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The failure of the Hardy inequality and interpolation of intersections

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Abstract. The main idea of this paper is to clarify why it is sometimes incorrect to interpolate inequalities in a "formal" way. For this we consider two Hardy type inequalities, which are true for each parameter $\alpha\neq 0$ but which fail for the "critical" point $\alpha=0$. This means that we cannot interpolate these inequalities between the noncritical points $\alpha=1$ and $\alpha=-1$ and conclude that it is also true at the critical point $\alpha=0$. Why? An accurate analysis shows that this problem is connected with the investigation of the interpolation of intersections $(N\cap L_p(w_0), N\cap L_p(w_1))$, where N is the linear space which consists of all functions with the integral equal to 0. We calculate the K-functional for the couple $(N\cap L_p(w_0), N\cap L_p(w_1))$, which turns out to be essentially different from the K-functional for $(L_p(w_0), L_p(w_1))$, even for the case when $N\cap L_p(w_i)$ is dense in $L_p(w_i)$ $(i=0,1)$. This essential difference is the reason why the "naive" interpolation above gives an incorrect result.

0. Introduction

It is well known (cf. [12]) that if $\alpha\in R$, $\alpha\neq 0$, then, the Hardy inequality implies the following estimate

$$\int_0^\infty |u(s)|s^{\alpha-1}\,ds \le C(\alpha)\int_0^\infty |u'(s)|s^{\alpha}\,ds \tag{0.1}$$

for all $u\in C_0^\infty(0,\infty)$, i.e., all infinitely differentiable functions u on $(0,\infty)$ with a compact support. Moreover, the inequality (0.1) is *not* true for $\alpha=0$ and the constant $C(\alpha)$ goes to $+\infty$ as $\alpha\to 0$.

It seems to be natural to ask why we cannot "interpolate" between $\alpha=1$ and $\alpha=-1$ in the inequality (0.1) and obtain it for $\alpha=0$.

There are many other inequalities for which such a phenomenon occurs. For example, in [7], in connection with the fractional Hardy inequality, it was proved

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that for all $\alpha \in R\setminus\{0\}$ and any locally integrable function u with compact support in $(0,\infty)$ the inequality

$$\int_0^\infty \left|\frac{u(t)}{t^\alpha}\right|^p \frac{dt}{t} \le B(\alpha) \int_0^\infty \left|\frac{u(t)-t^{-1}\int_0^t u(s)\,ds}{t^\alpha}\right|^p \frac{dt}{t} \tag{0.2}$$

is valid and, again, is not true for $\alpha=0$. Therefore, we also cannot "interpolate" (0.2) between $\alpha=1$ and $\alpha=-1$ to obtain it for $\alpha=0$.

One of the main purposes of this paper is to show that this phenomenon is deeply connected with the fact that the problem of *interpolation of intersections* can have a *negative answer* in some concrete situations.

We will formulate this problem in a more general setting. Let (X_0, X_1) be a Banach couple, i.e., X_0 and X_1 are two Banach spaces linearly and continuously imbedded in some Hausdorff topological vector space X and let $N\subset X$ be a linear space. We can then consider a *normed* couple $(N\cap X_0, N\cap X_1)$, where the norm in $N\cap X_i$ is just the restriction of the norm from X_i, $i=0,1$.

We say that the problem of *interpolation of intersections* has a positive solution (or answer) for the triple (X_0, X_1, N) and parameters $\theta\in(0,1)$, $p\in[1,\infty]$, if the formula

$$(N\cap X_0, N\cap X_1)_{\theta,p} = N\cap(X_0,X_1)_{\theta,p} \tag{0.3}$$

is true. In the opposite case we will say that the problem has a negative solution (or answer).

As we shall see, the examples for which the above problem has negative solution follows from the failure of the inequalities (0.1) and (0.2).

On the other hand, if X_0, X_1 are Banach function lattices and N has also the "lattice" structure, then the interpolation of intersections has a positive solution (see Remark 2).

The paper is organized in the following way: In Section 1 we show how the failure of the Hardy inequality leads to an example for which the problem of interpolation of intersections has a negative answer.

In Sections 2 and 3 we analyze this example from the interpolation point of view. For this purpose we calculate the K-functional for the couple $(N\cap L_p(w_0), N\cap L_p(w_1))$, where N is the linear space which consists of all functions with the integral equal to 0.

In order to avoid technical details and clarify the ideas, we begin our investigations in Section 2 by calculating the K-functional for the simple couple $(N\cap L_1(x), N\cap L_1(x^{-1}))$. It turns out that this K-functional contains two terms. The first term is just the K-functional for the couple $(L_1(x), L_1(x^{-1}))$ and gives no trouble. The second term contains the Hardy operator. Thus the K-functional is *essentially* different when we go from the couple $(L_1(x), L_1(x^{-1}))$ to the couple $(N\cap L_1(x), N\cap L_1(x^{-1}))$. This difference of the K-functionals is the reason for the negative answer to the problem of interpolation of intersections. Moreover, this second term, with the Hardy operator, leads naturally to the appearance of Cesàro function spaces of non-absolute type as interpolation spaces.

In Section 3, we generalize these interpolation results to the more general weighted L_p-couple $(N\cap L_p(w_0), N\cap L_p(w_1))_{\theta,p}$, $1\le p<\infty$. Moreover, we point out an example showing that when the weights are not power functions, then it is possible that the problem of intersections fails on the whole interval $[a,b]$, not only at one particular point.

Finally, in Section 4, we reformulate the inequalities (0.1) and (0.2) as the boundedness of some "inverse" operators and explain the reason why we could not interpolate them from $\alpha=1$ and $\alpha=-1$ to conclude that they also hold at the critical point $\alpha=0$.

The problem of interpolation of intersections is a particular case of the (important and rather difficult) problem of *interpolation of subspaces* (see [8], [14], [15], [10], [11], [16], [13], [6], [1] and [9]). Lions and Magenes wrote that the "main difficulties of the use of interpolation is that *the interpolated space between closed subspaces is not necessarily a closed subspace in the interpolated space*" (see [8, p. 107]).

It is still not completely clear under which conditions this problem has a positive or negative solution. In connection with this, it seems important to investigate concrete nontrivial cases. For example, it will be interesting to solve the following problem: *under which conditions does the problem of intersections*

$$(N\cap L_{p_0}(w_0), N\cap L_{p_1}(w_1))_{\theta,p} = N\cap (L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p'}$$

have a positive answer, where N is the linear space which consists of all functions with the integral equal to 0?

Let us note that the intersection with N sometimes appears in an interesting way. For example, the Hardy operator $Hf(x)=(1/x)\int_0^x f(s)\,ds$ is not bounded from $L_1(|\log x|)$ into L_1 but it is bounded from the intersection $N\cap L_1(|\log x|)$ into L_1 (see [12, pp. 70–72, 106–107] and our Proposition 3, and Remark 7).

Definitions and notation. For a normed couple (X_0, X_1), $f\in X_0+X_1$ and $t>0$, we define the K-functional by

$$K(t,f;X_0,X_1) = \inf\{\|f_0\|_{X_0}+t\|f_1\|_{X_1} : f=f_0+f_1,\ f_0\in X_0,\ f_1\in X_1\}.$$

For $0<\theta<1$ and $1\le p\le\infty$, the *real interpolation spaces* $(X_0, X_1)_{\theta,p}$ are then defined as the spaces of all $f\in X_0+X_1$ such that

$$\|f\|_{\theta,p}=\left(\int_0^\infty (t^{-\theta}K(t,f;X_0,X_1))^p\,\frac{dt}{t}\right)^{1/p}<\infty.$$

For $1\le p<\infty$ and a *weight* function w on $(0,\infty)$, i.e., w is a non-negative and locally integrable function on $(0,\infty)$, we shall denote by $L_p(w)$ the weighted L_p-spaces and by $C_p(w)$ the *weighted Cesàro function spaces of non-absolute type* given by the norms

$$L_p(w)=\left\{f \text{ on } (0,\infty):\|f\|_{L_p(w)}=\left(\int_0^\infty |f(x)|^p w(x)\,dx\right)^{1/p}<\infty\right\},$$
$$C_p(w)=\left\{f \text{ on } (0,\infty):\|f\|_{C_p(w)}=\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right|^p w(x)\,dx\right)^{1/p}<\infty\right\}.$$

For $w(x)=x^\alpha$ with $\alpha\in R$ we denote these spaces by $L_p(x^\alpha)$ and $C_p(x^\alpha)$, respectively; we also write, for simplicity, $L_p(x^0)=L_p$ and $C_p(x^0)=C_p$. The last spaces C_p we also call the *p-Cesàro function spaces of non-absolute type.*

Moreover, by N we denote the space of locally integrable functions on $(0,\infty)$ such that $\int_0^\infty f(s)\,ds=0$, or more precisely,

$$N=\left\{f \text{ on } (0,\infty):\int_a^b |f(s)|\,ds<\infty \text{ for all } 0<a<b<\infty \text{ and } \lim_{\substack{a\to 0\\ b\to\infty}}\int_a^b f(s)\,ds=0\right\}.$$

By $C_0^\infty=C_0^\infty(0,\infty)$ we denote all infinitely differentiable functions u on $(0,\infty)$ with compact support.

Remark 1. If $p<\infty$, then $N\cap C_0^\infty$ is dense in $N\cap L_p(x^\alpha)$ for all α and it is dense in $L_p(x^\alpha)$ for $\alpha\neq 0$.

Remark 2. Assume that X_0, X_1 are Banach function lattices and let N be a linear space of functions (on the same measure space) possessing the "lattice" property: if $g\in N$ and f is such that $|f|\le|g|$, then $f\in N$. Then the problem of intersections has a positive solution.

In fact, if we show that $K(t,f;N\cap X_0,N\cap X_1)\le K(t,f;X_0,X_1)$ for all functions $f\in N\cap(X_0+X_1)$, then we have a non-trivial imbedding $N\cap(X_0,X_1)_{\theta,p}\subset(N\cap X_0,N\cap X_1)_{\theta,p}$, which gives a positive solution to the problem of intersections.

The above estimate for the K-functional follows easily because in the computation of the K-functional for the couple of normed lattices (X_0,X_1) it is enough to take decompositions $f=f_0+f_1$ with the properties $|f_0|\le|f|$ and $|f_1|\le|f|$.

1. The failure of the Hardy inequality and interpolation of subspaces

Let us consider the Hardy inequality and its dual in the simplest case $p=1$. This means

$$\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right| x^\alpha\,dx \le \frac{1}{|\alpha|}\int_0^\infty |f(x)|x^\alpha\,dx, \quad \alpha<0, \tag{1.1}$$

and

$$\int_0^\infty \left|\frac{1}{x}\int_x^\infty f(s)\,ds\right| x^\alpha\,dx \le \frac{1}{|\alpha|}\int_0^\infty |f(x)|x^\alpha\,dx, \quad \alpha>0. \tag{1.2}$$

It is impossible to interpolate (1.1) and (1.2) directly because on the left-hand side we have two *different* operators

$$H_+f(x)=\frac{1}{x}\int_0^x f(s)\,ds, \quad \alpha<0, \tag{1.3}$$

and

$$H_-f(x)=-\frac{1}{x}\int_x^\infty f(s)\,ds, \quad \alpha>0. \tag{1.4}$$

Nevertheless, if we restrict the operators H_+ and H_- to the space N, where they coincide, then we will have the same operator and we can interpolate it. It is clear that H_+ and H_- coincide at

$$N=\left\{f\in L_1^{\mathrm{loc}}(0,\infty):\int_0^\infty f(s)\,ds=0\right\}. \tag{1.5}$$

Note that $N\cap L_1(x^\alpha)$ is dense in $L_1(x^\alpha)$ for $\alpha\neq 0$ and N is a subspace of codimension 1 in $L_1(x^0)$.

Denote by H the restriction of H_+ (or H_-) to the space N:

$$Hf(x)=\frac{1}{x}\int_0^x f(s)\,ds=-\frac{1}{x}\int_x^\infty f(s)\,ds, \quad f\in N. \tag{1.6}$$

Proposition 1. *The operator H is bounded from $N\cap L_1(x^\alpha)$ to $L_1(x^\alpha)$ if and only if $\alpha\in R\setminus\{0\}$.*

Proof. From (1.1) and (1.2) follows that the operator H is bounded from $N\cap L_1(x^\alpha)$ to $L_1(x^\alpha)$ for $\alpha\neq 0$. Moreover, direct calculations for the functions

$$f_n=\chi_{[1,2]}-\chi_{[n,n+1]}\in N, \quad n\ge 2,$$

show that

$$\frac{\|Hf_n\|_{L_1}}{\|f_n\|_{L_1}}=\frac{1}{2}\left[\log n-2\log 2+(n+1)\log\left(1+\frac{1}{n}\right)\right]\to\infty, \quad \text{as } n\to\infty,$$

i.e., H is not bounded from $N\cap L_1(x^\alpha)$ to $L_1(x^\alpha)$ for $\alpha=0$.

In particular, Proposition 1 implies the boundedness of the Hardy operator H from $N\cap L_1(x^\alpha)$ with the $\|\cdot\|_{L_1(x^\alpha)}$-norm into $L_1(x^\alpha)$, for example, for $\alpha=1$ and $\alpha=-1$. Interpolation of these two estimates shows only that the Hardy operator H is bounded from $(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}$ into $(L_1(x), L_1(x^{-1}))_{1/2,1}$, which leads to a problem in describing the space

$$(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}.$$

In view of Proposition 1 it is tempting to think that we have the equality

$$(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}=N\cap(L_1(x), L_1(x^{-1}))_{1/2,1},$$

but as we will see below this is not true.

Proposition 2. *The formula*

$$(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}=N\cap(L_1(x), L_1(x^{-1}))_{1/2,1} \tag{1.7}$$

is not valid.

Proof. Suppose that (1.7) is true. Then, by interpolation, H is bounded from $(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}$ into $(L_1(x), L_1(x^{-1}))_{1/2,1}$. Since, by the Stein–Weiss theorem (cf. [3, Theorem 5.4.1]), $(L_1(x), L_1(x^{-1}))_{1/2,1}=L_1$, it follows that H is bounded from $N\cap L_1$ into L_1, which contradicts the result in Proposition 1.

Let us observe that for the case $\theta\neq\frac{1}{2}$ the expected formula of type (1.7) is true

$$(N\cap L_1(x), N\cap L_1(x^{-1}))_{\theta,1}=N\cap(L_1(x), L_1(x^{-1}))_{\theta,1}=N\cap L_1(x^{1-2\theta}),$$

see our Theorem 2(b).

Remark 3. In the above discussion we notice an interesting phenomenon, namely that the operator H can be extended to a bounded operator H_+ in $L_1(x^{-1})$ and also to a bounded operator H_- in $L_1(x)$ but it cannot be extended to a bounded operator in L_1. This type of phenomenon was first discovered in [5].

Remark 4. All the above considerations can easily be extended to the case $p \geq 1$ and the Hardy inequalities corresponding to (1.1) and (1.2), are the following:

$$\left(\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right|^p x^\alpha\,dx\right)^{1/p} \leq \frac{p}{|p-\alpha-1|}\left(\int_0^\infty |f(x)|^p x^\alpha\,dx\right)^{1/p}, \quad \alpha < p-1,$$

and

$$\left(\int_0^\infty \left|\frac{1}{x}\int_x^\infty f(s)\,ds\right|^p x^\alpha\,dx\right)^{1/p} \leq \frac{p}{|p-\alpha-1|}\left(\int_0^\infty |f(x)|^p x^\alpha\,dx\right)^{1/p}, \quad \alpha > p-1.$$

If we denote, as before, by H the restriction of H_+ (or H_-) to the space N, then the operator H is bounded from $N \cap L_p(x^\alpha)$ to $L_p(x^\alpha)$ if and only if $\alpha \neq p-1$. In the sequence of functions $f_n \in N$, $n \geq 1$, given by

$$f_n(x) = \frac{1}{x}\chi_{[1,2]}(x) - \frac{1}{x}\chi_{[2^n,2^{n+1}]}(x)$$

we see that

$$\frac{\|Hf_n\|_{L_p(x^{p-1})}}{\|f_n\|_{L_p(x^{p-1})}} \geq \log 2\left(\frac{\log(\frac{1}{2}n)}{\log 4}\right)^{1/p} \to \infty, \quad \text{as } n \to \infty,$$

i.e., H is not bounded from $N \cap L_p(x^\alpha)$ to $L_p(x^\alpha)$ for $\alpha = p-1$.

In particular, the operator H is bounded from the space $N \cap L_p(x^\alpha)$ into $L_p(x^\alpha)$ for $\alpha = p$ and $\alpha = p-2$. Moreover, by interpolating we only find that H is bounded from $(N \cap L_p(x^p), N \cap L_p(x^{p-2}))_{1/2,p}$ into $(L_p(x^p), L_p(x^{p-2}))_{1/2,p} = L_p(x^{p-1})$, but the formula

$$(N \cap L_p(x^p), N \cap L_p(x^{p-2}))_{1/2,p} = N \cap (L_p(x^p), L_p(x^{p-2}))_{1/2,p}$$

is not valid.

2. Real interpolation of the couple $(N \cap L_1(x), N \cap L_1(x^{-1}))$

Technical difficulties can obscure the main idea and therefore we start by considering the couple $(N \cap L_1(x), N \cap L_1(x^{-1}))$. In the sequel we use the notation N_α for the space $N \cap L_1(x^\alpha)$ with the $\|\cdot\|_{L_1(x^\alpha)}$-norm.

The first main theorem is the following theorem.

Theorem 1. *For all* $f\in N_1+N_{-1}$ *and all* $t>0$ *we have*

$$K(t,f;N_1,N_{-1})\approx K(t,f;L_1(x),L_1(x^{-1}))+\sqrt{t}\left|\int_0^{\sqrt{t}} f(s)\,ds\right|. \tag{2.1}$$

Proof. We begin by establishing the estimate of $K(t,f;N_1,N_{-1})$ from below. Since $N_\alpha\subset L_1(x^\alpha)$ it follows that $K(t,f;N_1,N_{-1})\geq K(t,f;L_1(x),L_1(x^{-1}))$. Therefore it is enough to show that

$$\left|\int_0^{\sqrt{t}} f(s)\,ds\right|\leq \frac{2K(t,f;N_1,N_{-1})}{\sqrt{t}} \tag{2.2}$$

for all $f\in N_1+N_{-1}$ and $t>0$.

For a fixed $t>0$ and any $\varepsilon>0$, let $f=f_0+f_1$ be an almost optimal decomposition of $f\in N_1+N_{-1}$, i.e.,

$$\|f_0\|_{N_1}+t\|f_1\|_{N_{-1}}\leq(1+\varepsilon)K(t,f;N_1,N_{-1}).$$

Since $f_0\in N_1\subset N$ it follows that $\int_0^\infty f_0(s)\,ds=0$ and

$$\begin{aligned}\left|\int_0^{\sqrt{t}} f_0(s)\,ds\right|&=\left|\int_{\sqrt{t}}^\infty f_0(s)\,ds\right|\leq\int_{\sqrt{t}}^\infty |f_0(s)|\frac{s}{\sqrt{t}}\,ds\\&\leq\frac{\|f_0\|_{N_1}}{\sqrt{t}}\leq\frac{(1+\varepsilon)K(t,f;N_1,N_{-1})}{\sqrt{t}},\end{aligned}$$

and also

$$\left|\int_0^{\sqrt{t}} f_1(s)\,ds\right|\leq\int_0^{\sqrt{t}}|f_1(s)|\frac{\sqrt{t}}{s}\,ds\leq\sqrt{t}\,\|f_1\|_{N_{-1}}\leq\frac{(1+\varepsilon)K(t,f;N_1,N_{-1})}{\sqrt{t}}.$$

Thus

$$\left|\int_0^{\sqrt{t}} f(s)\,ds\right|\leq\left|\int_0^{\sqrt{t}} f_0(s)\,ds\right|+\left|\int_0^{\sqrt{t}} f_1(s)\,ds\right|\leq\frac{2(1+\varepsilon)K(t,f;N_1,N_{-1})}{\sqrt{t}}$$

and the inequality (2.2) holds.

To establish the estimate of $K(t,f;N_1,N_{-1})$ from above we need to construct a decomposition of $f\in N_1+N_{-1}$. For fixed $t>0$ we consider the decomposition $f=f_0+f_1$, where

$$f_0(s)=f(s)\chi_{(0,\sqrt{t})}(s)-c\chi_{[\sqrt{t}-\varepsilon,\sqrt{t}]}(s),$$

and

$$f_1(s)=f(s)-f_0(s)=f(s)\chi_{[\sqrt{t},\infty)}(s)+c\chi_{[\sqrt{t}-\varepsilon,\sqrt{t}]}(s),$$

with $c=\varepsilon^{-1}\int_0^{\sqrt{t}} f(u)\,du$ and $0<\varepsilon\le\frac{1}{2}\sqrt{t}$.

Since $f\in N_1+N_{-1}\subset N$ it follows that $\int_0^\infty f(s)\,ds=0$. The above definitions of f_0 and f_1 show that $\int_0^\infty f_0(s)\,ds=0$ and $\int_0^\infty f_1(s)\,ds=0$, i.e., $f_0, f_1\in N$. Moreover,

$$\begin{aligned}\|f_0\|_{N_1}&=\int_0^\infty |f_0(s)|s\,ds=\int_0^{\sqrt{t}-\varepsilon}|f(s)|s\,ds+\int_{\sqrt{t}-\varepsilon}^{\sqrt{t}}\left|f(s)-\frac{1}{\varepsilon}\int_0^{\sqrt{t}} f(u)\,du\right|s\,ds\\ &\le\int_0^{\sqrt{t}}|f(s)|s\,ds+\int_{\sqrt{t}-\varepsilon}^{\sqrt{t}}\left|\frac{1}{\varepsilon}\int_0^{\sqrt{t}} f(u)\,du\right|s\,ds\\ &\le\int_0^{\sqrt{t}}|f(s)|s\,ds+\frac{1}{\varepsilon}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\sqrt{t}\,\varepsilon\\ &=\int_0^{\sqrt{t}}|f(s)|s\,ds+\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\end{aligned}$$

and

$$\begin{aligned}t\|f_1\|_{N_{-1}}&=t\int_0^\infty |f_1(s)|s^{-1}\,ds\\ &=t\left|\frac{1}{\varepsilon}\int_0^{\sqrt{t}} f(u)\,du\right|\left|\int_{\sqrt{t}-\varepsilon}^{\sqrt{t}} s^{-1}\,ds\right|+t\int_{\sqrt{t}}^\infty |f(s)|s^{-1}\,ds\\ &\le t\frac{1}{\varepsilon}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\frac{1}{\sqrt{t}-\varepsilon}\varepsilon+t\int_{\sqrt{t}}^\infty |f(s)|s^{-1}\,ds\\ &\le 2\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|+t\int_{\sqrt{t}}^\infty |f(s)|s^{-1}\,ds,\end{aligned}$$

where in the last inequality we used the assumption $0<\varepsilon\le\frac{1}{2}\sqrt{t}$. Thus

$$\begin{aligned}K(t,f;N_1,N_{-1})&\le\|f_0\|_{N_1}+t\|f_1\|_{N_{-1}}\\ &\le\int_0^{\sqrt{t}}|f(s)|s\,ds+\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\\ &\quad+2\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|+t\int_{\sqrt{t}}^\infty |f(s)|s^{-1}\,ds\\ &=\int_0^\infty |f(s)|\min(s,t/s)\,ds+3\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\\ &=K(t,f;L_1(x),L_1(x^{-1}))+3\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|\end{aligned}$$

and this also means that the estimate from above in the equivalence (2.1) is proved.

Remark 5. Theorem 1 implies that

$$(N\cap L_1(x))+(N\cap L_1(x^{-1}))=N\cap(L_1(x)+L_1(x^{-1})). \tag{2.3}$$

In fact, the imbedding $(N\cap L_1(x))+(N\cap L_1(x^{-1}))\subset N\cap(L_1(x)+L_1(x^{-1}))$ is trivial. Moreover, if $f\in N\cap(L_1(x)+L_1(x^{-1}))$, then the functions f_0 and f_1, from the proof of Theorem 1, satisfy $f_0\in N\cap L_1(x)$, $f_1\in N\cap L_1(x^{-1})$ and $f_0+f_1=f$. This shows that $f\in(N\cap L_1(x))+(N\cap L_1(x^{-1}))$.

Remark 6. Our proof of Theorem 1 gives the estimates

$$\begin{aligned}\tfrac{1}{3}K(t,f;N_1,N_{-1})&\le K(t,f;L_1(x),L_1(x^{-1}))+\sqrt{t}\left|\int_0^{\sqrt{t}} f(s)\,ds\right| \\ &\le 3K(t,f;N_1,N_{-1})\end{aligned} \tag{2.4}$$

for all $f\in N_1+N_{-1}$ and $t>0$. Observe that we can prove the first inequality in (2.4) with constant $\frac{1}{2}$ instead of $\frac{1}{3}$. In fact, for $\eta>0$ we can take $0<\varepsilon\le\sqrt{t}\,\eta/(1+\eta)$, repeat our calculations and get

$$K(t,f;N_1,N_{-1})\le K(t,f;L_1(x),L_1(x^{-1}))+(2+\eta)\sqrt{t}\left|\int_0^{\sqrt{t}} f(u)\,du\right|.$$

We are now ready to present our announced interpolation result.

Theorem 2. (a) *If* $0\le\theta\le1$ *and* $\theta\ne\frac{1}{2}$, *then*

$$(N\cap L_1(x),N\cap L_1(x^{-1}))_{\theta,1}=N\cap L_1(x^{1-2\theta});$$

(b) $(N\cap L_1(x),N\cap L_1(x^{-1}))_{1/2,1}=C_1\cap L_1$.

Proof. (a) We have

$$(L_1(x),L_1(x^{-1}))_{\theta,1}=L_1(x^{1-\theta-\theta})=L_1(x^{1-2\theta})$$

and, according to Theorem 1, the norm of $f\in(N\cap L_1(x),N\cap L_1(x^{-1}))_{\theta,1}$ is equivalent to

$$\|f\|_{L_1(x^{1-2\theta})}+\int_0^\infty \frac{\sqrt{t}\,\left|\int_0^{\sqrt{t}} f(s)\,ds\right|}{t^\theta}\,\frac{dt}{t}.$$

Therefore, for $\theta\neq\frac{1}{2}$,

$$\|f\|_{(N_1,N_{-1})_{\theta,1}}\approx\|f\|_{L_1(x^{1-2\theta})}+\int_0^\infty x^{-2\theta}\left|\int_0^x f(s)\,ds\right|dx.$$

By using the Hardy inequality we can estimate the second term by the first one. In fact, for $\theta>\frac{1}{2}$ we have

$$\int_0^\infty x^{-2\theta}\left|\int_0^x f(s)\,ds\right|dx\leq\frac{1}{2\theta-1}\int_0^\infty x^{1-2\theta}|f(x)|\,dx,$$

and, for $\theta<\frac{1}{2}$ and $f\in N$ we obtain

$$\int_0^\infty x^{-2\theta}\left|\int_0^x f(s)\,ds\right|dx=\int_0^\infty x^{-2\theta}\left|\int_x^\infty f(s)\,ds\right|dx\leq\frac{1}{1-2\theta}\int_0^\infty x^{1-2\theta}|f(x)|\,dx.$$

Therefore,

$$\|f\|_{(N_1,N_{-1})_{\theta,1}}\approx\|f\|_{L_1(x^{1-2\theta})}.$$

(b) Now, if $\theta=\frac{1}{2}$, then

$$\|f\|_{(N_1,N_{-1})_{1/2,1}}\approx\|f\|_{L_1}+\int_0^\infty\left|\int_0^{\sqrt{t}} f(s)\,ds\right|\frac{dt}{t}$$

or, by changing variables,

$$\|f\|_{(N_1,N_{-1})_{1/2,1}}\approx\|f\|_{L_1}+\int_0^\infty\left|\frac{1}{x}\int_0^x f(s)\,ds\right|dx=\|f\|_{L_1}+\|f\|_{C_1}.$$

Observe that $C_1\cap L_1\subset N$. In fact, if $f\in C_1\cap L_1$, then $\int_0^\infty|f(x)|\,dx<\infty$. Moreover, for every $\varepsilon>0$ there exists $t_1>1$ such that $\int_{t_1}^\infty|f(x)|\,dx<\varepsilon$. Then, for $t_3>t_2>t_1$,

$$\left|\int_1^{t_3} f(x)\,dx-\int_1^{t_2} f(x)\,dx\right|=\left|\int_{t_2}^{t_3} f(x)\,dx\right|\leq\int_{t_2}^{t_3}|f(x)|\,dx\leq\int_{t_1}^\infty|f(x)|\,dx<\varepsilon,$$

i.e., $g(t)=\int_1^t f(x)\,dx$ satisfies the Cauchy condition and so $\lim_{t\to\infty}g(t)$ exists. Since $f\in C_1$ it follows that

$$\frac{1}{t}\left(\int_0^1 f(x)\,dx+\int_1^t f(x)\,dx\right)\in L_1$$

and this means that

$$\lim_{t\to\infty}\left(\int_0^1 f(x)\,dx+\int_1^t f(x)\,dx\right)=0,$$

i.e., $\int_0^\infty f(s)\,ds=0$ and so $f\in N$. The proof is complete.

Remark 7. If $\alpha>0$, then $L_1(x^{-\alpha})\subset C_1(x^{-\alpha})$ and $L_1(x^\alpha)\cap N\subset C_1(x^\alpha)$.

The following Hardy type estimate will illustrate the usefulness of the class N in the imbedding $N\cap L_1(|\log x|)\subset C_1$.

Proposition 3. *If* $f\in N\cap L_1(|\log x|)$, *then*

$$\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right| dx \le \int_0^\infty |f(x)|\,|\log x|\,dx. \tag{2.5}$$

Proof. By using the assumption $f\in N$ and changing an order of integration we obtain

$$\begin{aligned}
\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right| dx &= \int_0^1 \left|\frac{1}{x}\int_0^x f(s)\,ds\right| dx + \int_1^\infty \left|-\frac{1}{x}\int_x^\infty f(s)\,ds\right| dx \\
&\le \int_0^1 \frac{1}{x}\int_0^x |f(s)|\,ds\,dx + \int_1^\infty \frac{1}{x}\int_x^\infty |f(s)|\,ds\,dx \\
&= \int_0^1 \left(\int_s^1 \frac{1}{x}\,dx\right)|f(s)|\,ds + \int_1^\infty \left(\int_1^s \frac{1}{x}\,dx\right)|f(s)|\,ds \\
&= \int_0^1 |\log s|\,|f(s)|\,ds + \int_1^\infty |\log s|\,|f(s)|\,ds \\
&= \int_0^\infty |\log s|\,|f(s)|\,ds.
\end{aligned}$$

Remark 8. The inequality (2.5) means that the Hardy operator $Hf(x)=x^{-1}\int_0^x f(s)\,ds$ is bounded from the intersection $N\cap L_1(|\log x|)$ into L_1 or that we have the imbedding $N\cap L_1(|\log x|)\subset C_1$. Let us also recall that H is not bounded from all of the space $L_1(|\log x|)$ into L_1.

By using estimates from [12, Example 8.6(v) and Remark 8.7], we can prove the following more general result: If $f\in N\cap L_p(x^{p-1}|\log x|^p)$, $1\le p<\infty$, then

$$\int_0^\infty \left|\frac{1}{x}\int_0^x f(s)\,ds\right|^p x^{p-1}\,dx \le C\int_0^\infty |f(x)|^p x^{p-1}|\log x|^p\,dx. \tag{2.6}$$

This estimate says that the Hardy operator H is bounded from $N\cap L_p(x^{p-1}|\log x|^p)$ into $L_p(x^{p-1})$ or that we have the imbedding $N\cap L_p(x^{p-1}|\log x|^p)\subset C_p(x^{p-1})$, $p\ge 1$.

3. Computation of the spaces $(N\cap L_p(w_0), N\cap L_p(w_1))_{\theta,p}$

For $1\le p<\infty$ and a weight function w on $(0,\infty)$ we denote by $N_{p,w}$ the space $N_{p,w}=N\cap L_p(w)$ with the $\|\cdot\|_{L_p(w)}$-norm. We need the following technical assumptions about the weight functions w_0 and w_1:

(i) for $p=1$, w_0 is an increasing function and w_1 is a decreasing function with $w_1(\frac{1}{2}s)\le Aw_1(s)$ for all $s>0$,

(ii) for $p>1$, w_0 is an increasing function and

$$\int_x^\infty w_0(s)^{-1/(p-1)}\,ds\le Cxw_0(x)^{-1/(p-1)}\quad\text{for all } x>0;$$

w_1 is either a decreasing function with $w_1(\frac{1}{2}s)\le Aw_1(s)$ for all $s>0$ or w_1 is an increasing function such that $w_0(s)/w_1(s)$ is increasing and

$$\int_0^x w_1(s)^{-1/(p-1)}\,ds\le Bxw_1(x)^{-1/(p-1)}\quad\text{for all } x>0.$$

One important example here is the case when $w_0(x)=x^\alpha$ and $w_1(x)=x^\beta$, where $\beta<p-1<\alpha$ and $p\ge1$.

In the sequel we also use the notation $w_{01}(x)=w_0(x)/w_1(x)$ and $r(t)=w_{01}^{-1}(t^p)$. We are now ready to formulate the main result of this section.

Theorem 3. *Let $1\le p<\infty$. Assume that the weights w_0 and w_1 satisfy the above assumptions. Then*

(3.1)
$$K(t,f;N_{p,w_0},N_{p,w_1})\approx K(t,f;L_p(w_0),L_p(w_1))+r(t)^{1/p-1}w_0(r(t))^{1/p}\left|\int_0^{r(t)}f(s)\,ds\right|$$

for all f in $N_{p,w_0}+N_{p,w_1}$ and all $t>0$. If, in addition, $s(d/ds)w_{01}(s)\approx w_{01}(s)$, then

$$(N\cap L_p(w_0),N\cap L_p(w_1))_{\theta,p}=N\cap C_p(w_0^{1-\theta}w_1^\theta)\cap L_p(w_0^{1-\theta}w_1^\theta). \tag{3.2}$$

Proof. We first note that since $N_{p,w_i}\subset L_p(w_i)$, $i=0,1$, it follows that

$$K(t,f;N_{p,w_0},N_{p,w_1})\ge K(t,f;L_p(w_0),L_p(w_1)).$$

Therefore, in order to prove the lower estimate in (3.1) it is sufficient to prove that

$$\left|\int_0^{r(t)}f(s)\,ds\right|\le cr(t)^{1-1/p}w_0(r(t))^{-1/p}K(t,f;N_{p,w_0},N_{p,w_1}) \tag{3.3}$$

with the constant $c>0$ independent of $f\in N_{p,w_0}+N_{p,w_1}$ and $t>0$.

For a fixed $t>0$, let $f=f_0+f_1$ be an almost optimal decomposition of $f\in N_{p,w_0}+N_{p,w_1}$, i.e.,

$$\|f_0\|_{N_{p,w_0}}+t\|f_1\|_{N_{p,w_1}}\le 2K(t,f;N_{p,w_0},N_{p,w_1}).$$

Since $f_0\in N_{p,w_0}\subset N$ it follows that $\int_0^\infty f_0(s)\,ds=0$ and, by the Hölder inequality and the assumption on w_0, we find that

$$\begin{aligned}\left|\int_0^{r(t)} f_0(s)\,ds\right| &= \left|\int_{r(t)}^\infty f_0(s)\,ds\right| \le \int_{r(t)}^\infty |f_0(s)|w_0(s)^{1/p}w_0(s)^{-1/p}\,ds\\ &\le \left(\int_{r(t)}^\infty |f_0(s)|^p w_0(s)\,ds\right)^{1/p}\left(\int_{r(t)}^\infty w_0(s)^{-p'/p}\,ds\right)^{1/p'}\\ &\le C\|f_0\|_{N_{p,w_0}} r(t)^{1-1/p}w_0(r(t))^{-1/p}\\ &\le 2Cr(t)^{1-1/p}w_0(r(t))^{-1/p}K(t,f;N_{p,w_0},N_{p,w_1}).\end{aligned}$$

Similarly, we find that $f_1\in N$ and

$$\begin{aligned}\left|\int_0^{r(t)} f_1(s)\,ds\right| &\le \int_0^{r(t)} |f_1(s)|w_1(s)^{1/p}w_1(s)^{-1/p}\,ds\\ &\le \left(\int_0^{r(t)} |f_1(s)|^p w_1(s)\,ds\right)^{1/p}\left(\int_0^{r(t)} w_1(s)^{-p'/p}\,ds\right)^{1/p'}\\ &\le B\|f_1\|_{N_{p,w_1}} r(t)^{1-1/p}w_1(r(t))^{-1/p}\\ &\le 2Br(t)^{1-1/p}w_1(r(t))^{-1/p}K(t,f;N_{p,w_0},N_{p,w_1})/t\\ &= 2Br(t)^{1-1/p}w_0(r(t))^{-1/p}K(t,f;N_{p,w_0},N_{p,w_1}),\end{aligned}$$

where in the last equality we have used that $w_1(r(t))^{-1/p}/t=w_0(r(t))^{-1/p}$. Thus

$$\begin{aligned}\left|\int_0^{r(t)} f(s)\,ds\right| &\le \left|\int_0^{r(t)} f_0(s)\,ds\right|+\left|\int_0^{r(t)} f_1(s)\,ds\right|\\ &\le 2(C+B)r(t)^{1-1/p}w_0(r(t))^{-1/p}K(t,f;N_{p,w_0},N_{p,w_1})\end{aligned}$$

and the inequality (3.3) holds. Thus, the lower estimate in (3.1) is proved. In order to establish the upper estimate in (3.1) we fix $t>0$ and consider the decomposition $f=f_0+f_1$ of $f\in N_{p,w_0}+N_{p,w_1}$ with

$$f_0(s)=f(s)\chi_{(0,r(t))}(s)-c\chi_{[r(t)-\varepsilon,r(t)]}(s),$$

and

$$f_1(s)=f(s)-f_0(s)=f(s)\chi_{[r(t),\infty)}(s)+c\chi_{[r(t)-\varepsilon,r(t)]}(s),$$

where

$$c=\frac{1}{\varepsilon}\biggl(\int_0^{r(t)} f(u)\,du\biggr),\quad r(t)=w_{01}^{-1}(t^p)\quad\text{and}\quad \varepsilon=\tfrac{1}{2}r(t).$$

Since $f\in N_{p,w_0}+N_{p,w_1}\subset N$ it follows that $\int_0^\infty f(s)\,ds=0$. By using the definition of f_0 and f_1 we obtain $\int_0^\infty f_0(s)\,ds=0$ and $\int_0^\infty f_1(s)\,ds=0$. Therefore $f_0,f_1\in N$ and

$$\begin{aligned}
\|f_0\|_{N_{p,w_0}} &= \biggl(\int_0^\infty |f_0(s)|^p w_0(s)\,ds\biggr)^{1/p}\\
&= \biggl(\int_0^{r(t)-\varepsilon} |f(s)|^p w_0(s)\,ds+\int_{r(t)-\varepsilon}^{r(t)}\biggl|f(s)-\frac{1}{\varepsilon}\int_0^{r(t)} f(u)\,du\biggr|^p w_0(s)\,ds\biggr)^{1/p}\\
&\le 2^{1-1/p}\biggl(\int_0^{r(t)} |f(s)|^p w_0(s)\,ds+\biggl|\frac{1}{\varepsilon}\int_0^{r(t)} f(u)\,du\biggr|^p\int_{r(t)-\varepsilon}^{r(t)} w_0(s)\,ds\biggr)^{1/p}\\
&\le 2^{1-1/p}\biggl(\int_0^{r(t)} |f(s)|^p w_0(s)\,ds+\frac{1}{\varepsilon^p}\biggl|\int_0^{r(t)} f(u)\,du\biggr|^p w_0(r(t))\varepsilon\biggr)^{1/p}\\
&\le 2^{1-1/p}\biggl(\int_0^{r(t)} |f(s)|^p w_0(s)\,ds\biggr)^{1/p}\\
&\quad +2^{1-1/p}\varepsilon^{(1-p)/p}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(u)\,du\biggr|\\
&= 2^{1-1/p}\biggl(\int_0^{r(t)} |f(s)|^p w_0(s)\,ds\biggr)^{1/p}\\
&\quad +4^{1-1/p}r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(u)\,du\biggr|.
\end{aligned}$$

Next we note that

$$\begin{aligned}
t\|f_1\|_{N_{p,w_1}} &= t\biggl(\int_0^\infty |f_1(s)|^p w_1(s)\,ds\biggr)^{1/p}\\
&= t\biggl(\biggl|\frac{1}{\varepsilon}\int_0^{r(t)} f(u)\,du\biggr|^p\biggl|\int_{r(t)-\varepsilon}^{r(t)} w_1(s)\,ds\biggr|+\int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p}
\end{aligned}$$

Now, when w_1 is decreasing we have

$$\begin{aligned}
t\|f_1\|_{N_{p,w_1}} &\le t\biggl(\varepsilon^{-p}\biggl|\int_0^{r(t)} f(u)\,du\biggr|^p w_1(r(t)-\varepsilon)\varepsilon + \int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p} \\
&\le t\varepsilon^{1/p-1}A^{1/p}w_1(r(t))^{1/p}\biggl|\int_0^{r(t)} f(u)\,du\biggr| + t\biggl(\int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p} \\
&= 2^{1-1/p}A^{1/p}r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(u)\,du\biggr| \\
&\quad + t\biggl(\int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p},
\end{aligned}$$

and when w_1 is increasing we find that

$$\begin{aligned}
t\|f_1\|_{N_{p,w_1}} &\le t\biggl(\varepsilon^{-p}\biggl|\int_0^{r(t)} f(u)\,du\biggr|^p w_1(r(t))\varepsilon + \int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p} \\
&\le 2^{1-1/p}r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(u)\,du\biggr| \\
&\quad + t\biggl(\int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p}.
\end{aligned}$$

Thus

$$\begin{aligned}
K(t,f;N_{p,w_0},N_{p,w_1}) &\le \|f_0\|_{N_{p,w_0}} + t\|f_1\|_{N_{p,w_1}} \\
&\le c\biggl[\biggl(\int_0^{r(t)} |f(s)|^p w_0(s)\,ds + t^p\int_{r(t)}^\infty |f(s)|^p w_1(s)\,ds\biggr)^{1/p} \\
&\qquad + r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(s)\,ds\biggr|\biggr] \\
&= c\biggl[\biggl(\int_0^\infty |f(s)|^p \min(w_0(s), t^p w_1(s))\,ds\biggr)^{1/p} \\
&\qquad + r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(s)\,ds\biggr|\biggr] \\
&\le c\biggl[K(t,f;L_p(w_0),L_p(w_1)) \\
&\qquad + r(t)^{1/p-1}w_0(r(t))^{1/p}\biggl|\int_0^{r(t)} f(s)\,ds\biggr|\biggr]
\end{aligned}$$

and also the upper estimate in the equivalence (3.1) is proved. Moreover, the equivalence (3.1) for the K-functional gives an identification of the corresponding real interpolation spaces for $f\in N$. More exactly, we have

$$\begin{aligned}\|f\|_{(N\cap L_p(w_0),N\cap L_p(w_1))_{\theta,p}} &\approx \|f\|_{(L_p(w_0),L_p(w_1))_{\theta,p}}\\ &\quad+\left[\int_0^\infty \frac{1}{t^{\theta p}}\left(r(t)^{1/p-1}w_0(r(t))^{1/p}\left|\int_0^{r(t)} f(s)\,ds\right|\right)^p\frac{dt}{t}\right]^{1/p}\\ &\approx \|f\|_{(L_p(w_0),L_p(w_1))_{\theta,p}}\\ &\quad+\left(\int_0^\infty\left|\frac{1}{x}\int_0^x f(s)\,ds\right|^p w_0^{1-\theta}(x)w_1^\theta(x)\,dx\right)^{1/p}\\ &=\|f\|_{L_p(w_0^{1-\theta}w_1^\theta)}+\|f\|_{C_p(w_0^{1-\theta}w_1^\theta)}.\end{aligned}$$

Hence (3.2) is proved.

By applying Theorem 3 with $w_0(x)=x^\alpha$ and $w_1(x)=x^\beta$ we obtain the following formal generalization of Theorems 1 and 2.

Corollary 1. *If $p\geq1$ and $\beta<p-1<\alpha$, then*

$$(3.4)\quad K(t,f;N_{p,\alpha},N_{p,\beta})\approx K(t,f;L_p(x^\alpha),L_p(x^\beta))+t^{(\alpha+1-p)/(\alpha-\beta)}\left|\int_0^{t^{p/(\alpha-\beta)}} f(s)\,ds\right|$$

for all f in $N_{p,\alpha}+N_{p,\beta}$ and all $t>0$. Moreover,

$$(N\cap L_p(x^\alpha),N\cap L_p(x^\beta))_{\theta,p}=N\cap L_p(x^{(1-\theta)\alpha+\theta\beta}),\quad \text{if } \theta\neq(\alpha+1-p)/(\alpha-\beta)$$

and

$$(N\cap L_p(x^\alpha),N\cap L_p(x^\beta))_{\theta,p}=C_p(x^{p-1})\cap L_p(x^{p-1}),\quad \text{if } \theta=(\alpha+1-p)/(\alpha-\beta).$$

Finally, we present the following remarkable consequences of Theorem 3.

Corollary 2. *Let $w_0(x)=\max(x^{\alpha_0},x^{\alpha_1})$ and $w_1(x)=\min(x^{-\beta_0},x^{-\beta_1})$ with $0<\alpha_0\leq\alpha_1$, $0<\beta_0\leq\beta_1$ and $\alpha_0/\alpha_1\leq\beta_0/\beta_1$. If $\theta\in(0,1)\setminus[\alpha_0/(\alpha_0+\beta_0),\alpha_1/(\alpha_1+\beta_1)]$ and $f\in N$ we have both Hardy inequalities*

$$\int_0^\infty\left|\frac{1}{x}\int_0^x f(s)\,ds\right|w_0^{1-\theta}(x)w_1^\theta(x)\,dx\leq C\int_0^\infty|f(x)|w_0^{1-\theta}(x)w_1^\theta(x)\,dx$$

and

$$\int_0^\infty\left|\frac{1}{x}\int_x^\infty f(s)\,ds\right|w_0^{1-\theta}(x)w_1^\theta(x)\,dx\leq C\int_0^\infty|f(x)|w_0^{1-\theta}(x)w_1^\theta(x)\,dx,$$

and therefore

$$(N\cap L_1(w_0), N\cap L_1(w_1))_{\theta,1} = N\cap L_1(w_0^{1-\theta}w_1^{\theta}).$$

For $\theta\in[\alpha_0/(\alpha_0+\beta_0), \alpha_1/(\alpha_1+\beta_1)]$ *no one of these Hardy inequalities is true and*

$$(N\cap L_1(w_0), N\cap L_1(w_1))_{\theta,1} = N\cap C_1(w_0^{1-\theta}w_1^{\theta})\cap L_1(w_0^{1-\theta}w_1^{\theta}).$$

Remark 9. The result in Corollary 1 shows that if w_0 and w_1 are power weights, then we obtain the usual interpolation result except for *one* value of the parameter. This situation corresponds to the Hardy inequality for power weights and its failure for one value of the parameter. Corollary 2 shows that with other choices of weights we can even have an interval of parameters where the usual interpolation formula fails and also that this phenomenon is connected with the failure of the Hardy inequality. Moreover, our results give the appropriate interpolation results in all these exceptional cases.

4. Why we cannot interpolate some inequalities

We shall again consider the inequalities (0.1) and (0.2). First we consider (0.1),

$$\int_0^\infty |u(s)|s^{\alpha-1}\,ds \le C(\alpha)\int_0^\infty |u'(s)|s^{\alpha}\,ds, \quad u\in C_0^\infty,$$

which is true for $\alpha\neq 0$ and fails for $\alpha=0$.

We have to explain why it is impossible to interpolate it from $\alpha=1$ and $\alpha=-1$, and obtain it for $\alpha=0$. We note that the above inequality has the form

$$\|u\|_{L_1(x^{\alpha-1})} \le C(\alpha)\|Du\|_{L_1(x^{\alpha})}, \quad u\in C_0^\infty,\ \alpha\neq 0,$$

with the operator $Du=u'$.

If we wish to interpolate it, we, first of all, have to rewrite it as boundedness of the inverse operator,

$$\|D^{-1}u\|_{L_1(x^{\alpha-1})} \le C(\alpha)\|u\|_{L_1(x^{\alpha})}, \quad u\in D(C_0^\infty),\ \alpha\neq 0. \tag{4.1}$$

In fact, it is possible to do this because D has no kernel on C_0^∞. Moreover, as

$$D(C_0^\infty) = N\cap C_0^\infty, \tag{4.2}$$

which we will prove later on, it follows from Remark 1 that $D(C_0^\infty)$ is dense in $L_1(x^\alpha)$ for $\alpha\neq 0$ and in $N\cap L_1(x^\alpha)$ for $\alpha=0$. Therefore, for each $\alpha\neq 0$, (4.1) implies that D^{-1} has a unique extension to the bounded operator

$$D_\alpha^{-1}\colon L_1(x^\alpha) \longrightarrow L_1(x^{\alpha-1}), \quad \alpha\neq 0. \tag{4.3}$$

Furthermore, if the inequality (4.1) is true for $\alpha=0$, then we would have the bounded extension

$$D_0^{-1}: N\cap L_1 \longrightarrow L_1(x^{-1}). \tag{4.4}$$

Therefore the problem to interpolate (0.1) from $\alpha=1$ and $\alpha=-1$, and obtain it for $\alpha=0$, can be reformulated as follows: *Is it possible to interpolate* (4.3) *for* $\alpha=1$, $\alpha=-1$ *and obtain* (4.4)?

In our case the operators D_α^{-1}, $\alpha\neq 0$, can be written explicitly (and the boundedness of them follows from the Hardy inequality)

$$D_\alpha^{-1}v(x)=\int_0^x v(s)\,ds \text{ for } \alpha<0 \quad\text{and}\quad D_\alpha^{-1}v(x)=-\int_x^\infty v(s)\,ds \text{ for } \alpha>0.$$

We see that D_1^{-1} and D_{-1}^{-1} are two *different* operators and to interpolate them, as in Section 1, we have to restrict them to the subspace where they coincide. This space is exactly $N=\{v:\int_0^\infty v(s)\,ds=0\}$. By interpolation we obtain only that

$$D_0^{-1}: (N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1} \longrightarrow (L_1, L_1(x^{-2}))_{1/2,1}=L_1(x^{-1})$$

is bounded. From our Theorem 2 it follows that $(N\cap L_1(x), N\cap L_1(x^{-1}))_{1/2,1}=C_1\cap L_1$, and therefore we can only say that

$$D_0^{-1}: C_1\cap L_1 \longrightarrow L_1(x^{-1}) \tag{4.5}$$

is bounded, *instead of* (4.4).

In terms of inequalities (and going back to the operator D) the boundedness (4.5) of course only gives a trivial estimate.

We finish this part with the missing proof.

Proof of (4.2). If $u\in C_0^\infty$, then $\int_0^\infty Du(s)\,ds=\int_0^\infty u'(s)\,ds=0$ and so $D(C_0^\infty)\subset N\cap C_0^\infty$. The reverse imbedding follows from the fact that if $v\in N\cap C_0^\infty$, then the function

$$u(t)=\int_0^t v(s)\,ds=-\int_t^\infty v(s)\,ds$$

belongs to C_0^∞ and $Du=v$.

Let us now consider the inequality (0.2) which has the form

$$\int_0^\infty \left|\frac{u(t)}{t^\alpha}\right|^p \frac{dt}{t} \le B\int_0^\infty \left|\frac{u(t)-t^{-1}\int_0^t u(s)\,ds}{t^\alpha}\right|^p \frac{dt}{t}$$

and holds for functions $u \in L_0^{\text{loc}}$ (locally integrable functions on $(0,\infty)$ with a compact support) if $\alpha \neq 0$. For $\alpha=0$ it fails.

The situation here is quite analogous to the previous case. The key to understanding this analogy is to define the operator D by

$$Du(t) = \frac{u(t) - t^{-1}\int_0^t u(s)\,ds}{t}. \tag{4.6}$$

The reason for such a definition of D is that $Du \in N$ for $u \in L_0^{\text{loc}}$. Indeed, since the derivative of $t^{-1}\int_0^t u(s)\,ds$ is $Du(t)$ it follows that

$$\int_0^\infty Du(t) = \frac{1}{t}\int_0^t u(s)\,ds \Big|_{t=0}^{\infty} = 0.$$

In terms of the operator D the inequality (0.2) can be written as follows

$$\|u\|_{L_p(x^{-\alpha p-1})} \le B(\alpha)\|Du\|_{L_p(x^{-(\alpha-1)p-1})},$$

where $\alpha \neq 0$ and $u \in L_0^{\text{loc}}(0,\infty)$.

The inverse operator of D can be written explicitly (see [7, Remark 5] applied with $sv(s)$ instead of $v(s)$),

$$D_\alpha^{-1}v(x) = xv(x) + \int_0^x v(s)\,ds \quad \text{for } \alpha < 0,$$

and

$$D_\alpha^{-1}v(x) = xv(x) - \int_x^\infty v(s)\,ds \quad \text{for } \alpha > 0.$$

Moreover, the Hardy inequality implies that $D_\alpha^{-1}\colon L_p(x^{-(\alpha-1)p-1}) \to L_p(x^{\alpha p-1})$ is bounded for all $\alpha \neq 0$. Again, we see that D_1^{-1} and D_{-1}^{-1} are two *different* operators and to interpolate them we have to restrict them to the space of functions where they are equal. This happens exactly in the space $N = \{v : \int_0^\infty v(s)\,ds = 0\}$. By interpolation we get only that

$$D_0^{-1}\colon (N \cap L_p(x^{-1}), N \cap L_p(x^{2p-1}))_{1/2,p} \longrightarrow (L_p(x^{p-1}), L_p(x^{-p-1}))_{1/2,p} = L_p(x^{-1}).$$

Further, by our Theorem 3, it follows that

$$(N \cap L_p(x^{-1}), N \cap L_p(x^{2p-1}))_{1/2,p} = N \cap C_p(x^{p-1}) \cap L_p(x^{p-1}),$$

and therefore we can only say that

$$D_0^{-1}: N \cap C_p(x^{p-1}) \cap L_p(x^{p-1}) \longrightarrow L_p(x^{-1}) \tag{4.7}$$

is bounded, and that D_0^{-1} is *not* bounded from $N \cap L_p(x^{p-1})$ into $L_p(x^{-1})$, which corresponds to the invalidity of (0.2) for $\alpha=0$.

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