Singular solutions to $p$-Laplacian type equations

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Abstract. We construct singular solutions to equations
\[ \text{div } \mathcal{A}(x, \nabla u) = 0, \]
similar to the $p$-Laplacian, that tend to $\infty$ on a given closed set of $p$-capacity zero. Moreover, we show that every $G_\delta$-set of vanishing $p$-capacity is the infinity set of some $\mathcal{A}$-superharmonic function.

1. Introduction

Suppose that $u$ is a solution of the $p$-Laplacian equation
\[ \text{div}(|\nabla u|^{p-2}\nabla u) = 0 \]
in an open subset $\Omega$ of $\mathbb{R}^n$. If $u(x)$ tends continuously to $\infty$ as $x$ approaches the boundary $\partial \Omega$ of $\Omega$, then it is easily seen that the complement of $\Omega$ is of $p$-capacity zero, i.e.
\[ C_p(\mathbb{C}\Omega) = 0; \]
see e.g. [HK, 3.4], [HKM, 10.5 and 10.6], or [R, Theorem 5.9]. The $p$-capacity of a set $E$ is defined as
\[ C_p(E) = \inf \int_{\mathbb{R}^n} (|\nabla \varphi|^p + |\varphi|^p) \, dx, \]
where $\varphi$ runs through all $\varphi \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ such that $\varphi \geq 1$ on an open neighborhood of $E$. In this paper we are interested in the converse problem: given a set $E$ of $p$-capacity zero, can one construct a solution to the $p$-Laplacian whose singularity is the set $E$?

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In the precise treatment we use the language of nonlinear potential theory [HKM]. We consider more general equations
\begin{equation}
\text{div} \mathcal{A}(x, \nabla u) = 0
\end{equation}
that are similar to the \( p \)-Laplacian; see Section 2.1. The continuous solutions to (1.1) are called \( \mathcal{A} \)-harmonic and \( \mathcal{A} \)-superharmonic functions are defined via a comparison with the \( \mathcal{A} \)-harmonic functions. The precise definitions and properties of \( \mathcal{A} \)-harmonic and \( \mathcal{A} \)-superharmonic functions are listed in Section 2.1 below. Roughly speaking, \( \mathcal{A} \)-superharmonic functions \( u \) are solutions of
\begin{equation}
-\text{div} \mathcal{A}(x, \nabla u) = \mu
\end{equation}
with nonnegative Radon measures \( \mu \).

It has been known for about a decade that sets of \( p \)-capacity zero can be characterized as \( \mathcal{A} \)-polar sets; a set \( E \) is called \( \mathcal{A} \)-polar if there is an \( \mathcal{A} \)-superharmonic function \( u \) on \( \mathbb{R}^n \) such that \( u = \infty \) on \( E \). This was first established by Lindqvist and Martio for the \( p=n \) case in [LM] and later for all \( p \) in [HK] (see [HKM, Chapter 10]). Note that the definition of an \( \mathcal{A} \)-polar set does not require that it be exactly the infinity set for some \( \mathcal{A} \)-superharmonic function. Since each \( \mathcal{A} \)-superharmonic function \( u \) is lower semicontinuous, we have that its set of infinity is a \( G_\delta \)-set, a countable intersection of open sets:
\[
\{ x : u(x) = \infty \} = \bigcap_j \{ x : u(x) > j \}.
\]
Therefore it is natural to ask whether, for a given \( G_\delta \) set \( E \) of \( p \)-capacity zero, there exists an \( \mathcal{A} \)-superharmonic \( u \) that is \( \infty \) exactly on \( E \). The first result in this direction is Theorem 1.7 in [K] which states that an \( \mathcal{A} \)-superharmonic function can be chosen to be \( \infty \) on \( E \) but finite at a given point outside \( E \). In this paper we give a complete affirmative answer to the question and prove the following.

1.3. Theorem. Suppose that \( E \) is a \( G_\delta \)-set of \( p \)-capacity zero. Then there is an \( \mathcal{A} \)-superharmonic function \( u \) in \( \mathbb{R}^n \) such that
\[
E = \{ x : u(x) = \infty \}.
\]
Moreover, if \( 1 < p < n \), then \( u \) can be chosen to be positive.

We want to emphasize that according to Theorem 1.3 the “true \( \mathcal{A} \)-polarity” is independent of the actual operator: if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are two mappings that satisfy the assumptions listed in Section 2.1 and \( u_1 \) is \( \mathcal{A}_1 \)-superharmonic, then there is an \( \mathcal{A}_2 \)-superharmonic \( u_2 \) in \( \mathbb{R}^n \) such that
\[
\{ x : u_1(x) = \infty \} = \{ x : u_2(x) = \infty \}.
\]
Moreover, we show that the function \( u \) given by Theorem 1.3 can be chosen to be \( \mathcal{A} \)-harmonic outside \( E \) if \( E \) is closed.
1.4. Theorem. Let $E$ be a relatively closed subset of an open set $\Omega$. If $E$ is of $p$-capacity zero, then there is a continuous $A$-superharmonic function $u$ in $\Omega$ such that

$$E = \{x \in \Omega : u(x) = \infty\}$$

and $u$ is $A$-harmonic in $\Omega \setminus E$.

Since $A$-superharmonic functions solve equations like (1.2), we may interpret Theorem 1.4 as follows: there is a Radon measure $\mu$ supported on any given closed set $E$ of $p$-capacity zero so that $\mu$ is concentrated at each point of $E$. The precise meaning of this statement will be made clear later.

In the classical linear case Theorem 1.4 was first proven by Evans [E] whence such a function is often called an Evans potential. Later Choquet [C] extended it for a general $G_\delta$-set $E$ of capacity zero. In the case where $E$ is countable and compact Theorem 1.4 is established in Holopainen’s thesis [H]. In Section 3 we prove a slightly more general result than Theorem 1.4: any $G_\delta$-set $E$ of $p$-capacity zero that is also an $F_\sigma$-set (a countable union of compact sets) is the infinity set for some $A$-superharmonic function that is also $A$-harmonic in the complement of the closure of $E$.

Our method of proof is based on the potential estimate of the author and Malý [KM2] (see Theorem 2.12 below) that allows us to convert the construction of solutions of nonlinear equations into the construction of Radon measures with certain density properties. Indeed, there is a correspondence between Radon measures $\mu$ and $A$-superharmonic functions $u$ by

$$\mu = -\text{div} A(x, \nabla u).$$

Moreover, the local behavior of an $A$-superharmonic function $u$ whose “Riesz mass” is $\mu$ can be controlled in terms of a nonlinear potential, the Wolff potential of $\mu$,

$$W^{\mu}_{\frac{n}{p}-p}(x, r) = \int_0^r \left(\frac{\mu(B(x, t))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t}.$$ 

In particular, it was proven in [KM2] that $u(x) = \infty$ exactly when $W^{\mu}_{\frac{n}{p}-p}(x, r) = \infty$. So the proof of Theorem 1.3 reduces to constructing a measure $\mu$ such that $E$ is the set of infinity of its Wolff potential and then pick an $A$-superharmonic function whose Riesz mass $\mu$ is.

The main new trick in this paper is the “sweeping” of the nonlinear Riesz mass onto $E$ so that the Wolff potential does not get essentially smaller; this is done in Section 2.14 below.

I would like to thank Seppo Rickman whose questions persuaded me to reconsider these problems.
2. Measures, potentials, and $\mathcal{A}$-superharmonic functions

2.1. Preliminaries

Throughout the paper we let $\Omega$ denote an open set in $\mathbb{R}^n$ and $1 < p \leq n$ is a fixed number; note that the case $p > n$ is trivial since then no nonempty set is of $p$-capacity zero. Moreover, we assume that $\mathcal{A}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping which satisfies the following assumptions for some constants $0 < \alpha \leq \beta < \infty$:

\begin{align*}
\text{(2.2)} & \quad \text{the function } x \mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \text{ and} \\
\text{(2.3)} & \quad \text{the function } \xi \mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n,
\end{align*}

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$,

\begin{align*}
\text{(2.4)} & \quad \mathcal{A}(x, \xi) \cdot \xi \geq \alpha |\xi|^p, \\
\text{(2.5)} & \quad |\mathcal{A}(x, \xi)| \leq \beta |\xi|^{p-1}, \\
\text{(2.6)} & \quad (\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)) \cdot (\xi - \zeta) > 0,
\end{align*}

whenever $\xi \neq \zeta$, and

\[ \mathcal{A}(x, \lambda \xi) = \lambda |\lambda|^p \mathcal{A}(x, \xi) \]

for all $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

We define the divergence of $\mathcal{A}(x, \nabla u)$ in the sense of distributions, i.e. if $\varphi \in C_0^\infty(\Omega)$, then

\[ \text{div } \mathcal{A}(x, \nabla u)(\varphi) = - \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx, \]

where $u \in W^{1,p}_{\text{loc}}(\Omega)$. A solution $u \in W^{1,p}_{\text{loc}}(\Omega)$ to the equation

\[ \text{div } \mathcal{A}(x, \nabla u) = 0 \]

always has a continuous representative; we call continuous solutions $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ of (2.7) $\mathcal{A}$-harmonic in $\Omega$.

A lower semicontinuous function $u: \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$-superharmonic if $u$ is not identically infinite in each component of $\Omega$, and if for all open $D \subset \subset \Omega$ and all $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in $D$, $h \leq u$ on $\partial D$ implies $h \leq u$ in $D$.

The following connection between $\mathcal{A}$-superharmonic functions and supersolutions of (2.7) is fundamental.
2.8. Proposition. ([HKM, 7.25]) (i) If \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is such that

\[- \text{div} \mathcal{A}(x, \nabla u) \geq 0,\]

then there is an \( \mathcal{A} \)-superharmonic function \( v \) such that \( u = v \) a.e. Moreover,

\( v(x) = \text{ess lim inf}_{y \to x} v(y) \quad \text{for all } x \in \Omega. \)  \hspace{1cm} (2.9)

(ii) If \( v \) is \( \mathcal{A} \)-superharmonic, then (2.9) holds. Moreover, \(- \text{div} \mathcal{A}(x, \nabla v) \geq 0\) if \( v \in W^{1,p}_{\text{loc}}(\Omega).\)

(iii) If \( v \) is \( \mathcal{A} \)-superharmonic and locally bounded, then \( v \in W^{1,p}_{\text{loc}}(\Omega) \) and

\[- \text{div} \mathcal{A}(x, \nabla v) \geq 0.\]

Because an \( \mathcal{A} \)-superharmonic function does not necessarily belong to \( W^{1,p}_{\text{loc}}(\Omega) \), we extend the definition for the divergence of \( \mathcal{A}(x, \nabla u) \): If \( u \) is an \( \mathcal{A} \)-superharmonic function in \( \Omega \), then we define

\[- \text{div} \mathcal{A}(x, \nabla u)(\varphi) = \int_{\Omega} \lim_{k \to \infty} \mathcal{A}(x, \nabla \min(u, k)) \cdot \nabla \varphi \, dx, \quad \varphi \in C_0^\infty(\Omega).\]

By [KM1, 1.15]

\[ \lim_{k \to \infty} \mathcal{A}(x, \nabla \min(u, k)) \]

is locally integrable and hence \( \text{div} \mathcal{A}(x, \nabla u) \) is its divergence. (Since the truncations \( \min(u, k) \) are in \( W^{1,p}_{\text{loc}}(\Omega) \) and

\[ \nabla \min(u, k) = \nabla \min(u, j) \quad \text{a.e. in } \{ u < \min(k, j) \}, \]

the limit exists. It is equal to \( \mathcal{A}(x, \nabla u) \) if \( u \in W^{1,1}_{\text{loc}}(\Omega), \) which is always the case if \( p > 2 - 1/n. \) Our definition treats the difficulty that arises from the fact that for \( p \leq 2 - 1/n \) the distributional gradient \( \nabla u \) need not be a function. Indeed, the above definition of \( \text{div} \mathcal{A}(x, \nabla u) \) is merely a technical tool to treat all \( p \)'s simultaneously.

Since \(- \text{div} \mathcal{A}(x, \nabla u)\) is a nonnegative distribution in \( \Omega \) for an \( \mathcal{A} \)-superharmonic \( u \) it follows that there is a nonnegative Radon measure \( \mu \) such that

\[- \text{div} \mathcal{A}(x, \nabla u) = \mu \]

in \( \Omega \); this measure \( \mu \) is sometimes referred to as the \textit{Riesz mass of } \( u. \) Conversely, given a finite measure \( \mu \) in a bounded \( \Omega \), there is an \( \mathcal{A} \)-superharmonic function \( u \) such that \(- \text{div} \mathcal{A}(x, \nabla u) = \mu \) in \( \Omega \) and \( \min(u, k) \in W^{1,p}_{0}(\Omega) \) for all integers \( k. \) We refer to [KM1] and [HKM, Chapter 7] for details.

The existence of \( \mathcal{A} \)-superharmonic solutions to \(- \text{div} \mathcal{A}(x, \nabla u) = \mu \) in bounded \( \Omega \) is not adequate for us. Hence we establish the following theorem.
2.10. Theorem. Let $\mu$ be a finite Radon measure in $\mathbb{R}^n$. Then there is an $A$-superharmonic function $u$ in $\mathbb{R}^n$ such that

$$- \text{div } A(x, \nabla u) = \mu$$

in $\mathbb{R}^n$. If $1 < p < n$, then $u$ can be chosen to be positive.

Proof. The case $1 < p < n$ is easy and it follows by employing an argument similar to that used in [KM1]; an existence result is also proven in [BGPV, Theorem 8.1] except for the fact that $u$ is $A$-superharmonic. The details are left to the reader.

We outline how the argument of [DHM] should be modified to obtain our theorem in the case $p=n$. (I thank Stefan Müller for showing me an early draft of the paper [DHM].)

Choose a sequence $\mu_k \in C^\infty_0(B(0,k))$ of nonnegative functions (measures) such that $\mu_k \to \mu$ weakly in the sense of measures. Let $v_k$ be the $A$-superharmonic solution of the problem

$$\begin{cases} - \text{div } A(x, \nabla v_k) = \mu_k & \text{on } B(0,k), \\ v_k = 0 & \text{on } \partial B(0,k). \end{cases}$$

Using a rescaling argument similarly as in [DHM] we infer that for $u_k = v_k - c_k$, where $c_k$ is a constant, it holds that

$$\int_{B(0,1)} u_k \, dx = 0$$

and

$$[u_k]_{\text{BMO}} \leq c \|\mu_k\|^{1/(n-1)} \leq c(\|\mu\| + 1)^{1/(n-1)}.$$  

Since $u_k$ is bounded in BMO and since its mean value in the unit ball vanishes, it follows that $u_k$ is bounded in $L^n(B)$, where $B$ is any large ball. Now we obtain from the estimate [HKM, 3.36] that the sequence $u_k$ is uniformly bounded from below in $\frac{1}{2}B$. Hence it follows from [KM1, 1.15] that a subsequence of $u_k$ converges a.e. to an $A$-superharmonic function $u$ in $\mathbb{R}^n$. Moreover, $\nabla u_j \to \nabla u$ both a.e. pointwise and in $L^q_{\text{loc}}(\mathbb{R}^n)$ for $q < n$. In conclusion,

$$- \text{div } A(x, \nabla u) = \mu$$

in $\mathbb{R}^n$, as desired. \( \square \)

2.11. Wolff potentials

The fact that an $A$-superharmonic function can be locally estimated in terms of its Riesz mass is very useful. In our problem these estimates enable us to change
the construction of $\mathcal{A}$-superharmonic functions (solutions to nonlinear equations) to a much easier task: to construct certain Radon measures.

To make this precise we recall that \textit{the Wolff potential of the measure $\mu$ is}

$$W_{1,p}^\mu(x_0, r) = \int_0^r \left( \frac{\mu(B(x_0, t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}, \quad r > 0.$$ 

We next record the fundamental potential estimate.

\textbf{2.12. Theorem. ([KM2, 1.6]) Suppose that $u$ is a nonnegative $\mathcal{A}$-superharmonic function in $B(x_0, 3r)$. If $\mu = -\text{div}\mathcal{A}(x, \nabla u)$, then}

$$c_1 W_{1,p}^\mu(x_0, r) \leq u(x_0) \leq c_2 \inf_{B(x_0, r)} u + c_3 W_{1,p}^\mu(x_0, 2r),$$

where $c_1$, $c_2$, and $c_3$ are positive constants, depending only on $n$, $p$, and the structural constants $\alpha$ and $\beta$.

In particular, $u(x_0) < \infty$ if and only if $W_{1,p}^\mu(x_0, r) < \infty$ for some $r > 0$.

The following simple lemma will be used to estimate $\inf u$ that appears in the potential estimate above.

\textbf{2.13. Lemma. ([KM2, 3.9]) Suppose that $u \in W^{1,p}_0(\Omega)$ is $\mathcal{A}$-superharmonic in $\Omega$ and $\mu = -\text{div}\mathcal{A}(x, \nabla u)$. Then for $\lambda > 0$ it holds that}

$$\lambda^{p-1} \text{cap}_p(\{x \in \Omega: u(x) > \lambda\}, \Omega) \leq \frac{\mu(\Omega)}{\alpha}.$$ 

Recall that $\text{cap}_p(E, \Omega)$ stands for the relative $p$-capacity of $E$ in $\Omega$ which for $E \subseteq \Omega$ is defined as

$$\text{cap}_p(E, \Omega) = \inf_{G \subseteq \Omega \text{ open}} \sup_{K \subseteq G \text{ compact}} \text{cap}_p(K, \Omega),$$

where

$$\text{cap}_p(K, \Omega) = \inf \int_\Omega |\nabla u|^p \, dx;$$

here $u$ runs through all $u \in C_0^\infty(\Omega)$ with $u \geq 1$ on $K$.

\textbf{2.14. Sweeping the measure}

Let $K \subseteq \mathbb{R}^n$ be a closed set and let $\mu$ be a finite Radon measure. Our goal is to find a new Radon measure $\tilde{\mu}$, supported on $K$, such that the total mass of $\mu$ is preserved and the Wolff potential of $\tilde{\mu}$ is not essentially smaller than that of $\mu$ on $K$. More precisely, we prove the following theorem.
2.15. Theorem. Let $K \subset \mathbb{R}^n$ be a closed set and let $\mu$ be a finite Radon measure. Then there is a Radon measure $\tilde{\mu}$ such that

(i) the support of $\tilde{\mu}$ is contained in $K$ and thus $\tilde{\mu}(\partial K) = 0$,
(ii) $\mu(\mathbb{R}^n) = \tilde{\mu}(\mathbb{R}^n)$, and
(iii) there is $c = c(n,p) > 0$ such that

$$W^\tilde{\mu}(x, 7t) \geq c W^\mu(x, t)$$

for each $x \in K$ and $t > 0$.

Later we sometimes refer to the measure $\tilde{\mu}$ with the above properties as the swept out measure of $\mu$ into $K$.

Proof. Let $W$ be the Whitney decomposition of $\mathbb{R}^n$, i.e. $W$ is a countable collection of pairwise disjoint cubes $Q$ (with parts of the boundaries included) such that

$$\bigcup_{Q \in W} Q = \mathbb{R}^n$$

and

$$\text{diam}(Q) \leq \text{dist}(Q, K) \leq 4 \text{diam}(Q).$$

For each $Q \in W$ choose a point $x_Q \in K$ with

$$\text{dist}(x_Q, Q) \leq 5 \text{diam}(Q).$$

Define

$$\tilde{\mu} = \mu|_K + \sum_{Q \in W} \mu(Q) \delta_{x_Q},$$

where $\delta_y$ is the Dirac measure at $y$ and and $\mu|_K$ stands for the restriction to $K$ of the measure $\mu$, i.e.

$$\mu|_K(E) = \mu(E \cap K) \quad \text{for} \quad E \subset \mathbb{R}^n.$$

Then $\tilde{\mu}$ defines a finite Radon measure supported on $K$ with $\tilde{\mu}(K) = \mu(\mathbb{R}^n)$. Moreover, we have the estimate

$$\tilde{\mu}(B(x, 7r)) \geq \mu(B(x, r))$$

for $x \in K$ and $r > 0$. Indeed, fix $x \in K$ and $r > 0$, and let

$$W_r = \{Q \in W : Q \cap B(x, r) \neq \emptyset\}.$$

If $Q \in W_r$, choose $y \in Q \cap B(x, r)$. Then

$$|x_Q - y| \leq 6 \text{diam}(Q) \leq 6 \text{dist}(Q, K) \leq 6|x - y| < 6r,$$
and hence
\[ |x_Q - x| \leq |x_Q - y| + |x - y| < 7r, \]
or \( x_Q \in B(x, 7r) \). Consequently,
\[
\mu(B(x, r)) \leq \mu(B(x, r) \cap K) + \sum_{Q \in W_r} \mu(Q) \\
\leq \mu(B(x, 7r) \cap K) + \sum_{x_Q \in B(x, 7r)} \mu(Q) = \tilde{\mu}(B(x, 7r)).
\]

Next we write the estimate of the Wolff potential: if \( x \in K \) and \( t > 0 \), then by (2.16)
\[
W^{\tilde{\mu}}(x, 7t) = \int_0^{7t} \left( \frac{\mu(B(x, r))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \\
= \int_0^{7t} \left( \frac{\mu(B(x, r/7))}{r^{n-p}} \right)^{1/(p-1)} \frac{dr}{r} \\
= \int_0^{7t} \left( \frac{\mu(B(x, s))}{s^{n-p}} \right)^{1/(p-1)} \frac{ds}{s} = \frac{7(p-n)/(p-1)}{7(p-n)/(p-1)} W^{\mu}(x, t),
\]
and the theorem follows. \( \square \)

3. The \( F_\sigma \) case

In this section we prove the following.

3.1. Theorem. Suppose that \( E \) is a \( G_\delta \)-set of \( p \)-capacity zero. If \( E \) is also an \( F_\sigma \)-set, then there is an \( \mathcal{A} \)-superharmonic function \( u \) in \( \mathbb{R}^n \) such that
\[ E = \{ x : u(x) = \infty \} \]
and \( u \) is \( \mathcal{A} \)-harmonic in \( \overline{\mathbb{R}^n} \).

Proof. Choose an increasing sequence \( K_j \) of compact sets and a decreasing sequence \( G_k \) of open sets such that
\[ E = \bigcup_{j=1}^\infty K_j = \bigcap_{k=1}^\infty G_k. \]
Let \( u \) be an \( \mathcal{A} \)-superharmonic function in \( \mathbb{R}^n \) such that \( u = \infty \) on \( E \) and write
\[ \mu = - \text{div} \ A(x, \nabla u). \]
Sweep the measure $\mu$ onto $K_j$ by Theorem 2.15: obtain a Radon measure $\mu_j$ supported on $K_j$ such that $\mu_j(R^n) \leq 1$ and

$$W^{\mu_j}(x, 1) = \infty \quad \text{for each } x \in K_j.$$ 

By multiplying $\mu_j$ with a positive constant $\leq 1$ we may assume that

$$W^{\mu_j}(x, 1) < 4^{-j} \quad \text{whenever } x \notin G_j;$$

observe that $\text{dist}(K_j, CG_j) > 0$ and $\text{supp} \mu_j \subset K_j$.

Let

$$\sigma = \sum_{j=1}^{\infty} 2^{-j} \mu_j.$$ 

Then $\sigma$ is a finite Radon measure with $\sigma(\mathcal{L}E) = 0$. Moreover,

$$W^\sigma(x, 1) \geq 2^{-j/(p-1)} W^{\mu_j}(x, 1) = \infty \quad \text{for } x \in K_j$$

whence

$$W^\sigma(x, 1) = \infty \quad \text{for each } x \in E.$$ 

We next show that

$$W^\sigma(x, 1) < \infty \quad \text{if } x \notin E.$$ 

For this we use the estimate

$$\sigma(A)^{1/(p-1)} \leq \begin{cases} \sum_{j=1}^{\infty} 2^{-j/(p-1)} \mu_j(A)^{1/(p-1)}, & \text{if } p \geq 2, \\ \left(\sum_{j=1}^{\infty} 2^{-j/(2-p)}\right)^{(2-p)/(p-1)} \sum_{j=1}^{\infty} \mu_j(A)^{1/(p-1)}, & \text{if } p < 2. \end{cases}$$

Hence

$$W^\sigma(x, 1) \leq \begin{cases} \sum_{j=1}^{\infty} 2^{-j/(p-1)} W^{\mu_j}(x, 1), & \text{if } p \geq 2, \\ \left(\sum_{j=1}^{\infty} 2^{-j/(2-p)}\right)^{(2-p)/(p-1)} \sum_{j=1}^{\infty} W^{\mu_j}(x, 1), & \text{if } p < 2. \end{cases}$$

Next we observe that if $x \notin E$, then $x \notin G_j$ except possibly for finitely many, say $k_x$, $j$'s, and therefore $W^\sigma(x, 1)$ does not exceed

$$\sum_{j=1}^{k_x} 2^{-j/(p-1)} W^{\mu_j}(x, 1) + \sum_{j=k_x+1}^{\infty} 2^{-j/(p-1)} W^{\mu_j}(x, 1) \quad \text{if } p \geq 2.$$
and
\[
\left( \sum_{j=1}^{\infty} 2^{-j/(2-p)} \right)^{(2-p)/(p-1)} \left( \sum_{j=1}^{k_x} W^{\mu}(x, 1) + \sum_{j=k_x+1}^{\infty} W^{\mu}(x, 1) \right) \quad \text{if } p < 2.
\]

Finally since
\[
W^{\mu}(x, 1) < \begin{cases} 4^{-j}, & \text{if } x \notin G_j, \text{ i.e. } j > k_x, \\ \infty, & \text{if } x \notin E, \text{ i.e. } j < k_x, \end{cases}
\]
we infer that
\[
W^{\sigma}(x, 1) < \infty \quad \text{for all } x \notin E,
\]
as desired.

Now we are in the position to conclude the proof: by Theorem 2.10 there is an $A$-superharmonic function $v$ on $\mathbb{R}^n$ such that
\[- \operatorname{div} A(x, \nabla v) = \sigma.\]
The potential estimate Theorem 2.12 implies that $v(x) = \infty$ if and only if $x \in E$. Moreover, $v$ is $A$-harmonic in $\overline{E}$, for $\sigma(\overline{E}) = 0$ (see [M, 3.19]).

\section{4. Polar set as the set of infinity of an $A$-superharmonic function}

In this section we prove Theorem 1.3. We start with a lemma whose proof is displeasingly technical.

\textbf{4.1. Lemma.} Suppose that $\Omega$ is a bounded open set and $E \Subset \Omega$ is of $p$-capacity zero. Let $F \subset \overline{E}$ be closed. Then there is an $A$-superharmonic function $u$ in $\mathbb{R}^n$ such that $u \in W^{1,p}_0(\Omega)$, $u = \infty$ on $E$, and $u \leq 1$ on $F \cap \Omega$. Moreover, $u$ can be chosen so that $\mu(\Omega) = 1$, where $\mu = - \operatorname{div} A(x, \nabla u)$.

\textbf{Proof.} We assume, as we well may, that $F$ contains a neighborhood of $\partial \Omega$. Then choose open sets $G_1 \supset G_2 \supset \ldots \supset \bigcap_j G_j = F$. Let $v$ be an $A$-superharmonic function in $\mathbb{R}^n$ such that $v = \infty$ on $E$ and let $\sigma = - \operatorname{div} A(x, \nabla v)$. Now sweep the measure $\sigma$ into $\Omega \setminus G_j$ and let $\sigma_j$ stand for the swept out measure (see Theorem 2.15). As in the proof of the $F_\sigma$ case (Theorem 3.1) we find positive constants $c_j$ such that for the measure
\[
\mu = \sum_{j=1}^{\infty} c_j \sigma_j
\]
it holds that $\mu(\mathbb{R}^n) \leq 1$,
\[
W^{\mu}_{1,p}(x, 1) \leq 1 \quad \text{for all } x \in F,
\]
and
\[ W_{1,p}^{\mu}(x,1) \geq W_{1,p}^{c_{j}(x,1)} = \infty \quad \text{for all } x \in E \setminus G_j, \]
whence
\[ W_{1,p}^{\mu}(x,1) = \infty \quad \text{for all } x \in E. \]

To complete the proof we let \( w \) be an \( \mathcal{A} \)-superharmonic solution of the problem
\[
\begin{cases}
- \text{div} A(x, \nabla w) = \mu & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Now \( w \) is not quite that function we are looking for but close to it. To find the final function we choose an open set \( D \Subset \Omega \) such that \( E \Subset D, \partial D \subset F, \partial D = \partial \overline{D} \), and that all points on \( \partial D \) are regular points for the Dirichlet problem in \( \Omega \setminus \overline{D} \) (for instance, \( D \) may be a polyhedron; see [HKM]). Since the distance between \( D \) and \( \partial \Omega \) is positive, there is \( r_0 > 0 \) such that \( B(x,3r_0) \subset \Omega \) whenever \( x \in \overline{D} \). Now we infer from the estimate Lemma 2.13(2) that
\[
\inf_{B(x,r_0)} w \leq C \left( \frac{\mu(R^n)}{\text{cap}_p(B(x,r_0),\Omega)} \right)^{1/(p-1)} \leq C < \infty,
\]
where \( C \) is independent of the point \( x \in D \). Hence the potential estimate in Theorem 2.12 implies that \( w \leq c_0 \) in \( F \cap \overline{D} \).

Next we observe that the function \( \log(w+1) \) is a positive \( \mathcal{A} \)-superharmonic function in \( \Omega \), uniformly bounded from above in \( F \cap \overline{D} \), and \( \in W^{1,p}_{0}(\Omega) \) (see [HKM, 7.48]). If \( h \in W^{1,p}(\Omega \setminus \overline{B}) \) is the \( \mathcal{A} \)-harmonic function in \( \Omega \setminus \overline{B} \) that agrees with \( \log(w+1) \) on \( \partial D \) and with \( 0 \) on \( \partial \Omega \) (in the Sobolev sense), then the function
\[
\tilde{u} = \begin{cases}
\log(w+1) & \text{in } \overline{D}, \\
h & \text{in } \Omega \setminus \overline{D}
\end{cases}
\]
is a positive \( \mathcal{A} \)-superharmonic function in \( \Omega \) by the pasting lemma [HKM, 7.9], since \( h = \min(h, \log(w+1)) \) in \( \Omega \setminus D \). Moreover, \( \tilde{u} \in W^{1,p}_{0}(\Omega) \), \( \tilde{u} = \infty \) on \( E \), \( \tilde{u} \) is uniformly bounded from above in \( F \cap \Omega \), and its Riesz mass \( - \text{div} A(x, \nabla \tilde{u}) \) is finite in \( \Omega \). In conclusion, we may choose a constant \( \lambda > 0 \) such that the function \( u = \lambda \tilde{u} \) enjoys the desired properties of the lemma. \( \square \)

(2) Of course, \( w \notin W^{1,p}_{0}(\Omega) \) contrary to the assumptions of Lemma 2.13. However, following the standard construction of \( \mathcal{A} \)-superharmonic solutions to (4.2) as done e.g. in [KM1] one is easily convinced that there is a solution \( w \) of (4.2) for which the estimate of Lemma 2.13 holds. Let us pick such a function \( w \).
Proof of Theorem 1.3. Let $B_k = B(0, 2^k)$ and $E_k = E \cap B_{k-1}$. Suppose that $G_j$ are open sets such that $G_1 \supset G_2 \supset \ldots \supset \bigcap_j G_j = E$. Fix $k$. Let $u_j$ be an $A$-superharmonic function in $B_k$ with finite Riesz mass $-\text{div} A(x, \nabla u_j)$ such that $u_j = \infty$ on $E_k$, $u_j \in W_0^{1,p}(B_k)$, and $u_j \leq 1$ on $B_k \setminus (G_j \cap B_{k-1})$ (Lemma 4.1). Let

$$ v_j = \min\left(4^j/(p-1), \lambda_j u_j\right), $$

where $0 < \lambda_j < 1$ is chosen so that $v_j \leq 2^{-j}$ on $B_k \setminus (G_j \cap B_{k-1})$ and $\mu_j(B_k) \leq 1$, $\mu_j = -\text{div} A(x, \nabla v_j)$; for the last property observe that $\mu_j$ agrees with the Riesz measure of $\lambda_j^{p-1} u_j$ on $B_k \setminus B_{k-1}$ and

$$ \mu_j\left(\frac{3}{2} B_{k-1}\right) \leq \int_{B_k} \varphi \ d\mu_j = \int_{B_k} A(x, \nabla v_j) \cdot \nabla \varphi \ dx $$

$$ \leq c \left( \int_{B_k} |\nabla v_j|^p \ dx \right)^{(p-1)/p} \left( \int_{B_k} |\nabla \varphi|^p \ dx \right)^{1/p} \leq c \lambda_j^{p-1} \left( \int_{B_k} |\nabla u_j|^p \ dx \right)^{(p-1)/p}, $$

where $\varphi \in C_0^\infty(B_k)$ is any nonnegative function with $\varphi = 1$ on $\frac{3}{2} B_{k-1}$.

Now by the estimate Lemma 2.13 we have for $x \in B_{k-1}$ that

$$ \inf_{B(x,1)} v_j \leq c \left( \frac{\mu_j(B_k)}{\text{cap}_p(B(x,1), B_k)} \right)^{1/(p-1)} \leq M, $$

where $M$ is independent of $j$. Thus by the potential estimate in Theorem 2.12

$$ c W_{1,p}^{\mu_j}(x, 2) \geq v_j(x) - c \inf_{B(x,1)} v_j \geq 3^{j/(p-1)} $$

if $x \in E_k$ and $j$ is large enough.

Setting

$$ \mu^{(k)} = \sum_{j=1}^{\infty} 2^{-j} \mu_j $$

we obtain a finite Radon measure on $\mathbb{R}^n$ with the properties

\[
\begin{align*}
W_{1,p}^{\mu^{(k)}}(x, 1) &< \infty \quad \text{for all } x \notin E_k, \\
W_{1,p}^{\mu^{(k)}}(x, 1) &= \infty \quad \text{for all } x \in E_k.
\end{align*}
\]

Indeed, the finiteness of $W_{1,p}^{\mu^{(k)}}(x, 1)$ outside $E_k$ is proven similarly as in the $F_\sigma$ case (see 3.1). Further, if $x \in E_k$, then for $j$ large enough it holds that

$$ W_{1,p}^{\mu^{(k)}}(x, 2) \geq c 2^{-j/(p-1)} 3^{j/(p-1)} \geq \left(\frac{3}{2}\right)^{j/(p-1)}, $$
whence $W_{1,p}^{\mu^{(k)}}(x,1) = \infty$.

To complete the proof we write

$$
\mu = \sum_{k=1}^{\infty} \mu^{(k)}|_{B_k \setminus B_{k-2}},
$$

where $\mu^{(k)}|_{B_k \setminus B_{k-2}}$ stands for the restriction to $B_k \setminus B_{k-2}$ of the measure $\mu^{(k)}$, and $B_0 = B_{-1} = \emptyset$. Then $\mu$ is a finite Radon measure in $\mathbb{R}^n$ and

$$
W_{1,p}^{\mu}(x,1) = \infty \quad \text{if and only if} \quad x \in E.
$$

An $\mathcal{A}$-superharmonic solution of $-\operatorname{div} A(x, \nabla u) = \mu$ on $\mathbb{R}^n$ is the desired function $u$ (see Theorems 2.10 and 2.12). \qed

References


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