, and let $X' \in \mathbf{R}^{n-1}$. The following are equivalent*:

(a) *A' is thin at X'*;

(b) *$A' \times \mathbf{R}$ is thin at (X', x) for all $x \in \mathbf{R}$*;

(c) *$A' \times (0, +\infty)$ is minimally thin with respect to D at $(X', 0)$.*

For the equivalence of (a) and (b) above, see [11, Lemma 2] or [12]. It remains to establish the equivalence of (b) and (c). For this we recall the facts that, for a subset A of D and a cone $C_{X',a} = \{(Y', y) : y > a|Y' - X'|\}$, where $X' \in \mathbf{R}^{n-1}$ and $a > 0$, thinness of A at $(X', 0)$ implies minimal thinness of A with respect to D at

$(X',0)$, and thinness of $A\cap C_{X',a}$ at $(X',0)$ is equivalent to minimal thinness of $A\cap C_{X',a}$ with respect to D at $(X',0)$ (see Lelong-Ferrand [13, Section 6]).

If (b) holds, then $A'\times(0,+\infty)$ is thin at $(X',0)$ and (c) follows. Conversely, suppose that (c) holds, and let $C=C_{X',\sqrt{3}}$. Then $C\cap(A'\times(0,+\infty))$ is minimally thin, and hence thin, at $(X',0)$. By the integral form of Wiener's criterion (see [2, p. 81]),

$$\int_1^{+\infty} \mathcal{C}^*(\{Y\in C\cap(A'\times(0,+\infty)):|Y-(X',0)|^{2-n}\geq t\})\,dt<+\infty,$$

where $\mathcal{C}^*(\cdot)$ denotes outer Newtonian capacity for $\mathbf{R}^n$. If $b>0$, then

$$\{Y\in C:|Y-(X',0)|\leq b\}\supset\big\{(Y',y):|Y'-X'|\leq \tfrac{1}{4}b \text{ and } \tfrac{1}{4}\sqrt{3}\,b<y\leq\tfrac{1}{4}\sqrt{15}\,b\big\}.$$

Hence, by translational symmetry and the observation that $\sqrt{15}-\sqrt{3}>2$,

$$\begin{aligned}\mathcal{C}^*(\{Y\in C\cap(A'\times(0,+\infty)):|Y-(X',0)|\leq b\})&\\ \geq \mathcal{C}^*\big(\{Y\in A'\times\mathbf{R}:|Y-(X',0)|\leq\tfrac{1}{4}b\}\big).&\end{aligned}$$

Thus

$$\int_1^{+\infty} \mathcal{C}^*(\{Y\in A'\times\mathbf{R}:|Y-(X',0)|^{2-n}\geq t\})\,dt<+\infty,$$

and so $A'\times\mathbf{R}$ is thin at $(X',0)$, or indeed at (X',x) for any $x\in\mathbf{R}$ by translational symmetry. Hence (b) holds, and Lemma 1 is proved.

2.2. The simplest case of the following lemma, namely where $I=\mathbf{R}$, is partially covered by [6, Lemma 1].

Lemma 2. *Let $n\geq 3$, let I be an open interval in $\mathbf{R}$, and let V' be a finely open set in $\mathbf{R}^{n-1}$. Then the fine components of $V'\times I$ are precisely the sets of the form $W'\times I$, where W' is a fine component of V'.*

We will give the proof of Lemma 2 for $I=(a,b)$, where $a<b$; a similar argument applies to semi-infinite intervals I. We recall (see [6, Corollary 1(i)]) the fact that, if Ω is a finely open subset of $\mathbf{R}^n$, then there is a set $E\subset\mathbf{R}^{n-1}\times\{0\}$, which is polar in $\mathbf{R}^n$, such that the set $\{t\in\mathbf{R}:(X',t)\in\Omega\}$ is open in $\mathbf{R}$ whenever $(X',0)\notin E$.

Now let W be a fine component of the set $V'\times(a,b)$, which is finely open by Lemma 1, and let $(Y',y)\in W$. We will deduce that $\{Y'\}\times(a,b)\subset W$. To do this, let $0<\varepsilon<\min\{b-y,y-a\}$. By the local connectedness of the fine topology (see [8, p. 92]) there is a fine domain Ω_ε such that $(Y',y)\in\Omega_\varepsilon\subseteq V'\times(y-\varepsilon,y+\varepsilon)$.

Also, by the fact recalled in the preceding paragraph, there exists $Z' \in \mathbf{R}^{n-1}$ and an interval (c,d) such that $\{Z'\}\times(c,d)\subset\Omega_\varepsilon$. If $|\eta|<d-c$, then

$$\Omega_\varepsilon \cap \{(X', x+\eta) : (X',x)\in\Omega_\varepsilon\} \neq \emptyset$$

and so the set

$$\Omega_\varepsilon \cup \{(X', x+\eta) : (X',x)\in\Omega_\varepsilon\}$$

is a fine domain. By repeated application of this observation we see that

$$\bigcup_{a-y+\varepsilon<\eta<b-y-\varepsilon} \{(X', x+\eta) : (X',x)\in\Omega_\varepsilon\} \subset W,$$

and hence that $\{Y'\}\times(a+\varepsilon, b-\varepsilon)\subset W$. The number ε can be arbitrarily small, so $\{Y'\}\times(a,b)\subset W$. Since (Y',y) was an arbitrary point of W, we conclude that W can be written as $W'\times(a,b)$. Also, since W is finely open in $\mathbf{R}^n$ (see [8, p. 146]), it follows from Lemma 1 that W' is finely open in $\mathbf{R}^{n-1}$.

If W' could be expressed as the disjoint union of two non-empty finely open sets V_1' and V_2', then by Lemma 1 we would obtain the contradictory conclusion that $W'\times(a,b)$ is the disjoint union of the finely open sets $V_1'\times(a,b)$ and $V_2'\times(a,b)$. Hence W' is finely connected, and so is contained in some fine component U' of V'. On the other hand, if U' is a fine component of V', then it follows from the preceding paragraph that $U'\times(a,b)$ cannot have more than one fine component. The lemma is now established.

2.3. Let $B'(X',r)$ denote the open ball in $\mathbf{R}^{n-1}$ of centre X' and radius r. If $X\in\mathbf{R}^n$, $n\geq 3$, and $A\subseteq\mathbf{R}^n$, then let μ_X^A denote the balayage of the Dirac measure at X onto A relative to superharmonic functions on $\mathbf{R}^n$; that is, μ_X^A is the measure which satisfies

$$\widehat{R}^A_{|\cdot-X|^{2-n}}(Y) = \int_{\mathbf{R}^n} |Y-Z|^{2-n}\, d\mu_X^A(Z), \quad X\in\mathbf{R}^n,$$

where $\widehat{R}_v^A$ denotes the regularized reduced function (balayage) of v on A relative to superharmonic functions on $\mathbf{R}^n$. (See [5, 1.X] for the notion of a swept measure.) Also, let $C(a,b,\ldots)$ denote a positive constant, depending at most on $a,b,\ldots$, not necessarily the same on any two occurrences.

Lemma 3. *Let $n\geq 3$ and $W=W'\times(0,1)$, where W' is finely open in $\mathbf{R}^{n-1}$, and let $X\in W$. Then there is a set $F'\subseteq W'$, of Hausdorff dimension at most $n-2$, such that*

$$(4)\qquad \frac{1}{t}\int_{B'(Y',t)\times(t,3t)} |(Y',2t)-Z|^{2-n}\, d\mu_X^{\mathbf{R}^n\setminus W}(Z) \to 0, \quad \textit{as } t\to 0+,$$

for all Y' in $W'\backslash F'$.

To see this, let $0<\delta<\frac{1}{3}$ and $X=(X',x)$, and define

$$U=(W'\cup B'(X',\delta))\times\mathbf{R}.$$

If $Y'\in W'\backslash B'(X',2\delta)$ and $0<t<\delta$, then

$$\mu_X^{\mathbf{R}^n\backslash W}|_{B'(Y',t)\times(t,3t)}\le\mu_X^{\mathbf{R}^n\backslash U}|_{B'(Y',t)\times(t,3t)},\tag{5}$$

by a comparison of fine harmonic measures for the sets W and U (see [8, Section 14]). If E is a Borel set in $\mathbf{R}^n$, then by Harnack's inequality and the translational invariance of U,

$$\begin{aligned}\mu_X^{\mathbf{R}^n\backslash U}(\{(Z',z+\eta):(Z',z)\in E\})&=\mu_{(X',x-\eta)}^{\mathbf{R}^n\backslash U}(E)\\&\le C(n,\delta,\eta)\mu_X^{\mathbf{R}^n\backslash U}(E),\quad\eta\in\mathbf{R},\end{aligned}$$

where $C(n,\delta,\eta)\to1$, as $\eta\to0$. It follows that, if we define the finite measure μ' on $\overline{W}'\backslash B'(X',\delta)$ by

$$\mu'(E')=\mu_X^{\mathbf{R}^n\backslash U}(E'\times(0,1))$$

for any Borel set E' in $\mathbf{R}^{n-1}$, then

$$\begin{aligned}&\int_{B'(Y',t)\times(t,3t)}|(Y',2t)-Z|^{2-n}\,d\mu_X^{\mathbf{R}^n\backslash U}(Z)\\&\qquad\le C(n,\delta)\int_{B'(Y',t)}\int_t^{3t}|(Y',2t)-(Z',z)|^{2-n}\,dz\,d\mu'(Z')\\&\qquad=C(n,\delta)\int_{B'(Y',t)}|Z'-Y'|^{3-n}\int_0^{t/|Z'-Y'|}(1+s^2)^{1-n/2}\,ds\,d\mu'(Z')\\&\qquad\le\begin{cases}C(n,\delta)\int_{B'(Y',t)}|Z'-Y'|^{3-n}\,d\mu'(Z'), & n\ge4,\\ C(n,\delta)\int_{B'(Y',t)}\log(2t/|Z'-Y'|)\,d\mu'(Z'), & n=3.\end{cases}\end{aligned}\tag{6}$$

Now let $\varepsilon>0$ and $\alpha>n-2$, and suppose that Y' satisfies

$$m_{Y'}(r)\le r^\alpha,\quad 0<r<\varepsilon,\tag{7}$$

where

$$m_{Y'}(r)=\mu'(B'(Y',r)).$$

It follows from (6), (7) and integration by parts that, if $n\geq 4$, then

$$\begin{aligned}(8)\qquad \frac{1}{t}\int_{B'(Y',t)\times(t,3t)} |(Y',2t)-Z|^{2-n}\,d\mu_X^{\mathbf{R}^n\setminus U}(Z) &\leq \frac{C(n,\delta)}{t}\int_0^t r^{3-n}\,dm_{Y'}(r)\\ &\leq \frac{C(n,\delta)}{t}\int_0^t r^{2-n}m_{Y'}(r)\,dr+t^{2-n}m_{Y'}(t)\to 0,\quad t\to 0,\end{aligned}$$

and if $n=3$, then similarly

$$\begin{aligned}(9)\qquad \frac{1}{t}\int_{B'(Y',t)\times(t,3t)} |(Y',2t)-Z|^{-1}\,d\mu_X^{\mathbf{R}^n\setminus U}(Z) &\leq \frac{C(\delta)}{t}\int_0^t \log(2t/r)\,dm_{Y'}(r)\\ &\to 0,\quad t\to 0.\end{aligned}$$

The subset $S_{\varepsilon,\alpha}$ of points Y' of $\mathbf{R}^{n-1}$ where (7) fails to hold can be covered by a collection of open balls $B'(Y',r_{Y'})$, where $r_{Y'}<\varepsilon$ and $m_{Y'}(r_{Y'})>r_{Y'}^{\alpha}$. By a well-known covering lemma (see, for example, [18, pp. 9–10]) there is a countable disjoint subcollection $\{B'(Y_k',r_k):k\geq 1\}$ such that $S_{\varepsilon,\alpha}\subseteq\bigcup_k B'(Y_k',5r_k)$. Hence

$$\sum_k (5r_k)^\alpha < 5^\alpha \sum_k m_{Y_k'}(r_k)\leq 5^\alpha\mu'(\mathbf{R}^{n-1}).$$

Since ε can be arbitrarily small, it follows from (5), (8) and (9) that the set of all Y' in $W'\setminus B'(X',2\delta)$ for which (4) fails to hold has finite α-dimensional Hausdorff measure. Since δ can be arbitrarily small and α can be arbitrarily close to $n-2$, the lemma is proved.

2.4. For each Y' in $\mathbf{R}^{n-1}$, let $h_{Y'}$ denote the half-space Poisson kernel given by

$$h_{Y'}(X)=\frac{2\max\{1,n-2\}x}{(|X'-Y'|^2+x^2)^{n/2}},\quad X=(X',x)\in\bar{D}\setminus\{(Y',0)\}.$$

Also, let $B(X,r)$ denote the open ball in $\mathbf{R}^n$ with centre X and radius r.

Lemma 4. *Let $n\geq 3$ and $W=W'\times(0,1)$, where W' is finely open in $\mathbf{R}^{n-1}$, and let*

$$G_W(X,Y)=|X-Y|^{2-n}-\int_{\bar{D}}|Y-Z|^{2-n}\,d\mu_X^{\mathbf{R}^n\setminus W}(Z),\quad X,Y\in W.$$

Then there is a set $F'\subseteq W'$, of Hausdorff dimension at most $n-2$, such that

$$\frac{G_W(X,(Y',y))}{y}\to v_{Y'}(X),\quad y\to 0+,\ X\in W,\ Y'\in W'\setminus F',$$

where

$$v_{Y'}(X)=h_{Y'}(X)-\int_{\overline{D}} h_{Y'}(Z)\,d\mu_X^{\mathbf{R}^n\setminus W}(Z),\qquad X\in W,\ Y'\in\mathbf{R}^{n-1}.$$

To prove this, recall that the half-space Green function $G_D(\cdot\,,\,\cdot)$ satisfies

$$\begin{aligned} G_D((X',x),(Y',y)) &= |(X',x)-(Y',y)|^{2-n}-|(X',x)-(Y',-y)|^{2-n} \\ &\le 2(n-2)xy|(X',x)-(Y',y)|^{-n} \end{aligned} \tag{10}$$

and

$$\frac{G_D(X,(Y',y))}{y}\to h_{Y'}(X),\qquad y\to 0+,\ Y'\in\mathbf{R}^{n-1},\ X\in D.$$

Further, if $|Z-(Y',y)|\ge\frac12 y$, then

$$|Z-(Y',0)|\le|Z-(Y',y)|+y\le 3|Z-(Y',y)|,$$

so

$$\frac{G_D((Y',y),Z)}{y}\le 3^n h_{Y'}(Z),$$

by (10). Let $X\in W$. Since $h_{Y'}$ is integrable with respect to the measure $\mu_X^{\mathbf{R}^n\setminus W}$, it follows by dominated convergence that

$$\int_{\overline{D}\setminus B((Y',y),y/2)}\frac{G_D((Y',y),Z)}{y}\,d\mu_X^{\mathbf{R}^n\setminus W}(Z)\to\int_{\overline{D}} h_{Y'}(Z)\,d\mu_X^{\mathbf{R}^n\setminus W}(Z),\qquad y\to 0+,$$

for any Y' in $\mathbf{R}^{n-1}$. Also, since $G_D(Y,Z)\le|Y-Z|^{2-n}$, it follows from Lemma 3 (with $t=\frac12 y$) that there is a set $F'\subseteq W'$, of Hausdorff dimension at most $n-2$, such that

$$\int_{B((Y',y),y/2)}\frac{G_D((Y',y),Z)}{y}\,d\mu_X^{\mathbf{R}^n\setminus W}\to 0,\qquad y\to 0+,\ Y'\in W'\setminus F'.$$

Hence

$$\int_{\overline{D}}\frac{G_D((Y',y),Z)}{y}\,d\mu_X^{\mathbf{R}^n\setminus W}\to\int_{\overline{D}} h_{Y'}(Z)\,d\mu_X^{\mathbf{R}^n\setminus W}(Z),\qquad y\to 0+,\ Y'\in W'\setminus F'.$$

Since

$$\int_{\overline{D}}|(Y',-y)-Z|^{2-n}\,d\mu_X^{\mathbf{R}^n\setminus W}(Z)=|(Y',-y)-X|^{2-n},\qquad y>0,$$

we see that

$$G_W(X,Y)=G_D(X,Y)-\int_{\overline{D}} G_D(Y,Z)\,d\mu_X^{\mathbf{R}^n\setminus W}(Z),$$

and so

$$\frac{G_W(X,(Y',y))}{y}\to v_{Y'}(X),\qquad y\to 0+,\ Y'\in W'\setminus F',$$

as required.

3. Proof of Theorem 1

We first deal with the case where $n \geq 3$ and suppose to the contrary that the set E', defined by

$$E' = \left\{ X' \in U' : \liminf_{x \to 0+} u(X', x) > -\infty \right\},$$

is not of first category. Since $E' = \bigcup_{k=1}^{\infty} E'_k$, where

$$E'_k = \{X' \in E' : u(X', x) \geq -k \text{ for all } x \in (0, 1)\},$$

there exists k_0 such that the fine closure of E'_{k_0} has non-empty fine interior R'. The set V', given by $V' = R' \cap U'$, is thus a non-empty finely open subset of U'. Let W' be a fine component of V'. Then, by Lemma 2, the set $W = W' \times (0, 1)$ is a fine component of $V' \times (0, 1)$. It follows from Lemma 1 and the fine continuity of u that $u \geq -k_0$ on $R' \times (0, 1)$, and hence on W. Further, the fine domain W is contained in one (Euclidean) component, Ω say, of the open set $\{X \in D : u(X) > -k_0 - 1\}$.

Since W' is finely open, $\mathbf{R}^{n-1} \backslash W'$ is thin at each point of W' and so, by Lemma 1, $D \backslash W$ is minimally thin with respect to D at each point of $W' \times \{0\}$. For each $Y' \in W'$ it follows that $v_{Y'} \not\equiv 0$, where $v_{Y'}$ is the function defined in Lemma 4, so $v_{Y'} > 0$ on the fine domain W. Since the Green function, $G_\Omega(\cdot, \cdot)$, for Ω satisfies $G_W \leq G_\Omega$ on $W \times W$, it follows from Lemma 4 that there is a set $F' \subseteq W'$, of Hausdorff dimension at most $n-2$, such that

$$\liminf_{y \to 0+} \frac{G_\Omega(X, (Y', y))}{y} > 0, \quad X \in W, \ Y' \in W' \backslash F'.$$

Let X_0 be a point of W where u is finite, let A' be a (Euclidean) compact subset of W' which has non-empty fine interior and let

$$A'_{i,j} = \{Y' \in A' : G_\Omega(X_0, (Y', y)) \geq y/i \text{ whenever } 0 < y \leq j^{-1}\}, \quad i, j \in \mathbf{N}.$$

Then each $A'_{i,j}$ is a compact set and

$$A' \backslash F' \subseteq \bigcup_{i,j=1}^{\infty} A'_{i,j}. \tag{11}$$

We temporarily fix i and j and use w to denote the balayage of the positive superharmonic function $u + k_0 + 1$ relative to the set $A_{i,j} = A'_{i,j} \times (0, j^{-1}]$ in Ω. Then, by the Riesz decomposition, [5, 1.XII.(17.3)], [15, Théorèmes 12, 13] and the fact that $A'_{i,j}\subseteq A'\subseteq W'$, there are measures μ on $A_{i,j}$ and ν on $A'_{i,j}\times\{0\}$ such that

$$w(X)=\int_{A_{i,j}} G_\Omega(X,Y)\,d\mu(Y)+\int_{A'_{i,j}\times\{0\}} u_{Y'}(X)\,d\nu(Y',0),\quad X\in\Omega,$$

where

$$u_{Y'}(X)=h_{Y'}(X)-\int h_{Y'}(Z)\,d\mu_X^{\mathbf{R}^n\setminus\Omega}(Z),\quad X\in\Omega,\ Y'\in\mathbf{R}^{n-1}.$$

Also, by [15, p. 220],

$$u_{Y'}(X_0)\ge\limsup_{y\to0+}\frac{G_\Omega((Y',y),X_0)}{y}\ge\frac{1}{i},\quad Y'\in A'_{i,j}. \tag{12}$$

We define

$$m_X(r)=\int_{A_{i,j}\cap B(X,r)} y\,d\mu(Y',y)+\nu((A'_{i,j}\times\{0\})\cap B(X,r)),\quad X\in\Omega,\ r>0.$$

Let $(X',x)\in\Omega$ and suppose that

$$m_{(X',x)}(r)\le ar^\alpha,\quad r>0, \tag{13}$$

for some $a>0$. Then

$$\begin{aligned}\int_{B((X',x),x/2)} & G_\Omega((X',x),(Y',y))\,d\mu(Y',y)\\ &\le(2/x)\int_{B((X',x),x/2)}|(X',x)-(Y',y)|^{2-n}y\,d\mu(Y',y)\\ &=(2/x)\int_0^{x/2} r^{2-n}\,dm_{(X',x)}(r)\\ &\le(2/x)^{n-1}m_{(X',x)}(x/2)+(n-2)(2/x)\int_0^{x/2} r^{1-n}m_{(X',x)}(r)\,dr\\ &\le C(n,\alpha)ax^{\alpha+1-n},\end{aligned}$$

and, using (10),

$$\begin{aligned}
w(X',x)-\int_{B((X',x),x/2)} & G_\Omega((X',x),(Y',y))\,d\mu(Y',y)\\
&\le \int_{D\setminus B((X',x),x/2)} G_D((X',x),(Y',y))\,d\mu(Y',y)+\int_{\partial D} h_{Y'}(X)\,d\nu(Y',0)\\
&\le 2(n-2)x\int_{\overline{D}\setminus B((X',x),x/2)} |(X',x)-(Y',y)|^{-n}(y\,d\mu(Y',y)+d\nu(Y',y))\\
&= 2(n-2)x\int_{x/2}^{+\infty} r^{-n}\,dm_{(X',x)}(r)\\
&\le 2n(n-2)x\int_{x/2}^{+\infty} r^{-n-1}m_{(X',x)}(r)\,dr\\
&\le C(n,\alpha)ax^{\alpha+1-n}.
\end{aligned}$$

Thus

$$x^{n-1-\alpha}w(X',x)\le C(n,\alpha)a$$

for any (X',x) in Ω which satisfies (13). The subset S_a of $A_{i,j}$ where (13) fails to hold can be covered by a collection of open balls $B(X,r_X)$, such that $r_X<1+\operatorname{diam}(A')$ and $m_X(r_X)>ar_X^\alpha$. As in Section 2.3 there is a countable disjoint subcollection $\{B(X_k,r_k):k\ge 1\}$ such that $S_a\subset\bigcup_k B(X_k,5r_k)$. Hence

$$\begin{aligned}
\sum_k (5r_k)^\alpha &< \frac{5^\alpha}{a}\sum_k m_{X_k}(r_k)\\
&\le \frac{5^\alpha}{a}\left(\int_{A_{i,j}} y\,d\mu(Y',y)+\nu(A'_{i,j}\times\{0\})\right)\\
&\le \frac{5^\alpha i}{a}\left(\int_{A_{i,j}} G_\Omega(X_0,(Y',y))\,d\mu(Y',y)+\int_{A'_{i,j}\times\{0\}} u_{Y'}(X_0)\,d\nu(Y',0)\right)\\
&= \frac{5^\alpha i}{a}w(X_0)\le\frac{5^\alpha i}{a}(u(X_0)+k_0+1),
\end{aligned}$$

using (12) and the definition of $A_{i,j}$. Since $u(X_0)<+\infty$ and $w=u+k_0+1$ on $A_{i,j}$, apart from a polar set (which has Hausdorff dimension at most $n-2$), and since a can be arbitrarily large, we see that

$$\limsup_{x\to 0+} x^{n-1-\alpha}u(X',x)<+\infty,\qquad X'\in A'_{i,j}\setminus Z'_{i,j},$$

where $Z'_{i,j}$ has zero α-dimensional Hausdorff measure. Since this is true for any choice of i and j, we conclude from (11) that

$$\limsup_{x\to 0+} x^{n-1-\alpha}u(X',x)<+\infty,\qquad X'\in A'\setminus(F'\cup Z'),$$

where Z' has zero α-dimensional Hausdorff measure. This contradicts our hypothesis since A' has non-empty fine interior.

Theorem 1 is now proved in the case where $n\geq 3$. The case where $n=2$ is much easier: if E' is of second category (in $\mathbf{R}$) then there exists k_0 such that $\left(\overline{E'_{k_0}}\right)^\circ \neq \emptyset$, where E'_k is as defined above. Further, it follows easily from Wiener's criterion that $E'_{k_0}\times(0,1]$ is non-thin at each point of $\overline{E'_{k_0}}\times(0,1]$, so $u\geq -k_0$ on $\left(\overline{E'_{k_0}}\right)^\circ\times(0,1]$. Standard estimates for the Green function and Poisson kernel of a half-disc can now be used in conjunction with the argument following (12) above to obtain the result.

4. Proof of Theorem 2

To prove Theorem 2, we again suppose that the set (1) is not of first fine category and define k_0, W' and Ω as in the first paragraph of Section 3. Let F' denote the set (3). By hypothesis, the set $F'\cap W'$ has positive α-dimensional Hausdorff measure. Now

$$\mathbf{R}^{n-1}\backslash F' = \bigcup_{i=1}^{\infty}\bigcap_{j=1}^{\infty}\bigcup_{k=j}^{\infty} C'_{i,k},$$

where $C'_{i,k}$ is the canonical projection onto $\mathbf{R}^{n-1}$ of the closed set

$$\{(Y',y):(k+1)^{-1}\leq y\leq k^{-1} \text{ and } y^{n-1-\alpha}u(Y',y)\leq i\},$$

and since each $C'_{i,k}$ is closed, F' is a Borel subset of $\mathbf{R}^{n-1}$. Since the fine topology has a neighbourhood base consisting of Euclidean compact sets, and also has the quasi-Lindelöf property (see [5, 1.XI.11]), we can write W' as $A'\cup Z'$, where A' is a Euclidean F_σ set and Z' is polar in $\mathbf{R}^{n-1}$. The set Z' has zero α-dimensional Hausdorff measure because $\alpha>n-3$. Hence $F'\cap A'$ is a Borel subset of $F'\cap W'$ of positive α-dimensional measure, and it follows (see [3, II, Theorem 2]) that there is a compact subset K' of $F'\cap W'$ of positive α-dimensional measure. By Frostman's lemma (see [3, II, Theorem 1]), there is a non-zero measure ν on K' such that

$$\nu(B'(X',r))\leq r^\alpha,\quad r>0,\ X'\in\mathbf{R}^{n-1}.$$

The measure ν cannot charge any polar set in $\mathbf{R}^{n-1}$ since such a set has Hausdorff dimension at most $n-3$. Using integration by parts it follows that the Poisson integral

$$h(X)=\int_{\mathbf{R}^{n-1}} h_{Y'}(X)\,d\nu(Y'),\quad X\in D,$$

satisfies

$$
\begin{aligned}
h(X',x) &= 2n(n-2)x \int_0^{+\infty} \frac{r}{(r^2+x^2)^{n/2+1}} \nu(B'(X',r))\,dr \\
&\le 2n(n-2)x \int_0^{+\infty} \frac{r^{1+\alpha}}{(r^2+x^2)^{n/2+1}}\,dr \\
&\le 2n(n-2)x \left(\int_0^{x} \frac{r^{1+\alpha}}{x^{n+2}}\,dr + \int_x^{+\infty} r^{\alpha-1-n}\,dr \right) \\
&\le c x^{\alpha+1-n}
\end{aligned}
\tag{14}
$$

for $(X',x)\in D$, where $c=C(n,\alpha)$.

For each Y' in W', the set $D\backslash\Omega$ is minimally thin with respect to D at $(Y',0)$, so $u_{Y'}\not\equiv 0$, where $u_{Y'}$ is as defined in Section 3. Thus (see [15, Théorème 12]) $u_{Y'}$ is a minimal harmonic function on Ω. Let Y'_* denote the minimal Martin boundary point of Ω associated with $u_{Y'}$. Then, for any subset E of Ω, it follows from a result of Naïm [15, Théorème 15] that E is minimally thin with respect to D at $(Y',0)$ if and only if E is minimally thin with respect to Ω at Y'_*.

We define the sets

$$
F'_{i,j} = \{Y' \in K' : y^{n-1-\alpha} u(Y',y) \ge i \text{ for all } y \in (0,j^{-1})\}, \quad i,j \in \mathbf{N},
$$

which are finely closed, by Lemma 1 and the fine continuity of u. For each i, j there is a polar subset $Z'_{i,j}$ of $F'_{i,j}$ such that $F'_{i,j}$ is non-thin at each point of $F'_{i,j}\backslash Z'_{i,j}$ (see [5, 1.XI.6]). Hence, by Lemma 1 and the above result of Naïm, $F'_{i,j}\times(0,j^{-1})$ is not minimally thin with respect to Ω at Y'_* for each Y' in $F'_{i,j}\backslash Z'_{i,j}$. Since $u+k_0+1$ is a positive superharmonic function on Ω and

$$
u(X)+k_0+1 \ge \frac{i}{c} h(X), \quad X \in F'_{i,j}\times(0,j^{-1}),
$$

(see (14)), it follows (see [5, 1.XII.17, Application]) that

$$
u(X)+k_0+1 \ge \frac{i}{c} \int_{F'_{i,j}} u_{Y'}(X)\,d\nu(Y'), \quad X \in \Omega,
$$

(recall that ν does not charge the polar set $Z'_{i,j}$). Since $K'\subseteq F'$, we have $\bigcup_{j=1}^{\infty} F'_{i,j} = K'$, and hence

$$
u(X)+k_0+1 \ge \frac{i}{c} \int_{K'} u_{Y'}(X)\,d\nu(Y'), \quad X \in \Omega.
$$

Finally, since i can be arbitrarily large, we obtain the contradictory conclusion that $u\equiv+\infty$ on the open set Ω. Thus E' must be of first fine category and Theorem 2 is proved.

5. Proof of Theorem 3

The proof of Theorem 3 is similar in approach to that of Theorem 2. The set K' now has positive (Newtonian or logarithmic) capacity in $\mathbf{R}^{n-1}$, and hence is not σ-finite with respect to the Hausdorff measure associated with the measure function

$$\phi(t)=\begin{cases} t^{n-3}, & n\geq 4,\\ (\log^+ 1/t)^{-1}, & n=3,\end{cases}$$

(see [3, IV, Theorem 1]). It follows (see [17, pp. 83–84]) that there is an increasing continuous function $\Phi\colon(0,+\infty)\to(0,1]$ such that $\Phi(t)/\phi(t)\to 0$, as $t\to 0+$, and K' is not σ-finite with respect to the Hausdorff measure associated with the measure function Φ. By Frostman's lemma there is a measure ν on K' such that

$$\nu(B'(X',r))\leq\Phi(r),\quad r>0,\ X'\in\mathbf{R}^{n-1}.$$

Arguing as in (14), we see that the half-space Poisson integral h of ν satisfies

$$\begin{aligned}\frac{h(X',x)}{2n(n-2)x}&\leq\int_0^x\frac{r\Phi(r)}{x^{n+2}}\,dr+\int_x^{+\infty}r^{-n-1}\Phi(r)\,dr\\ &\leq\left(\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}\right)\left(\int_0^x\frac{r\phi(r)}{x^{n+2}}\,dr+\int_x^a r^{-n-1}\phi(r)\,dr\right)+\int_a^{+\infty}r^{-n-1}\,dr,\end{aligned}$$

when $0<x<a$. If $n\geq 4$, then

$$\frac{h(X',x)}{2n(n-2)x}\leq C(n)\left(\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}\right)\frac{1}{x^3}+\frac{a^{-n}}{n},$$

so

$$\limsup_{x\to 0+}x^2h(X',x)\leq C(n)\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}.$$

If $n=3$, then (provided $a\leq e^{-1/2}$)

$$\begin{aligned}\frac{h(X',x)}{6x}&\leq\left(\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}\right)\left(\int_0^x\frac{r}{x^5\log(1/x)}\,dr+\int_x^a\frac{3\log(1/r)-1}{r^4(\log(1/r))^2}\,dr\right)+\frac{1}{3a^3}\\ &\leq\left(\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}\right)\left(\frac{1}{2x^3\log(1/x)}+\left[\frac{-1}{r^3\log(1/r)}\right]_{r=x}^{r=a}\right)+\frac{1}{3a^3},\end{aligned}$$

so

$$\limsup_{x\to 0+}x^2\log(1/x)h(X',x)\leq C\sup_{0<t<a}\frac{\Phi(t)}{\phi(t)}.$$

In either case, since $\Phi(t)/\phi(t)\to 0$, as $t\to 0+$, and a can be arbitrarily small, we see that

$$\Psi_n(x)h(X',x)\to 0,\quad x\to 0+,\ X'\in\mathbf{R}^{n-1}.$$

The argument now proceeds as before except that we define

$$F'_{i,j}=\{Y'\in K':u(Y',y)\geq ih(Y',y)\ \text{for all}\ y\in(0,j^{-1})\}.$$

6. Sharpness of Theorems 1–3

6.1. The fine topology plays an essential role in Theorems 1–3, as the following example shows.

Example 1. If $n\geq 3$, then there is a superharmonic function u on D such that

$$x^{n-1}u(X',x)\to+\infty,\quad x\to 0+, \tag{15}$$

for $(n-1)$-almost every $X'\in\mathbf{R}^{n-1}$ and such that, for any non-empty open set $U'\subseteq\mathbf{R}^{n-1}$, the set

$$\left\{X'\in U':\liminf_{x\to 0+}u(X',x)>-\infty\right\} \tag{16}$$

is of second category with respect to the Euclidean topology on $\mathbf{R}^{n-1}$.

To see this, we slightly amend an argument of Rippon [16, Theorem 6]. Let $(E_k)_{k=1}^{\infty}$ be an increasing sequence of closed nowhere dense subsets of $\mathbf{R}$ such that the set $\bigcup_{k=1}^{\infty}E_k$ is of full measure in $\mathbf{R}$, and let

$$F=\bigcup_{k=1}^{\infty}(E_k\times(0,1/k]).$$

By a known approximation result (see [10, Corollary 3.21]) there is a harmonic function v on $\mathbf{R}\times(0,+\infty)$ such that

$$|v(x_1,x_2)-x_2^{-n}|<1,\quad (x_1,x_2)\in F.$$

Next, let s be a superharmonic function on $\mathbf{R}^{n-1}$ which is valued $+\infty$ on a dense set of points, and define

$$u(x_1,\dots,x_n)=v(x_1,x_n)+s(x_1,\dots,x_{n-1}),\quad (x_1,\dots,x_n)\in D.$$

Then (15) holds for almost every $X'\in\mathbf{R}^{n-1}$. Also, for any non-empty open set U' in $\mathbf{R}^{n-1}$, the set

$$U'\backslash\{X'\in\mathbf{R}^{n-1}:s(X')=+\infty\}=\bigcup_{k=1}^{\infty}\{X'\in U':s(X')\leq k\}$$

is of first category, in view of the lower semicontinuity of s, and so the set (16) is of second category in $\mathbf{R}^{n-1}$.

6.2. The next example demonstrates the sharpness of the growth condition in Theorem 1 when $n=2$. Let m_α denote α-dimensional Hausdorff measure.

Example 2. Let $D=\mathbf{R}\times(0,+\infty)$ and $0<\alpha\leq 1$. There is a positive harmonic function u on D such that, for every open interval I in $\mathbf{R}$,

$$m_\alpha\Big(\Big\{x\in I:\liminf_{y\to 0+} y^{1-\alpha}u(x,y)>0\Big\}\Big)>0.$$

If $\alpha=1$, then we can simply define $u\equiv 1$. Thus, in proving Example 2, we may suppose that $\alpha<1$. Let $\beta=2^{-1/\alpha}$. We construct a Cantor-type set as follows. Let $E_0=[0,1]$ and, for each j in $\mathbf{N}\cup\{0\}$, construct E_{j+1} by removing an open interval of proportion $1-2\beta$ from the centre of each interval of E_j. The set $E=\bigcap_{j=0}^{\infty}E_j$ then has finite positive α-dimensional Hausdorff measure (cf. [7, p. 15]). For each j, let μ_j be the unit measure consisting of 2^j equal point measures located at the midpoints of the intervals which comprise E_j. Then $(\mu_j)_{j=0}^{\infty}$ converges in the w^*-topology to a measure μ on E.

The constituent intervals of E_j are of length β^j. Hence, if $x\in E$,

$$\mu([x-\beta^j,x+\beta^j])\geq 2^{-j},\qquad j\geq 0,$$

so

$$\mu([x-r,x+r])\geq 2^{-j-1},\quad \beta^{j+1}\leq r<\beta^j,\ j\geq 0,$$

and thus

$$\mu([x-r,x+r])\geq 2^{-1}r^\alpha,\quad 0<r\leq 1.$$

Let h denote the Poisson integral of μ in D. Then

$$\begin{aligned}h(x,y)&=2y\int_{-\infty}^{+\infty}\frac{d\mu(z)}{(z-x)^2+y^2}\\&\geq y^{-1}\mu([x-y,x+y])\geq \tfrac{1}{2}y^{\alpha-1},\quad x\in E,\ 0<y\leq 1.\end{aligned}\tag{17}$$

Now let

$$v(x,y)=\sum_{i=-\infty}^{+\infty}2^{-|i|}h(x+i,y),\quad (x,y)\in D.$$

Since

$$\limsup_{y\to+\infty} yv(0,y)<+\infty,$$

it follows that a positive harmonic function u on D is defined by

$$u(x,y)=\sum_{k=1}^{\infty}v(2^kx,2^ky),\quad (x,y)\in D,$$

and it is clear from (17) that u has the property asserted in Example 2.

6.3. We now demonstrate the sharpness of the growth conditions in Theorems 1 and 2 when $n \geq 3$.

Example 3. Let $n \geq 3$.

(a) If $n-3<\alpha \leq n-1$, then there is a positive harmonic function u on D such that, for every non-empty finely open set U' in $\mathbf{R}^{n-1}$, $m_\alpha(E')>0$, where

$$E' = \left\{ X' \in U' : \liminf_{x \to 0+} x^{n-1-\alpha} u(X', x) > 0 \right\}.$$

(b) If $n-3<\alpha \leq n-2$, then there is a positive superharmonic function u on D such that, for every non-empty finely open set U' in $\mathbf{R}^{n-1}$,

$$m_\alpha \left(\left\{ X' \in E' : \limsup_{x \to 0+} x^{n-1-\alpha} u(X', x) = +\infty \right\} \right) > 0,$$

where E' is as in (a).

To see this, let $n-3<\alpha<n-1$. (We again dispense with the trivial case where $\alpha = n-1$.) We first note that there is a measure ν with support $E' \subseteq [0,1]^{n-1}$ such that $m_\alpha(E')>0$ and such that the Poisson integral h of ν in D satisfies

$$\liminf_{x \to 0+} x^{n-1-\alpha} h(X', x) > 0, \quad X' \in E'. \tag{18}$$

(This can be seen by taking ν to be a product measure formed from the measure μ in the proof of Example 2, using $\alpha/(n-1)$ in place of α, and arguing as in (17).) Now let

$$v(X', x) = \sum_{i' \in \mathbf{Z}^{n-1}} 2^{-|i'|} h(X'+i', x), \quad (X', x) \in D,$$

and define a positive harmonic function u on D by writing

$$u(X) = \sum_{k=1}^{\infty} v(2^k X), \quad X \in D.$$

If U' is any finely open set in $\mathbf{R}^{n-1}$ and $Z' \in U'$, then $\mathbf{R}^{n-1} \backslash U'$ is thin at Z'. Since $\alpha > n-3$, it follows (see [9, Theorem A]) that there is a sequence of balls $(B'(Z'_k, r_k))_{k=1}^{\infty}$ in $\mathbf{R}^{n-1}$ such that $r_k < \frac{1}{2}|Z'-Z'_k|$ for each k,

$$B'(Z', 1) \backslash U' \subseteq \bigcup_{k=1}^{\infty} B'(Z'_k, r_k)$$

and

$$\sum_{k=1}^{\infty} \left(\frac{r_k}{|Z'-Z'_k|} \right)^{\alpha} < +\infty.$$

Hence

$$2^{k\alpha}m_\alpha(B'(Z',2^{-k})\backslash U')\to 0, \quad \text{as } k\to\infty,$$

whereas (18) and our construction of u ensure that

$$\liminf_{k\to\infty} 2^{k\alpha}m_\alpha\Big(\Big\{X'\in B'(Z',2^{-k}):\liminf_{x\to 0+} x^{n-1-\alpha}u(X',x)>0\Big\}\Big)>0.$$

This establishes (a).

Further, if $n-3<\alpha\leq n-2$, then $E'\times\{0\}$ is polar in $\mathbf{R}^n$ since it has σ-finite m_α-measure. Thus there is a positive superharmonic function w on $\mathbf{R}^n$ such that $w=+\infty$ on the polar set $E'\times\mathbf{Q}$. If we add w to the harmonic function in (a), we obtain a positive superharmonic function on D with the desired properties.

6.4. Finally, we indicate the sharpness of the growth condition in Theorem 3.

Example 4. Let $n\geq 3$. If $0<\gamma<2$, then there is a positive harmonic function u on D such that, for every non-empty finely open set U' in $\mathbf{R}^{n-1}$, the set

$$\{X'\in U': x^\gamma u(X',x)\to+\infty, \text{ as } x\to 0+\} \tag{19}$$

is non-polar.

To see this, let $\alpha=n-2-\frac{1}{2}\gamma$. Then $n-3<\alpha<n-2$. Let u be the positive harmonic function of Example 3(a) and E' the set defined there. Then $m_\alpha(E')>0$ and hence E' is non-polar in $\mathbf{R}^{n-1}$. Since $\gamma<\frac{1}{2}(\gamma+2)=n-1-\alpha$, the set (19) contains E' and so is also non-polar.

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