Ark. Mat., **37** (1999), 305–322

On Kneser solutions of higher order nonlinear ordinary differential equations

Vladimir A. Kozlov(1)

Dedicated to Professor Vladimir Maz'ya on the occasion of his 60th birthday.

Abstract. The equation $x^{(n)}(t)=(-1)^n|x(t)|^k$ with $k>1$ is considered. In the case $n\leq 4$ it is proved that solutions defined in a neighbourhood of infinity coincide with $C(t-t_0)^{-n/(k-1)}$, where C is a constant depending only on n and k. In the general case such solutions are Kneser solutions and can be estimated from above and below by a constant times $(t-t_0)^{-n/(k-1)}$. It is shown that they do not necessarily coincide with $C(t-t_0)^{-n/(k-1)}$. This gives a negative answer to two conjectures posed by Kiguradze that Kneser solutions are determined by their value in a point and that blow-up solutions have prescribed asymptotics.

1. Introduction

In [KM1] V. Maz'ya and the author studied a linear equation of the form

$$\mathcal{M}\,(d/dt)x(t)-\omega(t)x(t)=g(t),$$

where $\mathcal{M}$ is an ordinary differential operator with positive Green's function and ω is a nonnegative function. This equation plays an important role in the theory of linear differential equations with operator coefficients developed in [KM2]. An attempt to extend the linear theory to the nonlinear case leads to equations of the above form with $\omega(t)x(t)$ replaced by a nonlinear term $f(t,x(t))$, where f is a nonnegative function. This paper deals with one such type of equations,

$$x^{(n)}(t)=(-1)^n|x(t)|^k, \tag{1}$$

with $k>1$. Here $x^{(n)}$ denotes the nth derivative of x with respect to t.

(1) The author was supported by the Swedish Natural Science Research Council (NFR) grant M-AA/MA 10879-304.

One can directly verify that the functions

$$x_a(t)=(\alpha(\alpha+1)\dots(\alpha+n-1))^{1/(k-1)}(t-a)^{-\alpha} \tag{2}$$

and

$$x_b(t)=(-1)^n(\alpha(\alpha+1)\dots(\alpha+n-1))^{1/(k-1)}(b-t)^{-\alpha}, \tag{3}$$

where

$$\alpha=\frac{n}{(k-1)}, \tag{4}$$

satisfy (1) on the semiaxes (a,∞) and $(-\infty,b)$, respectively. One of the main questions studied here is:

Do solutions of (1), *defined in a neighbourhood of* $\pm\infty$, *coincide with* x_a *or* x_b?

In Section 6 the existence of n and k, $k>1$, such that equation (1) has solutions of the form

$$(t-a)^{-\alpha}h(\log(t-a)) \tag{5}$$

where $h(\tau)$ is a nonconstant periodic function, is proved. This gives, in particular, a negative answer to the following conjectures by I. T. Kiguradze.

I. Consider the equation

$$x^{(n)}(t)=(-1)^n f(t,x) \quad \text{for } t>0, \tag{6}$$

where f is a continuous nonnegative function of $t\geq 0$ and $x\geq 0$ such that $f(t,0)=0$ and $f(t,x)\geq f(t,y)$ if $x\geq y$. In the case $n=2$ this equation was studied by A. Kneser [Kn], and it was proved that for every positive c there exists exactly one Kneser solution (i.e., a solution whose derivatives change sign) satisfying $x(0)=c$. Kiguradze [K] (see also [KC, Section 13.5]) conjectured that the same is valid for arbitrary n, possibly, if it is additionally assumed that $x(t)\to 0$, as $t\to+\infty$.

It is clear that $f(t,x)=|x|^k$ satisfies the above assumptions, and, by Theorem 1.1(i) below, all solutions defined in a neighbourhood of infinity are Kneser solutions. The functions (5) and (2) with an appropriate choice of a have the same value at a point.

II. Consider the Emden–Fowler equation of order n,

$$x^{(n)}=p(t)|x|^k\operatorname{sgn} x, \tag{7}$$

with $k>1$ and p being a positive continuous function in a neighborhood of b. The conjecture on blow-up solutions is the following: If $|x(t)|\to+\infty$ as $t\to b$, $t<b$, then

$$|x(t)|=c(b-t)^{-n/(k-1)}(1+o(b-t)),$$

where c is a constant (see [KC, Section 16.4]). For $n=2$, this asymptotic representation is justified by Kiguradze and Chanturia [KC, Section 20], and for $n=3$, 4 by Astashova [A].

If one takes $p(t)=1$ in (7) and changes the variable t to $-t$, then x satisfies (1) provided x is positive. Thus the above counterexample shows that this conjecture also fails.

Despite that both conjectures fail, the following assertion (proved in Section 3) on two-side estimates for solutions to (1) is still valid.

Theorem 1.1. (i) *Let x be a nontrivial solution of* (1) *defined in a neighbourhood of $+\infty$. Then the maximal interval of existence of x is a semiaxis (a,∞) with some finite a, where x satisfies*

$$c_1(t-a)^{-\alpha-m}\le(-1)^m x^{(m)}(t)\le c_2(t-a)^{-\alpha-m},\quad m=0,\dots,\ n-1, \tag{8}$$

for $t>a$.

(ii) *Let x be a nontrivial solution of* (1) *defined in a neighbourhood of $-\infty$. Then the maximal interval of existence of x is a semiaxis $(-\infty,b)$ with some finite b and the following estimates hold*

$$c_1(b-t)^{-\alpha-m}\le(-1)^n x^{(m)}(t)\le c_2(b-t)^{-\alpha-m},\quad m=0,\dots,\ n-1, \tag{9}$$

for $t<b$.

(iii) *Let x be a solution of* (1) *with finite maximal interval of existence (a,b). Then x satisfies* (8) *in a right neighbourhood of a, and the estimates* (9) *hold in a left neighbourhood of b.*

In both inequalities (8) *and* (9), *c_1 and c_2 are positive constants depending only on n and k.*

In the case $n\le4$, Theorem 1.1 can be improved. The inequalities in (i) and (ii) can be replaced by the equalities $x=x_a$ and $x=x_b$, and the two-sided estimates in (iii) can be replaced by an asymptotic representation for x near a and b. This is done in the last section. Thus the first conjecture is true for $n\le4$.

2. Two lemmas

We begin with the derivation of an integral equation for solutions to (1) defined on a semiaxis.

Lemma 2.1. *Let $x=x(t)$ satisfy* (1) *on the semiaxis* (a,∞). *Then*

$$x(t)=\int_t^\infty \frac{(\tau-t)^{n-1}}{(n-1)!}|x(\tau)|^k\,d\tau \tag{10}$$

for $t>a$.

Proof. For $n=1$ the assertion is trivial. Therefore we shall suppose that $n\geq 2$.

From (1) it follows that only one of the following alternatives for the solution x hold:

(i) $(-1)^n x^{(k)}(t)\geq 0$, $k=0,1,\dots,n$, on a semiaxis $t\geq t_1$, where $t_1>a$;

(ii) $(-1)^k x^{(k)}(t)\geq 0$, $k=0,1,\dots,n$, on a semiaxis $t\geq t_1$ and $x^{(k)}(t)\to 0$, as $t\to\infty$, $k=0,1,\dots,n$.

Consider first (i). Integrating (1) n times over the interval (t_1,t) and using positiveness of derivatives of the function $(-1)^n x$ we arrive at

$$(-1)^n x(t)\geq c_1+\int_{t_1}^t \frac{(t-\tau)^{n-1}}{(n-1)!}|x(\tau)|^k\,d\tau,$$

with a positive constant c_1. Suppose that we have constructed a positive function y satisfying the opposite inequality, i.e.,

$$y(t)\leq c_1+\int_{t_1}^t \frac{(t-\tau)^{n-1}}{(n-1)!}|y(\tau)|^k\,d\tau \tag{11}$$

for $t\in[t_1,T)$ and $y(t_1)<c_1$. Then

$$(-1)^n x(t)\geq y(t)\quad \text{for } t\in[0,T). \tag{12}$$

If, additionally,

$$y(t)\to\infty,\quad \text{as } t\to T, \tag{13}$$

then $(-1)^n x(t)\to\infty$, as $t\to T$, which contradicts the fact that $x(t)$ is a locally bounded function on $[t_1,\infty)$. Thus, it suffices to construct y subject to (11) and (13). We are looking for y of the form

$$y(t)=q(T-t)^{-\alpha},$$

where α is given by (4), and q and T are constants. Inserting this expression into (11) and changing variables

$$t=t_1+(T-t_1)\eta,\quad \tau=t_1+(T-t_1)\xi$$

we get that (11) is equivalent to

$$q^{1-k}(1-\eta)^{-\alpha}\leq c_1q^{-k}(T-t_1)^{\alpha}+\int_0^{\eta}\frac{(\eta-\xi)^{n-1}}{(n-1)!}(1-\xi)^{-k\alpha}\,d\xi, \tag{14}$$

where $0\leq\eta<1$. Since the last integral is estimated by $\mathrm{Const}(1-\eta)^{-\alpha}$, we can satisfy (14) for all $\eta\in[0,1)$ by choosing q and then T sufficiently large. Thus (12) is proved and hence the alternative (i) is impossible.

Suppose that (ii) is valid. Then

$$(-1)^{n-1}x^{(n-1)}(t_2)+\int_{t_1}^{t_2}|x(\tau)|^k\,d\tau=(-1)^{n-1}x^{(n-1)}(t_1).$$

Since both terms on the left have the same sign, we obtain

$$(-1)^{n-1}x^{(n-1)}(t)=\int_t^{\infty}|x(\tau)|^k\,d\tau\quad\text{for } t>a.$$

Successively integrating this equality $n-1$ times and using the positiveness of $(-1)^kx^{(k)}(t)$ we arrive at (10). □

In the following lemma we give two-sided estimates for solutions with a finite maximal interval of existence.

Lemma 2.2. *Let x be a solution to* (1) *defined in a right neighbourhood of* 0 *and let x be noncontinuable to the left from* 0. *Then $(-1)^kx^{(k)}(t)\to\infty$, as $t\to0$, and*

$$\frac{1}{2}\int_t^{\varepsilon}\frac{(\tau-t)^{n-1-k}}{(n-1-k)!}|x(\tau)|^k\,d\tau\leq(-1)^kx^{(k)}(t)\leq2\int_t^{\varepsilon}\frac{(\tau-t)^{n-1-k}}{(n-1-k)!}|x(\tau)|^k\,d\tau \tag{15}$$

for $t\in(0,\delta)$, where ε and δ are positive numbers, $\delta<\varepsilon$.

Proof. From (1) it follows that

$$x(t)=\int_t^{\varepsilon}\frac{(\tau-t)^{n-1}}{(n-1)!}|x(\tau)|^k\,d\tau+p(t) \tag{16}$$

for $t\in(0,\varepsilon)$, where ε is a sufficiently small positive number and $p(t)$ is a polynomial of degree $\leq n-1$. Let us show first that $x(t)\to\infty$, as $t\to0$. Suppose that x is bounded. By (16), $x^{(k)}$, $k=1,\dots,n$, are bounded also. Hence x can be extended to the left from 0. This contradiction shows that x should be unbounded. By (16), $(-1)^kx^{(k)}(t)\to\infty$, as $t\to0$. Since the integral in (16) diverges, (15) follows from (16). □

3. Proof of Theorem 1.1(i)

The maximal interval of existence of x

Suppose that x can be extended to $(-\infty, \infty)$ as a solution to (1). Then by (10), x is a positive and strictly decreasing function. The function $t \mapsto (-1)^n x(-t)$ also satisfies (1) and, using (10) again, we obtain

$$(-1)^n x(t) = \int_{-\infty}^{t} \frac{(t-\tau)^{n-1}}{(n-1)!} |x(\tau)|^k \, d\tau,$$

which implies that the function x is negative or increasing. This contradiction proves that the maximal interval of existence is (a, ∞) with a finite a. Furthermore, $x(t) \to \infty$, as $t \to a$. In fact, if x is bounded in a neighbourhood of a then, by (10), $x^{(k)}$, $k=1, \dots, n$, is bounded there also. Hence x can be extended to the left from a.

Since equation (1) is invariant with respect to translations we shall suppose, in what follows, that $a=0$, and the semiaxis $(0, \infty)$ is the maximal interval of existence of x, and

$$x(t) \to \infty, \quad \text{as } t \to 0. \tag{17}$$

The upper estimates

Let us first prove the estimate

$$x(t) \le ct^{-\alpha} \quad \text{for } t > 0, \tag{18}$$

where α is given by (4) and c depends only on n and k. Formula (10) implies that the function x is bounded, nonnegative and decreasing. So, if $0<a<b$ then one has

$$x(at) \ge \int_{at}^{bt} \frac{(\tau - at)^{n-1}}{(n-1)!} \, d\tau |x(bt)|^k = \frac{(b-a)^n t^n}{n!} |x(bt)|^k. \tag{19}$$

Now let

$$a_0 = 2, \quad a_{j+1} = a_j - 2^{-j-1}, \quad j = 0, \ 1, \dots .$$

Using (19) with $b=a_j$ and $a=a_{j+1}$ we get

$$x(a_j t) \le (2^{n(j+1)} n! t^{-n})^{1/k} x(a_{j+1} t)^{1/k}.$$

Hence

$$x(2t) \le (2^n n! t^{-n})^S 2^{nQ},$$

where

$$S=\sum_{j=1}^{\infty} k^{-j}=\frac{1}{k-1}, \quad Q=\sum_{j=0}^{\infty} jk^{-j}.$$

This yields (18).

Differentiating (10) one obtains

$$(-1)^m x^{(m)}(t)=\int_t^{\infty} \frac{(\tau-t)^{n-m-1}}{(n-m-1)!}|x(\tau)|^k\, d\tau, \tag{20}$$

for $m=1, \ldots, n-1$. This together with (18) gives

$$0<(-1)^m x^{(m)}(t) \leq ct^{-\alpha-m} \quad \text{for } t>0, \tag{21}$$

where $m=0, 1, \ldots, n$ and c is a positive constant depending only on n and k.

Transformation of equation (1)

We represent x as

$$x(t)=t^{-\alpha} y(t). \tag{22}$$

Inserting this in (1) and using

$$(-1)^n x^{(n)}(t)=t^{-n} L\left(-t \frac{d}{dt}\right) x(t),$$

where

$$L(z)=z(z+1) \ldots (z+n-1), \tag{23}$$

we obtain

$$L\left(\alpha-t \frac{d}{dt}\right) y(t)=|y(t)|^k \quad \text{for } t>0.$$

Changing the variable $t=e^\tau$ and introducing

$$u(\tau)=L(\alpha)^{-1/(k-1)} y(e^\tau) \tag{24}$$

we arrive at

$$L\left(\alpha-\frac{d}{d\tau}\right) u(\tau)=L(\alpha)|u(\tau)|^k \quad \text{for } \tau \in \mathbf{R}. \tag{25}$$

From (21) it follows that

$$|u^{(m)}(\tau)| \leq C \quad \text{for } \tau \in \mathbf{R} \tag{26}$$

for $m=0, 1, \ldots, n$, where C is a constant depending only on k and n.

The lower estimate

Let us show that there exists a positive constant c_0, depending only on n and k, such that

$$u(\tau) \geq c_0 \quad \text{for all } \tau \in \mathbf{R}. \tag{27}$$

Inserting (22) into (10) and changing the variable in the integral we obtain

$$y(t) = \int_1^\infty |y(ts)|^k \frac{s^{-k\alpha}(s-1)^{n-1}}{(n-1)!}\, ds. \tag{28}$$

Therefore

$$\left(t\frac{d}{dt}\right)^m y(t) = (-1)^m \int_1^\infty |y(ts)|^k \left(1+s\frac{d}{ds}\right)^m \frac{s^{-k\alpha}(s-1)^{n-1}}{(n-1)!}\, ds \tag{29}$$

for $m=1,\dots,\ n-1$. For every positive $\delta \in (0,1)$ the estimate

$$\left|\left(1+s\frac{d}{ds}\right)^m s^{-k\alpha}(s-1)^{n-1}\right| \leq C\delta^{-m} s^{-k\alpha}(s-1)^{n-1} \tag{30}$$

holds for $s \geq 1+\delta$, where C depends only on n and k. Using this estimate together with (28), (29) and the boundedness of y one gets

$$\left|\left(t\frac{d}{dt}\right)^m y(t)\right| \leq C(\delta^{-m} y(t) + \delta) \tag{31}$$

for $m=1,\dots,\ n-1$. Let $y(t)<1$. Taking $\delta = y(t)^{1/(m+1)}$ we obtain

$$\left|\left(t\frac{d}{dt}\right)^m y(t)\right| \leq C y(t)^{1/(m+1)} \quad \text{for } m=1,\dots,\ n-1, \tag{32}$$

or, what is the same,

$$|u^{(m)}(\tau)| \leq C u(\tau)^{1/(m+1)} \quad \text{for } m=1,\dots,\ n-1. \tag{33}$$

Thus, if $u(\tau)$ is small at a certain point then all its derivatives up to the order $n-1$ are also small at the same point.

Furthermore, equation (25) can be written as the first order system

$$-\frac{d}{d\tau}U + AU = L(\alpha)\operatorname{col}(0,\dots,0,|U_1|^k), \tag{34}$$

where $U_1=u$, $U_j=(\alpha+j-2-d/d\tau)U_{j-1}$ for $j=2,\dots,n$ and A is the matrix with the elements α, $\alpha+1,\dots,\alpha+n-1$ on the main diagonal, with -1 on the diagonal above the main one and with zero otherwise. Since all eigenvalues of the matrix A are positive, there exists a positive matrix B such that the matrix $BA+A^*B$ is also positive. If u is sufficiently small at t_* then by (33) the norm of $U(t_*)$ is also small and we derive from (34) that

$$\frac{d}{d\tau}(BU(\tau),U(\tau))\geq\varepsilon(BU(\tau),U(\tau))\quad\text{for }\tau<t_*,$$

with a positive ε depending only on n and k. This implies that

$$(BU(\tau),U(\tau))\leq Ce^{\varepsilon\tau}\quad\text{for }\tau<t_*,$$

and hence

$$u(\tau)\leq ce^{\varepsilon\tau/2}$$

for the same τ.

Now we rewrite (25) as

$$L\left(\alpha-\frac{d}{d\tau}\right)u(\tau)=p(\tau)u(\tau),$$

where $p(\tau)=O(e^{\varepsilon(k-1)\tau/2})$ for $\tau<0$. Hence the asymptotics of u at $-\infty$ are described by the zeros of $L(\alpha-d/d\tau)v(\tau)=0$. This implies, in particular, that

$$u(\tau)\leq ce^{\alpha\tau},$$

which gives the boundedness of $x(t)$, as $t\to0$. This contradicts (17).

Proof of (8)

The upper estimate in (8) is a consequence of (21). From (27) one derives the lower estimate in (21) for $m=0$. The lower estimates for $m=1,\dots,n-1$ follows from the one just proved and from (20).

4. Proof of Theorem 1.1(ii) and (iii)

One can verify directly that if $x=x(t)$ is a solution to (1) then the functions $x(t+q)$ and $(-1)^n x(-t)$, where q is real, are also solutions to (1). Therefore, (ii) follows from (i) and in part (iii) it is sufficient to only prove the estimate (8), where $a=0$. This inequality is proved in the same way as the estimate (8) (in Theorem 1.1(i)), but instead of the representation (10) one should use the inequalities (15).

5. On zeros of the polynomial $L(\alpha+z)-L(\alpha+1)$

Here we study roots of the equation

$$L(\alpha+z)-L(\alpha+1)=0, \tag{35}$$

with $\alpha>0$. The information obtained here will be used in the next section, where we shall construct a solution to (1) of the form (2) with nonconstant h.

It is clear that $z=1$ satisfies (35) and there are no other real roots z on the semiaxis $z<-2\alpha-n$. In the next lemma we collect some simple properties of pure imaginary roots of equation (35).

Lemma 5.1. (i) *Equation* (35) *has at most one root on the positive part of the imaginary axis.*

(ii) *If* $z=iq$, $q>0$, *is a root of* (35) *and* $\alpha\geq n$ *then*

$$\alpha+n<q^2<2\alpha+2n-1 \tag{36}$$

(*the right-hand inequality is valid for all* $\alpha>0$).

(iii) *For pure imaginary* z

$$\frac{d}{dz}L(\alpha+z)\neq 0.$$

In particular, all pure imaginary roots of (35) *are simple.*

Proof. (i) Let $z=iq$, $q>0$, be a root of equation (35). Then

$$(\alpha^2+q^2)((\alpha+1)^2+q^2)\ldots((\alpha+n-1)^2+q^2)=(\alpha+1)^2(\alpha+2)^2\ldots(\alpha+n)^2.$$

Since the left-hand side is a monotone function of q this equation has exactly one positive root q.

(ii) If the right-hand estimate in (36) fails then

$$(\alpha+j)^2+q^2\geq(\alpha+j+1)^2$$

for $j=0,1,\ldots,n-1$. Moreover the inequality is strict for $j<n-1$. Hence $|L(\alpha+z)|>L(\alpha+1)$. This contradiction proves the right-hand estimate in (36).

Suppose now that $q^2\leq\alpha+n$. Then

$$(\alpha+x)^2+q^2\leq(\alpha+x)^2+\alpha+n<(\alpha+x+1)^2$$

provided $\alpha\geq n$ and $x\geq 0$. These estimates imply $|L(\alpha+z)|<L(\alpha+1)$. This proves the left-hand inequality in (36).

(iii) Let $\mathcal{L}(z)=L(\alpha+z)-L(\alpha+1)$. Then $\mathcal{L}'(z)=L'(\alpha+z)$. The zeros of the polynomial $L(\alpha+z)$ are $-\alpha, -\alpha-1,\ldots, -\alpha-n+1$. Therefore the zeros of $\mathcal{L}'(z)$ lie in the interval $(-\alpha,-\alpha-n+1)$. This completes the proof. $\square$

The existence of pure imaginary roots to equation (35) will be obtained from the following two lemmas.

Lemma 5.2. *Let n be a given positive integer. For sufficiently large positive α all roots of equation* (35) *except $z=1$ lie in the half-plane* $\operatorname{Re} z<0$.

Proof. We are looking for solutions of (35) in the form

$$z_j(\alpha)=(-1+\varepsilon_j)\alpha+c_{0j}+\sum_{k=1}^{\infty} c_{kj}\alpha^{-k}, \tag{37}$$

where $\varepsilon_j=e^{2j\pi i/n}$ and $j=1, \dots, n-1$. Since $z=1$ is always the root of (35), $z_j(\alpha)$ should give all other roots of (35). We rewrite (35) as

$$(z+\alpha)^n+\frac{(n-1)n}{2}(z+\alpha)^{n-1}+\sum_{k=2}^{n} a_k(z+\alpha)^{n-k}=\alpha^n+\frac{n(n+1)}{2}\alpha^{n-1}+\sum_{k=2}^{n} b_k\alpha^{n-k}, \tag{38}$$

where a_k and b_k are positive constants. Inserting (37) in (38) one can find all terms in the left-hand side of (37). In particular,

$$c_{0j}=\tfrac{1}{2}(n+1)\varepsilon_j-\tfrac{1}{2}(n-1).$$

The proof of the fact that the series in (37) is convergent is standard. Since all roots $z_j(\alpha)$, $j=1, \dots, n-1$, are located in the plane $\operatorname{Re} z<0$, the proof is complete. □

Lemma 5.3. *Let $\alpha=cn$, where c is a fixed positive constant. Then for a sufficiently large n equation* (35) *has roots in the half-plane* $\operatorname{Re} z>0$ *different from* $z=1$.

Proof. We rewrite (35) as

$$F(\alpha, z)=1, \tag{39}$$

where

$$F(\alpha, z)=\frac{L(\alpha+z)}{L(\alpha+1)}=\frac{\Gamma(z+\alpha+n)\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)\Gamma(z+\alpha)}. \tag{40}$$

Using Stirling's formula we obtain

$$F(\alpha, z)=e^{\Phi(\alpha,z)}\left(1+O\left(\frac{1}{z+\alpha+n}\right)+O\left(\frac{1}{\alpha+n+1}\right)+O\left(\frac{1}{\alpha+1}\right)+O\left(\frac{1}{z+\alpha}\right)\right), \tag{41}$$

where

$$\begin{aligned}\Phi(\alpha, z)=&\left(z+\alpha+n-\tfrac{1}{2}\right)\log(z+\alpha+n)-\left(\alpha+n+\tfrac{1}{2}\right)\log(\alpha+n+1)\\&+\left(\alpha+\tfrac{1}{2}\right)\log(\alpha+1)-\left(z+\alpha-\tfrac{1}{2}\right)\log(z+\alpha).\end{aligned} \tag{42}$$

We are looking for solutions of (39) which are bounded by a constant independent of n and α. From (42) we derive

$$\Phi(\alpha,z)=(z-1)\log\left(\frac{c+1}{c}\right)+O\left(\frac{1}{n}\right). \tag{43}$$

We introduce the rectangle

$$\Pi=\left\{z:-(2\varkappa+1)\pi<\operatorname{Im} z\log\left(\frac{c+1}{c}\right)<(2\varkappa+1)\pi,\ |\operatorname{Re} z-1|<\frac{1}{2}\right\},$$

where $\varkappa$ is a positive integer, and denote by $\partial\Pi$ its boundary. Using (41) and (43) one verifies that

$$|F(\alpha,z)-e^{(z-1)\log((c+1)/c)}|\geq\tfrac{1}{2}|e^{(z-1)\log((c+1)/c)}-1|$$

on $\partial\Pi$. Therefore equation (39) has the same number of roots in the rectangle Π as the equation

$$e^{(z-1)\log((c+1)/c)}=1.$$

This number is greater than 1 for large $\varkappa$. □

Theorem 5.4. *For every n_0 and c_0 there exist $n\geq n_0$ and $\alpha\geq c_0 n$ such that equation* (35) *has a pure imaginary root.*

Proof. By Lemma 5.3 equation (35) has a root in the half-plane $\operatorname{Re} z>0$ for a certain $n>n_0$ and $\alpha=c_0 n$. By increasing α and using Lemma 5.2 we obtain the existence $\alpha>c_0 n$ such that there exists a root of (35) on the imaginary axis. □

We conclude this section by the following technical assertion.

Lemma 5.5. *Let n and α/n be sufficiently large. Let also z be a pure imaginary number such that $|z|\leq C\sqrt{\alpha}$, where C is a constant independent of n and α. Then*

$$\partial_z F(\alpha,z)=\left(\log\frac{\alpha+n}{\alpha}-\frac{nz}{\alpha(\alpha+n)}+O\left(\frac{n}{\alpha^2}\right)\right)e^{\Phi(\alpha,z)} \tag{44}$$

and

$$\partial_\alpha F(\alpha,z)=\left(\frac{n}{\alpha(\alpha+n)}\left(1-z+\frac{z^2}{\alpha}\right)+O\left(\frac{n^2+\alpha+n|z|}{\alpha^3}\right)\right)e^{\Phi(\alpha,z)}, \tag{45}$$

where F is introduced by (40).

Proof. By (41) we have

$$\partial_z F(\alpha,z)=\left(\partial_z\Phi(\alpha,z)+O\left(\frac{1}{\alpha^2}\right)\right)e^{\Phi(\alpha,z)} \tag{46}$$

and

$$\partial_\alpha F(\alpha,z)=\left(\partial_\alpha\Phi(\alpha,z)+O\left(\frac{1}{\alpha^2}\right)\right)e^{\Phi(\alpha,z)}. \tag{47}$$

Furthermore, from (42) we derive

$$\partial_z\Phi(\alpha,z)=\log\frac{\alpha+n+z}{\alpha+z}-\frac{1}{2(\alpha+n+z)}+\frac{1}{2(\alpha+z)}=\log\frac{\alpha+n}{\alpha}-\frac{nz}{\alpha(\alpha+n)}+O\left(\frac{n}{\alpha^2}\right)$$

and

$$\begin{aligned}\partial_\alpha\Phi(\alpha,z)&=\log\left(1-\frac{(z-1)n}{(\alpha+z)(\alpha+n+1)}\right)+\frac{n}{2(\alpha+z)(\alpha+n+z)}-\frac{n}{2(\alpha+1)(\alpha+n+1)}\\&=\frac{n}{\alpha(\alpha+n+1)}\left(1-z+\frac{z^2}{\alpha+z}-\frac{2z}{\alpha}\right)+O\left(\frac{n^2}{\alpha^3}\right).\end{aligned}$$

These relations together with (46) and (47) give (44) and (45). □

6. Existence of periodic solutions

Theorem 6.1. *For every number N and K, $K>1$, there exist an integer $n\geq N$ and a real number $k\in(1,K)$ such that equation (1) has a solution*

$$x(t)=t^{-n/(k-1)}h(\log t),\quad t>0, \tag{48}$$

where h is a periodic nonconstant function on $\mathbf{R}$.

Proof. We seek an h on the form

$$h(\tau)=L(\alpha)^{1/(k-1)}(1+v(\tau)).$$

If (48) solves (1) then v should satisfy

$$(L(\alpha-\partial_t)-L(\alpha+1))v(t)=L(\alpha)f(v(t))v^2(t), \tag{49}$$

where

$$f(y)=\frac{(1+y)^k-1-ky}{y^2}. \tag{50}$$

Clearly, f is real analytic for $y>-1$.

According to Theorem 5.4, for every positive integer N and real number $K>1$ there exist $n\geq N$ and $k_0\in(1,K)$ such that equation (35) with

$$\alpha=\alpha_0=\frac{n}{k_0-1}$$

has a root on the imaginary axis. We denote this root by iq, $q>0$. By the same theorem we can suppose that n and α_0/n are sufficiently large. It is clear that $-iq$ is also a root of (35) and by Lemma 5.1(i) there are no other pure imaginary roots.

Let $z(\alpha)$ be the root of (35) which is situated near iq for α close to α_0 and such that $z(\alpha_0)=iq$. Clearly, the function $z(\alpha)$ is analytic in a neighbourhood of α_0. Since z satisfies (39) we get

$$z'(\alpha)=-\frac{\partial_\alpha F(\alpha,z)}{\partial_z F(\alpha,z)}.$$

By Lemma 5.1(ii) all assumptions of Lemma 5.5 are fulfilled (where $z=iq$). Therefore we can use formulae (44) and (45) for calculation of $z'(\alpha_0)$. Hence

$$z'(\alpha_0)=\frac{-\dfrac{1}{\alpha_0+n}\Big(1-z+\dfrac{z^2}{\alpha_0}\Big)+O\Big(\dfrac{n}{\alpha_0^2}+\dfrac{1}{n\alpha_0}+\dfrac{|z|}{\alpha_0^2}\Big)}{\dfrac{\alpha_0}{n}\log\Big(1+\dfrac{n}{\alpha_0}\Big)-\dfrac{z}{\alpha_0+n}+O\Big(\dfrac{1}{\alpha_0}\Big)},$$

which implies

$$z'(\alpha_0)=-\frac{1-z}{\alpha_0+n}+o\Big(\frac{1}{\alpha_0}\Big).$$

This yields

$$\operatorname{Re} z'(\alpha_0)\neq 0.$$

Now the application of the Hopf bifurcation theorem (see, for example, [GH, Section 3.4]) completes the proof. □

7. The case $n=1, 2, 3, 4$

Here we improve Theorem 1.1 for the case $n=1, 2, 3, 4$.

Theorem 7.1. *Let $n\leq 4$. Then the following assertions hold.*

(i) *Let x be a nontrivial solution of* (1) *defined in a neighbourhood of $+\infty$. Then the maximal interval of existence of x is a semiaxis (a,∞) with some finite a and $x=x_a$, where x_a is given by* (2).

(ii) *Let x be a nontrivial solution of* (1) *defined in a neighbourhood of $-\infty$. Then the maximal interval of existence of x is a semiaxis $(-\infty,b)$ with some finite b and $x=x_b$, where x_b is given by* (3).

Theorem 7.2. *Let x be a solution of* (1) *with finite maximal interval of existence* (a,b). *Then*

$$x(t)=x_a(t)(1+O(t^\sigma)) \tag{51}$$

in a right neighbourhood of a and

$$x(t)=x_b(t)(1+O(t^\sigma)) \tag{52}$$

in a left neighbourhood of b. Here σ is a positive constant, which depends only on n and k. The asymptotic formulae for $x^{(k)}$, $k=1,\dots,\ n-1$, are obtained from (51) *and* (52) *by differentiation.*

We start with an auxiliary assertion which is valid for all n.

Lemma 7.3. *Let all the roots of equation* (35), *except $z=1$, lie in the half-plane* $\operatorname{Re} z<0$.

(i) *If v is a solution of* (49) *on* $\mathbf{R}$ *such that*

$$v(t)\to 0, \quad \textit{as } t\to+\infty,$$

then $v=0$.

(ii) *If v is a solution of* (49) *in a neighbourhood of $-\infty$ such that*

$$v(t)\to 0, \quad \textit{as } t\to-\infty,$$

then

$$v(t)=O(e^{\sigma t}), \quad \textit{as } t\to-\infty, \tag{53}$$

where σ is a positive number.

Proof. (i) We suppose that $v\neq 0$. The function v is bounded since

$$x(t)=t^{-\alpha}L(\alpha)^{1/(k-1)}(1+v(\log t)) \tag{54}$$

solves equation (1) on $(0,\infty)$. We rewrite (49) as

$$(L(\alpha-\partial_t)-L(\alpha+1))v(t)=p(t)v(t), \tag{55}$$

where

$$p(t)=L(\alpha)f(v(t))v(t),$$

and, therefore, $p(t)\to 0$, as $t\to+\infty$. Since there are no roots of (35) on the imaginary axis, we obtain $v(t)=O(e^{-\delta t})$, $\delta>0$, for large positive t. Therefore, v satisfies (55) with $p(t)=O(e^{-\delta t})$. Using the assumptions on the roots of (35) we derive from (55) that

$$v(t)=ce^{-t}+O(e^{-2t}), \quad \text{as } t\to+\infty,$$

with a nonzero constant c. This together with (54) implies that

$$x(t)=C_\alpha t^{-\alpha}(1+ct^{-1}+O(t^{-2})), \quad \text{as } t\to+\infty,$$

where $C_\alpha=L(\alpha)^{1/(k-1)}$.

(a) Let $c>0$. Then

$$x(t)>C_\alpha(t-c_1)^{-\alpha} \tag{56}$$

for large t with $0<c_1<c$. From (10) it follows that (56) holds for all $t>t_1$. This contradicts the fact that the maximal interval of existence of x is $(0,\infty)$.

(b) Let $c<0$. Then

$$x(t)<C_\alpha(t-c_1)^{-\alpha} \tag{57}$$

for large t, where $c<c_1<0$. From (10) we obtain that (57) is valid for all positive t. This implies that $u(t)\to 0$, as $t\to 0$, which contradicts (27).

(ii) We rewrite equation (49) as (55). Since $p(t)\to 0$ and $v(t)\to 0$, as $t\to-\infty$, and since there are no pure imaginary roots of $L(\alpha-z)-L(\alpha+1)=0$ we arrive at the estimate (53). $\square$

The main step in the proof of Theorems 7.1 and 7.2 is contained in the following lemma.

Lemma 7.4. *Let $n\le 4$.*

(i) *If v is solution of (49) on $\mathbf{R}$ then $v=0$.*

(ii) *If v solves (49) in a neighbourhood of $-\infty$ then*

$$v(t)=O(e^{\sigma t}) \quad \textit{for large negative } t. \tag{58}$$

Proof. One can verify that in the case $n=1, 2, 3, 4$ all roots of the equation (35) except $z=1$ lie in the half-plane $\operatorname{Re} z<0$.

We rewrite equation (49) as

$$(L(\alpha-\partial_t)-L(\alpha))v(t)=L(\alpha)g(v)v, \tag{59}$$

where

$$g(\tau)=\frac{(1+\tau)^k-1-\tau}{\tau}.$$

It is easy to check that $g(\tau)\geq c_\varepsilon>0$ for $\tau\geq-1+\varepsilon$, $\varepsilon>0$. By (54) and (8) $-1+\delta< v(t)<C$ with some positive δ, therefore

$$c_0\leq g(v(t))\leq c_1 \quad \text{for all } t,$$

where c_0 and c_1 are positive constants.

For $n=1$, 2 the assertion is trivial. Let $n=3$. Multiplying (59) by v, integrating over an interval (a,b) we obtain, after partial integration, that

$$-3(\alpha+1)\int_a^b \dot{v}^2(t)\,dt=L(\alpha)\int_a^b g(v(t))v^2(t)\,dt+C(a,b),$$

where the constant $C(a,b)$ is bounded uniformly with respect to a and b because of the boundedness of $v^{(k)}$, $k=0, 1,\dots, n-1$. This implies that

$$\int_{\mathbf{R}}(v^2(t)+\dot{v}^2(t))\,dt<\infty.$$

Hence $v(t)\to 0$, as $t\to+\infty$, in the case (i) and $v(t)\to 0$, as $t\to-\infty$, in the case (ii). Referring to Lemma 7.3 completes the proof for $n=3$.

Let $n=4$. Multiplying again (59) by v, integrating from a to b and making partial integration we get

$$\int_a^b(\ddot{v}^2(t)-c_2\dot{v}^2(t))\,dt=L(\alpha)\int_a^b g(v(t))v^2(t)\,dt+C_1(a,b), \tag{60}$$

where

$$c_2=\alpha(3\alpha+6)+(\alpha+1)(2\alpha+5)+(\alpha+2)(\alpha+3)$$

and $C_1(a,b)$ is a bounded function of a and b. Multiplying (59) by $\dot{v}$, integrating from a to b and making partial integration we get

$$\int_a^b(c_1\ddot{v}^2(t)-c_3\dot{v}^2(t))\,dt=C_2(a,b), \tag{61}$$

where

$$c_1=4\alpha+6 \quad \text{and} \quad c_3=\alpha(\alpha+1)(2\alpha+5)+(\alpha+1)(\alpha+2)(\alpha+3)$$

and $C_2(a,b)$ is a bounded function. From (60) and (61) we derive that

$$-(c_1c_2-c_3)\int_a^b \dot{v}(t)\,dt=c_1L(\alpha)\int_a^b g(v(t))v^2(t)\,dt+C(a,b).$$

Since $c_1c_2>c_3$ we obtain

$$\int_{\mathbf{R}}(v^2(t)+\dot v^2(t))\,dt<\infty.$$

This implies $v(t)\to 0$, as $t\to+\infty$, in the case (i) and $v(t)\to 0$, as $t\to-\infty$, in the case (ii). Referring to Lemma 7.3 completes the proof. $\square$

Proof of Theorems 7.1 *and* 7.2. We can suppose that $a=0$ and represent x as (54). Then the function v satisfies (49) and by Lemma 7.4(i) $v=0$ provided x is a solution to (1) for $t>0$ and v is subject to (58) if x solves (1) in a right neighbourhood of 0. This proves Theorem 7.1(i) and (51). The assertion (ii) in Theorem 7.1 and (52) follow from the just proved lemma due to the invariance of equation (1) with respect to the transformations

$$x(t)\longmapsto x(t+q) \quad\text{and}\quad x(t)\longmapsto(-1)^n x(-t). \quad \square$$

References

[A] ASTASHOVA, I. V., Asymptotic behavior of solutions of certain nonlinear differential equations, in *Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics* **1**:3 (Kiguradze, I. T., ed.), pp. 9–11, Tbilis. Gos. Univ., Tbilisi, 1985 (Russian).

[GH] GUCKENHEIMER, J. and HOLMES, P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Appl. Math. Sci. **42**, Springer-Verlag, New York, 1983.

[K] KIGURADZE, I. T., On Kneser solutions of ordinary differential equations, *Uspekhi Mat. Nauk* **41**:4 (1986), 211 (Russian).

[KC] KIGURADZE, I. T. and CHANTURIA, T. A., *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer, Dordrecht, 1993.

[Kn] KNESER, A., Untersuchung und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei grossen reelen Werten des Arguments, *J. Reine Angew. Math.* **116** (1896), 178–212.

[KM1] KOZLOV, V. and MAZ'YA, V., *Theory of a Higher-Order Sturm–Liouville Equation,* Lecture Notes in Math. **1659**, Springer-Verlag, Berlin–Heidelberg, 1997.

[KM2] KOZLOV, V. and MAZ'YA, V., *Differential Equations with Operator Coefficients* (*with Applications to Boundary Value Problems for Partial Differential Equations*), Monogr. Math., Springer-Verlag, Berlin–Heidelberg, 1999.

Received October 9, 1997
in revised form October 20, 1998

Vladimir A. Kozlov
Department of Mathematics
Linköping University
SE-581 83 Linköping
Sweden
email: vlkoz@mai.liu.se