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Interpolating sequences in the ball of $\mathbf{C}^n$

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Abstract. Let $\mathbf{B}$ be the unit ball of $\mathbf{C}^n$, I give necessary conditions on a sequence S of points in $\mathbf{B}$ to be $H^\infty(\mathbf{B})$ interpolating in term of a $\mathbf{C}^n$ valued holomorphic function zero on S (a substitute for the interpolating Blaschke product).

These conditions are sufficient to prove that the sequence S is interpolating for $\bigcap_{p>1} H^p(\mathbf{B})$ and is also interpolating for $H^p(\mathbf{B})$ for $1 \le p < \infty$.

1. Introduction

Let $\mathbf{B}$ be the unit ball of $\mathbf{C}^n$ and $S := \{a_j\}_{j\in\mathbf{N}}$ be a sequence of points in $\mathbf{B}$. We shall say that S is $H^\infty(\mathbf{B})$ interpolating if for every $\lambda = \{\lambda_j\}_{j\in\mathbf{N}} \in l^\infty(\mathbf{N})$, there exists $f \in H^\infty(\mathbf{B})$ such that $f(a_j) = \lambda_j$ for all $j \in \mathbf{N}$.

We shall say that S is $\bigcap_{p>1} H^p(\mathbf{B})$ interpolating if for every $\lambda = \{\lambda_j\}_{j\in\mathbf{N}} \in l^\infty(\mathbf{N})$, there exist $f \in \bigcap_{p>1} H^p(\mathbf{B})$ such that $f(a_j) = \lambda_j$ for all $j \in \mathbf{N}$.

Finally we shall say that S is $H^p(\mathbf{B})$ interpolating if for every $\lambda = \{\lambda_j\}_{j\in\mathbf{N}}$ with $\|\lambda\|_p^p := \sum_{j=0}^\infty |\lambda_j|^p (1-|a_j|^2)^n < +\infty$, there exists $f \in H^p(\mathbf{B})$ such that $f(a_j) = \lambda_j$ for all $j \in \mathbf{N}$.

If S is $H^\infty(\mathbf{B})$ interpolating then the closed graph theorem gives the existence of a constant C such that for any bounded sequence λ there exists a function $f \in H^\infty(\mathbf{B})$ such that for all $j \in \mathbf{N}$, $f(a_j) = \lambda_j$ with the control $\|f\|_\infty \le C\|\lambda\|_\infty$. The smallest such C is called the interpolating constant of S.

The $H^\infty(\mathbf{B})$ interpolating sequences are precisely characterized for $n=1$ in the theorem of L. Carleson [8] and they are the same as the $H^p(\mathbf{B})$ interpolating sequences in that case [11]. Such a sequence is the set of zeros of an interpolating Blaschke product.

Let for $a \in \partial\mathbf{B}$ and $h > 0$, $Q := Q(a,h) := \{\eta \in \mathbf{B} \mid |1 - \bar{a}\eta| < h\}$ be a pseudoball. We say that a measure μ on $\mathbf{B}$ is a Carleson measure if there exist $C > 0$ such that

$$|\mu|(Q(a,h)) \le Ch^n \quad \text{for all } a \in \partial\mathbf{B} \text{ and } h > 0.$$

In the case $n>1$, N. Varopoulos [13], proved that if S is interpolating for $H^\infty(\mathbf{B})$ then the measure $\mu:=\sum_{j=1}^{\infty}\delta_{a_j}(1-|a_j|^2)^n$ is Carleson.

In [2], I proved that if S is H^2 interpolating, then again the measure $\mu:=\sum_{j=0}^{\infty}\delta_{a_j}(1-|a_j|^2)^n$ is Carleson and in [1], we proved that there is a sequence S in the ball of $\mathbf{C}^2$ which is H^2 interpolating but not H^∞ interpolating, which means that the Varopoulos' condition is not sufficient for H^∞ interpolation.

On the other hand B. Berndtsson [7] proved that if the product of the Gleason distances of the points of S is bounded below, the sequence S is H^∞ interpolating. He also showed that this condition, which characterizes interpolating sequences when $n=1$, is not necessary for $n>1$.

The aim of this work is to give a generalization of the interpolating Blaschke product in the case of the ball in $\mathbf{C}^n$.

Let B be a $\mathbf{C}^n$ valued bounded holomorphic function in $\mathbf{B}$.

Definition 1.1. Let $a\in\mathbf{B}$, and Φ_a be a biholomorphic map exchanging a and 0. We shall say that B is equivalent to Φ_a near a if $B=M_a\cdot\Phi_a$ with the matrix M_a invertible near a. More precisely, we require that there is a $\delta>0$ and $C_B>0$ such that M_a is invertible in $|\Phi_a|<\delta$ and, with $A_a:=M_a^{-1}$, $|A_a|<C_B$ in $|\Phi_a|<\delta$.

Now we can give the definition of an interpolating function for S.

Definition 1.2. Let $S:=\{a_j\}_{j\in\mathbf{N}}$ be a sequence of points in $\mathbf{B}$ and B be a $\mathbf{C}^n$ valued bounded holomorphic function in $\mathbf{B}$. We say that B is interpolating for S if B is equivalent to $\Phi_j:=\Phi_{a_j}$ near a_j uniformly with respect to a_j, i.e. the constants δ and C_B are independent of a_j.

Of course, if B is interpolating for S then it is zero on S.

This is a characterization of the interpolating Blaschke products up to multiplication by a unit in $H^\infty(\mathbf{D})$, if we add that S are the only zeros of B.

The fact that this is a "possible" generalization in several variables is supported by the following theorems.

Theorem 1.3. *Let* $\mathbf{B}$ *be the unit ball of* $\mathbf{C}^n$, *if the sequence* $S:=\{a_j\in\mathbf{B}\}_{j\in\mathbf{N}}$ *is interpolating for* $H^\infty(\mathbf{B})$ *then there is an interpolating function* B *for* S.

Theorem 1.4. *Let* $\mathbf{B}$ *be the unit ball of* $\mathbf{C}^2$, *if there is an interpolating function* B *for the sequence* S, *then the sequence* S *is* $\bigcap_{p>1}H^p(\mathbf{B})$ *interpolating.*

Theorem 1.5. *Let* $\mathbf{B}$ *be the unit ball of* $\mathbf{C}^2$, *if there is an interpolating function* B *for the sequence* S, *then the sequence* S *is* $H^p(\mathbf{B})$ *interpolating for* $1\le p<\infty$.

The sufficient results are stated and proved in $\mathbf{C}^2$. No doubt they are true in $\mathbf{C}^n$, but at the price of non-trivial technical new results.

I want to thank B. Berndtsson for giving me simpler proofs of some lemmas.

2. Necessary conditions

In order to prove Theorem 1.3, we shall need the following.

2.1. Linear extension

Let us recall Drury's lemma as used by A. Bernard in [6]:

Let $S=\{a_j\}_{j=1}^N \subset \mathbf{B}$ be a finite sequence and let $\lambda := e^{2i\pi/N}$ be an Nth root of 1.

Suppose S is interpolating for $H^\infty(\mathbf{B})$ with constant C, then we can find functions $\beta_j \in H^\infty(\mathbf{B})$, $j=1,\dots,N$ such that $\beta_j(a_k)=\lambda^{jk}$ and $\|\beta_j\|_\infty \le C$.

Put on the group G of the Nth roots of 1, its Haar measure

$$d\mu := \frac{1}{N}\sum_{j=1}^N \delta_{\lambda^j},$$

and on the dual group $\Gamma=\mathbf{Z}/N\mathbf{Z}$, the dual measure $d\nu := \sum_{j=1}^N \delta_j$.

We may consider the β_j as functions on G, depending on the parameter z, and take their Fourier transform,

$$\hat{\beta}_j := \frac{1}{N}\sum_{k=1}^N \lambda^{-jk}\beta_k.$$

We get easily that $\hat{\beta}_j \in H^\infty(\mathbf{B})$, $\hat{\beta}_j(a_k)=\delta_{jk}$ and, using the Plancherel formula,

$$\sum_{j=1}^N |\hat{\beta}_j|^2 = \frac{1}{N}\sum_{j=1}^N |\beta_j|^2 \le C^2.$$

Hence we have proved the following proposition.

Proposition 2.1. ([6]) *If $S:=\{a_j \in \mathbf{B}\}_{j=1}^N$ is a finite set of points in $\mathbf{B}$, with H^∞ interpolating constant C, then there are functions β_j in $H^\infty(\mathbf{B})$ such that $\sum_{j=1}^N |\beta_j(z)|^2 \le C^2$ and $\beta_j(a_k)=\delta_{jk}$.*

Using this fact, we can set

$$B(z) := \sum_{j=1}^N \Phi_j(z)\beta_j^2(z)$$

and it remains to show that this $\mathbf{C}^2$ valued function fulfills the conclusion of Theorem 1.3.

The following is a simple generalization of a result of Ahern and Schneider [10, p. 115].

Theorem 2.2. *Let $\beta:=\{\beta_j\in H^\infty(\mathbf{B})\}_{j=1}^K$ be such that $\sum_{j=1}^K|\beta_j(z)|^2\le A^2$ for all $z\in\mathbf{B}$, and for all $j=1,\dots,K$, $\beta_j(a)=0$ for an $a\in\mathbf{B}$, then $\beta_j=\gamma_j\cdot\Phi_a$, where Φ_a is a biholomorphic mapping exchanging a and 0 and the γ_j's are $\mathbf{C}^n$ valued bounded holomorphic functions in $\mathbf{B}$ with $\sum_{j=1}^K|\gamma_j|^2\le CA^2$ and C independent of K.*

Proof. Using the invariance of the sup norm under a biholomorphic mapping, it suffices to prove the theorem for $a=0$.

Let us introduce the "big Hankel" operator as in [10] but for $\mathbf{C}^K$ valued holomorphic functions:

$$f=(f_1,\dots,f_K),\quad V_\varphi f:=\varphi\cdot f-T_\varphi f,$$

where $\varphi\cdot f:=(\varphi f_1,\dots,\varphi f_K)$ and $T_\varphi f:=(P(\varphi f_1),\dots,P(\varphi f_K))$ and P is the projection of $L^2(\partial\mathbf{B})$ on $H^2(\mathbf{B})$; V_φ is the projection on the orthogonal complement of $H^2(\mathbf{B})$ called "big Hankel" of symbol φ.

Now we put the euclidian norm on $\mathbf{C}^K$ valued functions,

$$|f(z)|^2:=\sum_{j=1}^K|f_j(z)|^2,$$

and we want to show that if $f\in H^\infty(\mathbf{B})$, then $V_\varphi f$ is also bounded with norm depending only on the norm of f and not on K.

Let $F:=(F_1,\dots,F_K)$, be such that $|F|\in L^1(\partial\mathbf{B})$. We have

$$\langle V_\varphi f,F\rangle=\int_{\partial\mathbf{B}}\int_{\partial\mathbf{B}}\sum_{j=1}^K f_j(\zeta)\overline{F}_j(z)\Gamma_z(\zeta)\,d\sigma(\zeta)\,d\sigma(z),$$

with $\Gamma_z(\zeta):=C(z,\zeta)(\varphi(z)-\varphi(\zeta))$, where $C(z,\zeta)$ is the Cauchy kernel in $\mathbf{B}$.

Using Schwarz' inequality in the sum we get,

$$|\langle V_\varphi f,F\rangle|\le\int_{\partial\mathbf{B}}\int_{\partial\mathbf{B}}|f(\zeta)|\,|F(z)|\,|\Gamma_z(\zeta)|\,d\sigma(\zeta)\,d\sigma(z),$$

but $|f|$ is bounded and $|\Gamma_z(\zeta)|$ is uniformly integrable in ζ ([10]). Hence we get $|\langle V_\varphi f,F\rangle|\le C\|f\|_\infty\|F\|_1$, which proves that the sup norm of $V_\varphi f$ is bounded by a fixed constant times the sup norm of f.

Now we can use exactly the end of the proof in [10] to conclude that we can factorize the identity if the vector β is 0 at 0 with control of the norm of γ, hence the theorem. □

It remains to finish the proof of Theorem 1.3. Recall that

$$B(z):=\sum_{j=1}^N\Phi_j(z)\beta_j^2(z),$$

hence B is a $\mathbf{C}^n$ valued holomorphic function in $\mathbf{B}$, which is obviously bounded, as the Φ_j send $\mathbf{B}$ on $\mathbf{B}$.

Clearly B is zero on the sequence S and we have to show that $B\simeq\Phi_j$ near a_j.

Applying Theorem 2.2, we have the existence, for all indices j and k, of a $\mathbf{C}^n$ valued holomorphic function γ_{kj} such that

$$\beta_k=\gamma_{kj}\cdot\Phi_j,\ k\neq j,\quad \text{and}\quad \sum_{k\neq j}|\gamma_{kj}|^2\leq C^2.$$

Then $B(z)=\Phi_j\beta_j^2(z)+\sum_{k\neq j}\Phi_k(\gamma_{kj}\cdot\Phi_j)^2$. Put $\alpha_{jk}:=(\gamma_{kj}\cdot\Phi_j)\gamma_{kj}$. This is also a $\mathbf{C}^n$ valued holomorphic function and $m_{jk}:=\Phi_k\cdot\alpha_{jk}$ may be seen as an $n\times n$ matrix, defined by the identity $(\Phi_k\cdot\alpha_{jk})\cdot v=(\alpha_{jk}\cdot v)\Phi_k$, $v\in\mathbf{C}^n$, hence we can define $m_j:=\sum_{k\neq j}m_{jk}$ as an $n\times n$ matrix.

We have $|m_j|\leq\sum_{j\neq k}|(\gamma_{kj}\cdot\Phi_j)\gamma_{kj}|\,|\Phi_k|\leq|\Phi_j|\sum_{j\neq k}|\gamma_{kj}|^2\leq|\Phi_j|C^2$.

With this notation $B(z)$ can be written

$$B(z)=(\beta_j^2(z)I+m_j)\cdot\Phi_j\quad\text{with } I \text{ being the identity } n\times n \text{ matrix.}$$

Since $\beta_j(a_j)=1$ it follows that $|\beta_j|\geq\frac{1}{2}$ in $|\Phi_j|\leq\delta$, if δ is small enough, and putting $B_j:=\beta_j^2(z)I+m_j$ we have that $B=M_j\cdot\Phi_j$ and that the holomorphic matrix M_j is bounded in $\mathbf{B}$ by a constant independent of j. It is also invertible in $|\Phi_j|\leq\delta$, if δ is small enough, and its inverse A_j is also bounded independently of j in $|\Phi_j|\leq\delta$, hence the theorem. □

3. Sufficient conditions

Let $S:=\{a_j\in\mathbf{B}\}_{j\in\mathbf{N}}$ be a sequence in $\mathbf{B}$ and $\chi(t)$ be the usual cut-off function, $\chi\in\mathcal{C}^\infty(\mathbf{R}^+)$ satisfying $\chi(t)=0$ if $t\geq1$ and $\chi(t)=1$ if $0\leq t\leq\frac{1}{2}$, and let $\chi_j(z):=\chi(|\Phi_j|^2/\delta^2)$.

We say that the sequence S is uniformly separated if the sets $\{|\Phi_j|<\delta\}$ are disjoint.

To prove Theorem 1.4 we shall need a proposition.

Proposition 3.1. *If there is an interpolating function B for S, then the sequence S is uniformly separated and $\sum_{j=0}^{\infty}\chi_j|\partial\Phi_j|^2(1-|z|^2)\,d\lambda$ is a Carleson measure in $\mathbf{B}$.*

Proof. That B is equivalent to Φ_j near a_j means that $B=M_j\cdot\Phi_j$ with the matrix M_j invertible near a_j; precisely, there are $\delta>0$ and $C_B>0$ such that for all $j\in\mathbf{N}$, M_j is invertible in $|\Phi_j|<\delta$ and, with $A_j:=M_j^{-1}$ and $|A_j|<C_B$ in $|\Phi_j|<\delta$.

Using the Ahern and Schneider theorem, we already know that $|M_j|\leq C\|B\|_\infty$, and, since B is zero on a_k, we still get that $B_j=M_{jk}\cdot\Phi_k$, again with $|M_{jk}|\leq C^2\|B\|_\infty$.

Now suppose that $z\in\{|\Phi_j|<\eta\}\cap\{|\Phi_k|<\eta\}$, then $M_j(z)=M_{jk}(z)\cdot\Phi_k(z)$, hence $|M_j(z)|\leq C^2\eta\|B\|_\infty$, which leads to a contradiction if η is less than $1/C_BC^2\|B\|_\infty$, hence the separation.

To prove that the associated measure is Carleson we take advantage of the fact that, since B is in $H^\infty(\mathbf{B})$,

$$|\partial B(z)|^2(1-|z|^2)\,d\lambda \text{ is a Carleson measure;}$$

but $\partial B=M_j\cdot\partial\Phi_j+\partial M_j\cdot\Phi_j$, and as M_j is bounded, we get

$$|\partial M_j|\leq\frac{\|M_j\|_\infty}{1-|z|^2}\leq\frac{C\|B\|_\infty}{1-|z|^2}$$

hence on $|\Phi_j|<\delta$, $|\partial M_j|\leq C|\partial\Phi_j|$ with a constant C independent of j.

From $\partial\Phi_j+A_j\cdot\partial M_j\cdot\Phi_j=A_j\cdot\partial B$ in $|\Phi_j|<\delta$, we get for a δ such that $\delta C\|A_j\|<\frac{1}{2}$,

$$\frac{1}{2|\partial\Phi_j|}\leq|\partial\Phi_j|-|A_j\cdot\partial M_j\cdot\Phi_j|\leq C_B|\partial B|\quad\Longrightarrow\quad\chi_j|\partial\Phi_j|\leq C\chi_j|\partial B|,$$

with the usual cut-off function $\chi_j(z):=\chi(|\Phi_j|^2/\delta^2)$.

Hence we get that $\sum_{j=0}^\infty\chi_j|\partial\Phi_j|^2(1-|z|^2)\leq C|\partial B(z)|^2(1-|z|^2)$, since the sets $|\Phi_j|<\delta$ are disjoint, and the measure $\sum_{j=0}^\infty\chi_j|\partial\Phi_j|^2(1-|z|^2)\,d\lambda$ is Carleson in $\mathbf{B}$, which concludes the proof. □

Lemma 3.2. *Let $a\in\mathbf{B}$ and Φ_a be a biholomorphic mapping exchanging a and 0, then on $|\Phi_a|<\delta$, $|\partial\Phi_a|\simeq1/(1-|a|^2)$ the constants being independent of $a\in\mathbf{B}$.*

Proof. First suppose $a:=(a_1,0)\in\mathbf{B}$ and let

$$\Phi_a:=\Big(\frac{z_1-a_1}{1-\bar a_1z_1},\sqrt{1-|a_1|^2}\,\frac{z_2}{1-\bar a_1z_1}\Big)$$

be a biholomorphic mapping exchanging a and 0. If $|\Phi_a|<\delta$ then

$$|z_1-a_1|<\delta|1-\bar a_1z_1|\leq\delta\big|1-|a_1|^2+\bar a_1(a_1-z_1)\big|\leq\delta(1-|a_1|^2+\delta|1-\bar a_1z_1|),$$

and this implies

$$|z_1-a_1|\leq\frac{\delta}{1-\delta}(1-|a_1|^2);$$

in the same way

$$|z_2| \le \frac{\delta}{1-\delta}\sqrt{1-|a_1|^2}.$$

We then get

$$\partial\Phi_a = \left(\frac{1-|a_1|^2}{(1-\bar{a}_1 z_1)^2}\,dz_1, \sqrt{1-|a_1|^2}\,\frac{\bar{a}_1 z_2}{1-\bar{a}_1 z_1}\,dz_1 + \frac{\sqrt{1-|a_1|^2}}{1-\bar{a}_1 z_1}\,dz_2\right),$$

hence on $|\Phi_a|<\delta$,

$$|\partial\Phi_a| \simeq \left|\frac{1-|a_1|^2}{(1-\bar{a}_1 z_1)^2}\right| \simeq \frac{1}{1-|a_1|^2}.$$

This is invariant by rotations, hence for any $a\in\mathbf{B}$ we have on $|\Phi_a|<\delta$, $|\partial\Phi_a|\simeq 1/(1-|a|^2)$, the constants being independent of $a\in\mathbf{B}$. □

Remark 3.3. This implies that the measure $\mu:=\sum_{j=0}^{\infty}\delta_{a_j}(1-|a_j|^2)^2$ is also Carleson.

Proof of the remark. Let $Q(\zeta,r):=\{z\in\mathbf{B}\,|\,|1-\bar{\zeta}\cdot z|<r\}$ be a Carleson set. We have

$$a_j\in Q \quad\Longrightarrow\quad (1-|a_j|^2)^2 \lesssim \int_Q \chi_j|\partial\Phi_j|^2(1-|z|^2)\,d\lambda(z),$$

since the volume of $\{|\Phi_j|<\delta\}\cap Q$ is of order $(1-|a_j|^2)^3$ and $|\partial\Phi_j|\gtrsim 1/(1-|a_j|^2)$ there, hence adding

$$\mu(Q) = \sum_{a_j\in Q}(1-|a_j|^2)^2 \lesssim \int_Q \sum_{j=0}^{\infty}\chi_j|\partial\Phi_j|^2(1-|z|^2) \lesssim r^2,$$

by the proposition. □

Beginning of the proof of Theorem 1.4. First we solve the problem smoothly. Let $B=(B_1,B_2)$ be the $\mathbf{C}^2$ valued function given in the theorem, let $\lambda:=\{\lambda_j\}_{j\in\mathbf{N}}$ be a sequence in $l^\infty(\mathbf{N})$ and set

$$F(z) := \sum_{j=0}^{\infty}\lambda_j\chi\left(\frac{|\Phi_j|^2}{\delta^2}\right).$$

Since the sets $\{z\in\mathbf{B}\,|\,|\Phi_j|<\delta\}$ are disjoint we get $F(a_j)=\lambda_j$ for all $j\in\mathbf{N}$, hence F solves the problem in the $\mathcal{C}^\infty(\mathbf{B})$ class.

We shall correct it to make it holomorphic, so let us compute its $\bar{\partial}$:

$$\bar{\partial}F = \sum_{j=0}^{\infty}\lambda_j\chi'\frac{1}{\delta^2}\langle\Phi_j,\partial\Phi_j\rangle.$$

We have $B=M_j\cdot\Phi_j \Rightarrow \Phi_j=A_j\cdot B$ in $|\Phi_j|<\delta$ for all j by assumption, with the A_j uniformly bounded, hence

$$\bar\partial F=\frac{1}{\delta^2}\sum_{j=0}^{\infty}\lambda_j\chi'\langle B, A_j^*\cdot\partial\Phi_j\rangle.$$

The form $A_j^*\cdot\partial\Phi_j$ is a $\mathbf{C}^2$ valued $(1,0)$ form and its complex conjugate ${}^tA_j\cdot\overline{\partial\Phi_j}$ has 2 components denoted by $({}^tA_j\cdot\overline{\partial\Phi_j})_k$, $k=1,\ 2$. We thus get

$$\bar\partial F=B_1\omega_1+B_2\omega_2,\quad \omega_k:=\frac{1}{\delta^2}\sum_{j=0}^{\infty}\lambda_j\chi'({}^tA_j\cdot\overline{\partial\Phi_j})_k,\ k=1,\ 2.$$

We have to generalize the notion of Carleson measures of order α defined in [5] to forms.

Definition 3.4. A measure μ in $\mathbf{B}$ is a Carleson measure of order α, $\mu\in W^\alpha(\mathbf{B})$, if it belongs to the intermediate space $(W^0(\mathbf{B}),W^1(\mathbf{B}))_\alpha$, where $W^0(\mathbf{B})$ is the space of bounded measures and $W^1(\mathbf{B})$ the space of Carleson measures.

Definition 3.5. A $(0,1)$ form ω with continuous coefficients is in the class $W^\alpha_{(0,1)}(\mathbf{B})$ if the measure $(|\omega|+|\omega\wedge\bar\partial\varrho/\sqrt{-\varrho}\,|)\,d\lambda$ is a Carleson measure of order α in $\mathbf{B}$, with $\varrho(z):=|z|^2-1$, a defining function for the ball.

Lemma 3.6. *The* $(0,1)$ *forms* ω_k *belong to the class* $W^1_{(0,1)}(\mathbf{B})$.

In order to see this, we must prove that the coefficients of ω_k are Carleson measures

$$|\omega_k|\le\frac{\|\lambda\|_\infty}{\delta^2}\sum_{j=0}^{\infty}|\chi'|\,|{}^tA_j|\,|\partial\Phi_j|.$$

Now integrating over a Carleson set Q in $\mathbf{B}$, we get

$$\int_Q|\omega_k|\le C\sum_{a_j\in Q}(1-|a_j|^2)^3\frac{1}{1-|a_j|^2},$$

since the volume of $|\Phi_j|<\delta$ is equivalent to $(1-|a_j|^2)^3$, we have $|{}^tA_j|\le C$ and $|\partial\Phi_j|\lesssim(1-|a_j|^2)^{-1}$ by Lemma 3.2.

We also have to check that $\omega_k\wedge\bar\partial\varrho/\sqrt{-\varrho}$ is still Carleson, but again,

$$\left|\frac{\bar\partial\Phi_j\wedge\bar\partial\varrho}{\sqrt{-\varrho}}\right|\lesssim\frac{1}{1-|a_j|^2},$$

which proves the lemma. □

Unfortunately these forms are not closed and we have to modify this decomposition in order to get closed forms.

Let us take the $\bar\partial$ of these forms,

$$\bar\partial\omega_k=\sum_{j=0}^{\infty}\frac{\lambda_j}{\delta^2}\chi''\langle\Phi_j,\partial\Phi_j\rangle\wedge({}^tA_j\cdot\overline{\partial\Phi_j})_k,\quad k=1,\ 2,$$

and again, since the support of χ'' is in $|\Phi_j|<\delta$, we can write Φ_j in terms of B, $\Phi_j=A_j\cdot B$, hence again,

$$\bar\partial\omega_1=B_2\sum_{j=0}^{\infty}\frac{\lambda_j}{\delta^2}\chi''({}^tA_j\overline{\partial\Phi_j})_2\wedge({}^tA_j\cdot\overline{\partial\Phi_j})_1,$$

and

$$\bar\partial\omega_2=-B_1\sum_{j=0}^{\infty}\frac{\lambda_j}{\delta^2}\chi''({}^tA_j\overline{\partial\Phi_j})_2\wedge({}^tA_j\cdot\overline{\partial\Phi_j})_1.$$

Now to close the ω_j's in the Carleson class we have to solve

$$\bar\partial R=\omega_3:=\sum_{j=0}^{\infty}\frac{\lambda_j}{\delta^2}\chi''({}^tA_j\overline{\partial\Phi_j})_2\wedge({}^tA_j\cdot\overline{\partial\Phi_j})_1$$

with R a $(0,1)$ form in the Carleson class.

Lemma 3.7. *Let $a\in\mathbf{B}$ and let Φ be a biholomorphic map exchanging a and 0, let A be a bounded matrix, then*

$$|(A\cdot\overline{\partial\Phi})_2\wedge(A\cdot\overline{\partial\Phi})_1|\lesssim\|\det A\|_\infty\frac{(1-|a|^2)^{3/2}}{|1-a\cdot\bar z|^3}.$$

Proof. We can assume without loss of generality that $a=(a_1,0)$, then

$$\overline{\partial\Phi_1}=\frac{1-|a|^2}{(1-a_1\bar z_1)^2}\,d\bar z_1,$$
$$\overline{\partial\Phi_2}=\frac{\sqrt{1-|a|^2}}{1-a_1\bar z_1}a_1\,d\bar z_2+\frac{a_1\bar z_2}{(1-a_1\bar z_1)^2}a_1\,d\bar z_1.$$

Hence

$$(A\cdot\overline{\partial\Phi})_2\wedge(A\cdot\overline{\partial\Phi})_1=(A_{11}\overline{\partial\Phi_1}+A_{12}\overline{\partial\Phi_2})\wedge(A_{21}\overline{\partial\Phi_1}+A_{22}\overline{\partial\Phi_2})=\det A\cdot\overline{\partial\Phi_1}\wedge\overline{\partial\Phi_2},$$

since $\overline{\partial\Phi_j}\wedge\overline{\partial\Phi_j}=0$. □

Lemma 3.8. *The coefficient of the* $(0,2)$ *form* $\omega_3\sqrt{1-|\zeta|^2}$ *is a Carleson measure in* $\mathbf{B}$ *of order* $\alpha=1-1/p$.

Proof. Recall that

$$\omega_3:=\sum_{j=0}^{\infty}\frac{\lambda_j}{\delta^2}\chi''({}^tA_j\overline{\partial\Phi}_j)_2\wedge({}^tA_j\cdot\overline{\partial\Phi}_j)_1,$$

the matrices ${}^tA_j$ are uniformly bounded on the support of $\chi''(|\Phi_j|^2/\delta^2)$ and so are the determinant of these matrices, hence applying the previous lemma, we get

$$|\omega_3|\lesssim\sum_{j=0}^{\infty}|\lambda_j|\left|\chi''\left(\frac{|\Phi_j|^2}{\delta^2}\right)\right|\frac{(1-|a_j|^2)^{3/2}}{|1-a_j\cdot\bar{\zeta}|^3}.$$

Let

$$\gamma:=\sum_{j=0}^{\infty}|\lambda_j|\left|\chi''\left(\frac{|\Phi_j|^2}{\delta^2}\right)\right|\frac{(1-|a_j|^2)^{3/2}}{|1-a_j\cdot\bar{\zeta}|^3},$$

multiplying it by $\sqrt{1-|\zeta|^2}$ and integrating over a pseudoball $Q(a,h)$ leads to

$$\int_Q\gamma\sqrt{1-|\zeta|^2}\lesssim\sum_{a\in Q\cap S}(1-|a|^2)^2\lesssim h^2,$$

as we already know that the sequence S is Carleson.

Now clearly $\lambda\in L^p(\gamma)$, hence using [5] we get that ω_3 is Carleson α. □

In order to finish the proof of Theorem 1.4 we shall use the following theorem which is proved in the last section.

Theorem 3.9. *Let* ω_3 *be a* $(0,2)$ *form defined in* $\mathbf{B}$ *in* $\mathbf{C}^2$ *and such that the coefficient of* $\omega_3\sqrt{1-|\zeta|^2}$ *is Carleson of order* α, *then there is a* $(0,1)$ *form* ω *in* $\mathbf{B}$ *with* $\bar{\partial}\omega=\omega_3$ *and* $\omega\in W^{\alpha}_{(0,1)}(\mathbf{B})$.

We solve the equation $\bar{\partial}R=\omega_3$ using the previous theorem and we correct the ω_j's the usual way,

$$\mu_1:=\omega_1-B_2R,\qquad \mu_2:=\omega_2+B_1R.$$

The μ_j's are still in the same Carleson class, and now they are $\bar{\partial}$ closed and we still have

$$\bar{\partial}F=B_1\mu_1+B_2\mu_2,$$

hence we can solve the equations

$$\bar{\partial}S_j=\mu_j,\quad j=1,\ 2,$$

with the S_j's in $\mathrm{BMO}(\partial\mathbf{B})$ if $\alpha=1$ and in $L^p(\partial\mathbf{B})$ with $p=1/(1-\alpha)$, if $0\leq\alpha<1$, [5], hence the function $H:=F-B_1S_1-B_2S_2$ is in $\bigcap_{p>1}H^p(\mathbf{B})$ if $\alpha=1$ and in $H^p(\mathbf{B})$ with $p=1/(1-\alpha)$, if $0\leq\alpha<1$, and solve the interpolation problem. □

4. Kernels

We want kernels solving the $\bar\partial$ equation for $(0,2)$ forms in the unit ball $\mathbf{B}$ of $\mathbf{C}^2$. We shall use Skoda's kernels but lifted by one dimension to get interior values instead of boundary ones as we already did in [4] for $(0,1)$ forms.

Theorem 4.1. *Let γ be a $(0,2)$ form in $\mathbf{B}$, there are kernels solving $\bar\partial\omega=\gamma$ in the ball with*

$$\omega_j = A_j\cdot\gamma + B_j\cdot\gamma, \qquad j=1,\ 2,$$
$$\omega = \omega_1\, d\bar z_1 + \omega_2\, d\bar z_2,$$

and

$$|A_j| \lesssim \frac{\mu}{r^6}\Gamma_{2,5}(\alpha), \qquad j=1,\ 2,$$
$$|B_j| \lesssim \frac{s}{\mu^{1/2} r^4}\Gamma_{1,9/2}(\alpha), \quad j=1,\ 2,$$

with the notation

$$r := \sqrt{1-|z|^2}, \quad s := \sqrt{1-|\zeta|^2}, \quad \mu := |1-\zeta\cdot\bar z|, \quad \alpha := \frac{rs}{\mu^2};$$

for any $0<\delta<1$, let $\chi(t)=1$ if $t<\delta$, $\chi(t)=0$ if $t\geq\delta$, then

$$\Gamma_{p,q}(\alpha) \lesssim \chi(\alpha)\alpha^{p+1} + (1-\chi(\alpha))\frac{1}{q-p-2}\left(\frac{1}{1-\alpha}\right)^{q-p-2}.$$

Proof. Let us take Skoda's kernels in $\mathbf{C}^3$ for $(0,2)$ forms [12]

$$D(z,\zeta) := [-\varrho + \langle P, \zeta - z\rangle]^3 \langle Q, \zeta - z\rangle^2,$$

and for the unit ball, we have

$$\varrho(\zeta) := |\zeta|^2 - 1, \quad P_j := \frac{\partial\varrho}{\partial\zeta_j} = \bar\zeta_j, \quad Q_j := \bar z_j,$$

hence

$$D(z,\zeta) := (1-\bar\zeta\cdot z)^3 (1-\zeta\cdot\bar z)^2,$$

and

$$N_j := (-1)^{j-1}(1-|\zeta|^2)^2 \bar z_j \bigwedge_{k\neq j} (d\bar z_k + d\bar\zeta_k)\wedge\beta_0, \quad j=1,\ 2,\ 3,$$
$$M_l := (-1)^{j+k}(1-|\zeta|^2)(\bar z_j\bar\zeta_k - \bar z_k\bar\zeta_j)\bar\partial(|\zeta|^2)\wedge d\bar z_l\wedge\beta_0, \quad j<k,\ j\neq l,\ k\neq l,$$

with

$$\beta_0 := \bigwedge_{k=1}^{3} d\zeta_k,$$

the kernels are

$$A_j = \frac{N_j}{D},\ j=1,\ 2,\ 3 \quad \text{and} \quad B_l = \frac{M_l}{D},\ l=1,\ 2,\ 3,$$

and if ω is a $(0,2)$ form in $\mathbf{C}^3$, the solution of $\bar{\partial}_b u = \omega$ is

$$u(z) = \sum_{j=1}^{3} \int_{\mathbf{B}} A_j(z,\zeta) \wedge \omega(\zeta) + \sum_{l=1}^{3} \int_{\mathbf{B}} B_l(z,\zeta) \wedge \omega(\zeta).$$

Now if ω depends only on the 2 first variables, we have

$$\omega(\zeta_1,\zeta_2) = \mu(\zeta_1,\zeta_2)\, d\bar{\zeta}_1 \wedge d\bar{\zeta}_2,$$

hence

$$\begin{aligned}
&N_1 \wedge \omega = (1-|\zeta|^2)^2 \bar{z}_1 \mu(\zeta)\, d\bar{z}_2 \wedge \beta(\zeta) \wedge \beta(\bar{\zeta}),\\
&N_2 \wedge \omega = -(1-|\zeta|^2)^2 \bar{z}_2 \mu(\zeta)\, d\bar{z}_1 \wedge \beta(\zeta) \wedge \beta(\bar{\zeta}),\\
&N_3 \wedge \omega = 0,\\
&M_1 \wedge \omega = -(1-|\zeta|^2)(\bar{z}_2\bar{\zeta}_3 - \bar{z}_3\bar{\zeta}_2)\mu(\zeta)\zeta_3\, d\bar{z}_1 \wedge \beta(\zeta) \wedge \beta(\bar{\zeta}),\\
&M_2 \wedge \omega = (1-|\zeta|^2)(\bar{z}_1\bar{\zeta}_3 - \bar{z}_3\bar{\zeta}_1)\mu(\zeta)\zeta_3\, d\bar{z}_2 \wedge \beta(\zeta) \wedge \beta(\bar{\zeta}),\\
&M_3 \wedge \omega = -(1-|\zeta|^2)(\bar{z}_1\bar{\zeta}_2 - \bar{z}_2\bar{\zeta}_1)\mu(\zeta)\zeta_3\, d\bar{z}_3 \wedge \beta(\zeta) \wedge \beta(\bar{\zeta}).
\end{aligned}$$

The solution $u(z)$ verifies $\bar{\partial}_b u = \omega$, hence if $U = U_1\, d\bar{z}_1 + U_2\, d\bar{z}_2 + U_3\, d\bar{z}_3$ is an extension of u in $\mathbf{B}_3$, then

$$\frac{\partial U_1}{\partial \bar{z}_2} - \frac{\partial U_2}{\partial \bar{z}_1} = \mu(z_1, z_2),$$

therefore we can take $U_3 \equiv 0$. Moreover for any fixed w, $U_w := U(z_1, z_2, w)$ still verifies $\bar{\partial}_z U_w = \omega$, we can take the mean value of U_w on the circle C of center $(z_1, z_2, 0)$ and of radius $r = \sqrt{1-|z_1|^2-|z_2|^2}$, this circle C is on $\partial\mathbf{B}_3$, hence $U_w = u(z_1, z_2, w)$ is well defined there and we get

$$v_j(z_1, z_2) := \frac{1}{2\pi} \int_0^{2\pi} u_j(z_1, z_2, re^{i\theta})\, d\theta, \quad j = 1,\ 2,$$

and with $v(z_1, z_2) := v_1\, d\bar{z}_1 + v_2\, d\bar{z}_2$, we have $\bar{\partial} v = \omega$.

This way we have an interior solution in $\mathbf{B}_2$. It remains to estimate the associated kernels.

5. Computations

We shall use the following simple lemma.

Lemma 5.1. *If $p>1$, then*

$$\int_0^{2\pi} \frac{d\theta}{|1-e^{-i\theta}z|^p} \lesssim \frac{1}{1-|z|^{p-1}} \quad \textit{for all } z\in\mathbf{D}.$$

Let

$$M_{p,q} := \int_{\substack{|z_3|^2=1-|z'|^2\\ |\zeta_3|^2<1-|\zeta'|^2}} \frac{(1-|\zeta|^2)^p}{|1-\zeta\cdot\bar z|^q}\, d|z_3|\, d\lambda(\zeta_3).$$

In order to estimate $M_{p,q}$ we shall first integrate with respect to z_3. Let

$$J := \int_0^{2\pi} \frac{d\theta}{|1-\zeta'\cdot\bar z'-\zeta_3 r e^{-i\theta}|^q} = \frac{1}{\mu^q}\int_0^{2\pi}\frac{d\theta}{|1-r\gamma e^{-i\theta}|^q}$$

with the notation

$$\zeta' := (\zeta_1,\zeta_2), \quad z' := (z_1,z_2), \quad r^2 := 1-|z'|^2, \quad \mu := |1-\zeta'\cdot\bar z'|, \quad \gamma := \frac{\zeta_3}{\mu}.$$

Hence using Lemma 5.1, we get

$$J \lesssim \frac{1}{\mu^q\left|1-r^2|\gamma|^2\right|^{q-1}} = \frac{1}{\mu^q\left|1-r^2|\zeta_3|^2/\mu^2\right|^{q-1}}.$$

Now we have to integrate with respect to ζ_3,

$$M_{p,q} \lesssim \int_{|\zeta_3|^2<1-|\zeta'|^2} (1-|\zeta|^2)^p J\, d\lambda(\zeta_3),$$

hence, with

$$L_{p,q} := \int_{|\zeta_3|^2<1-|\zeta'|^2} \frac{(1-|\zeta'|^2-|\zeta_3|^2)^p}{(1-r^2|\zeta_3|^2/\mu^2)^{q-1}}\, d\lambda(\zeta_3),$$

we have $M_{p,q} \lesssim L_{p,q}/\mu^q$. Let

$$\alpha := \frac{(1-|\zeta'|^2)(1-|z'|^2)}{\mu^2} \quad \text{and} \quad u := \frac{1-|z'|^2}{\mu^2}t^2,$$

then, passing to polar coordinates, we get

$$L_{p,q} \lesssim \frac{\mu^2}{r^{2(p+1)}}\int_0^\alpha \frac{(r^2s^2-\mu^2u)^p}{(1-u)^{q-1}}\, du.$$

Let $v=1-u$, then

$$L_{p,q}\lesssim\frac{\mu^{2+2p}}{r^{2+2p}}\int_{1-\alpha}^{1}\frac{(\alpha-1+v)^p}{v^{q-1}}\,dv.$$

Let us compute the integral

$$\Gamma_{p,q}:=\frac{1}{(1-\alpha)^{q-p-2}}\int_{1}^{1/(1-\alpha)}\frac{(u-1)^p}{u^{q-1}}\,du,$$

after the change $v=(1-\alpha)u$ we have two cases.

If $\alpha<\frac{1}{2}$, then we majorize $u-1$ by $1/(1-\alpha)-1$ and $1/u$ by 1, to get

$$\Gamma_{p,q}(\alpha)\le C(p,q)\alpha^{p+1},$$

and if $\alpha\ge\frac{1}{2}$, then we majorize $u-1$ by u, to get

$$\Gamma_{p,q}(\alpha)\le\frac{1}{q-p-2}\left(\frac{1}{1-\alpha}\right)^{q-p-2};$$

provided that $q-p-2>0$, which will be the case for us. This can be summarized by

$$\Gamma_{p,q}(\alpha)\le\chi(\alpha)C(p,q)\alpha^{p+1}+(1-\chi(\alpha))\frac{1}{q-p-2}\left(\frac{1}{1-\alpha}\right)^{q-p-2}$$

with χ the characteristic function of $\left[0,\frac{1}{2}\right[$.

Now back to $L_{p,q}$ and $M_{p,q}$,

$$L_{p,q}\lesssim\frac{\mu^{2+2p}}{r^{2+2p}}\Gamma_{p,q}(\alpha),$$
$$M_{p,q}\lesssim\frac{\mu^{2+2p-q}}{r^{2+2p}}\Gamma_{p,q}(\alpha).$$

We can apply this to our kernels, with $s=\sqrt{(1-|\zeta|^2)}$,

$$|A_j(z,\zeta)|\lesssim M_{2,5}\lesssim\frac{\mu}{r^6}\Gamma_{2,5}(\alpha),\qquad j=1,\ 2,$$
$$|B_j(z,\zeta)|\lesssim sM_{1,9/2}\frac{s}{\mu^{1/2}r^4}\Gamma_{1,9/2}(\alpha),\quad j=1,\ 2.$$

This finishes the proof of Theorem 4.1.

Corollary 5.2. *Let Φ be a biholomorphic map exchanging ζ and 0, then we have*

(1) *if* $|\Phi(z)|<\delta$ *then*

$$|A_j|\lesssim\frac{|1-\zeta\cdot\bar z|}{(1-|z|^2)^3}\frac{1}{|\Phi(z)|^2},$$
$$|B_j|\lesssim\frac{\sqrt{1-|\zeta|^2}}{\sqrt{|1-\zeta\cdot\bar z|}\,(1-|z|^2)^2}\frac{1}{|\Phi(z)|^3};$$

(2) *if* $|\Phi(z)|\geq\delta$ *then*

$$|A_j|\lesssim\frac{(1-|\zeta|^2)^3}{|1-\zeta\cdot\bar z|^5},$$
$$|B_j|\lesssim\frac{(1-|\zeta|^2)^{5/2}}{|1-\zeta\cdot\bar z|^{9/2}}.$$

Proof. We just remark that $1-\alpha=|\Phi|^2$. □

6. Application

We are now in position to prove Theorem 3.9 and in order to prove this theorem, we shall use the following lemma.

Lemma 6.1. *Let $I_p:=\int_{|\Phi|<\delta}(1/|\Phi|^p)\,dm(z)$, where Φ is a biholomorphic map exchanging ζ and 0, then we have $p<4\Rightarrow I_p\lesssim(1-|\zeta|^2)^3$.*

Proof. We make the change of variables $w=\Phi(z)$. With $\zeta=(\zeta_1,0)$ we have already computed $\partial\Phi$ for Lemma 3.2 and we have

$$|\det\partial\Phi(z)|^2=\frac{(1-|\zeta_1|^2)^3}{|1-\bar\zeta_1 z_1|^6}\simeq\frac{1}{(1-|\zeta_1|^2)^3}\quad\text{on } |\Phi|<\delta.$$

Hence for any $\zeta\in\mathbf{B}$ by rotation we get

$$|\det\partial\Phi|^2\simeq\frac{1}{(1-|\zeta|^2)^3}\quad\text{on } |\Phi|<\delta$$

and the Jacobian in w is its inverse, $\operatorname{Jac}(w)\simeq(1-|\zeta|^2)^3$ and we get

$$I_p:=\int_{|\Phi|<\delta}\frac{1}{|\Phi|^p}\,dm(z)=\int_{|w|<\delta}\frac{1}{|w|^p}\operatorname{Jac}(w)\,dm(w)\lesssim\frac{\delta^{4-p}}{4-p}(1-|\zeta|^2)^3,$$

if we integrate using polar coordinates. □

6.1. Proof of Theorem 3.9. The case of Carleson measures ($\alpha=1$)

We set

$$R:=(A_1(\gamma)+B_1(\gamma))\,d\bar z_1+(A_2(\gamma)+B_2(\gamma))\,d\bar z_2.$$

We have to show that the coefficients of R and the coefficient of $R\wedge\bar\partial|z|^2/\sqrt{1-|z|^2}$ are Carleson measures provided that this is the case for $\sqrt{1-|z|^2}\,\omega$.

Hence we shall be done if we do so for the kernels divided by $\sqrt{1-|z|^2}$, which are the worst cases.

Let us call any of these divided kernels $K(z,\zeta)$ and compute the integral over a pseudoball $Q:=Q(a,h):=\{\eta\in\mathbf{B}\,|\,|1-\bar a\eta|<h\}$,

$$I:=\int_Q\int_{\mathbf{B}}|K(z,\zeta)|\,|\omega(\zeta)|\,dm(\zeta)\,dm(z).$$

By Fubini we can exchange the order of integration,

$$I:=\int_{\mathbf{B}}\int_Q|K(z,\zeta)|\,|\omega(\zeta)|\,dm(z)\,dm(\zeta).$$

Define $Q_n:=Q(a,2^nh)$, then

$$I=\sum_{n=0}^{\infty}I_n,$$

with

$$I_1:=\int_{Q_1}\int_Q|K(z,\zeta)|\,dm(z)|\omega(\zeta)|\,dm(\zeta),$$

and

$$I_n:=\int_{Q_{n+1}\setminus Q_n}\int_Q|K(z,\zeta)|\,dm(z)|\omega(\zeta)|\,dm(\zeta),\quad n\geq 2.$$

Let us look at I_1. Let $\tilde\zeta:=\zeta/|\zeta|$. Since $\zeta\in Q_1$, we have $Q\subset\widetilde Q:=Q(\tilde\zeta,\gamma h)$ with a γ independent of a and of h, hence we have that

$$I_1\leq\int_{Q_1}\int_{\widetilde Q}|K(z,\zeta)|\,|\omega(\zeta)|\,dm(z)\,dm(\zeta).$$

The inner integral becomes

$$J:=\int_{\widetilde Q}|K(z,\zeta)|\,dm(z),$$

and

$$J_1 := \int_{|\Phi|<\delta} |K(z,\zeta)|\, dm(z),$$
$$J_2 := \int_{\widetilde{Q}\setminus\{|\Phi|<\delta\}} |K(z,\zeta)|\, dm(z)$$

are such that $J \lesssim J_1 + J_2$.

On $\{|\Phi|>\delta\}$ the kernels satisfy, because of Corollary 5.2,

$$|A_j| \lesssim \frac{(1-|\zeta|^2)^3}{|1-\zeta\cdot\bar{z}|^5},$$
$$|B_j| \lesssim \frac{(1-|\zeta|^2)^{5/2}}{|1-\zeta\cdot\bar{z}|^{9/2}},$$

hence in any case

$$|K| \lesssim \frac{(1-|\zeta|^2)^2}{(1-|z|^2)^{1/2}|1-\zeta\cdot\bar{z}|^4},$$

because we have to divide by $\sqrt{1-|z|^2}$.

Let us first look at J_2,

$$J_2 \lesssim \int_{\widetilde{Q}} \frac{(1-|\zeta|^2)^2}{(1-|z|^2)^{1/2}|1-\zeta\cdot\bar{z}|^4}\, dm(z),$$

and by invariance under rotations we may suppose that $\zeta_2=0$ and $\zeta_1=r>0$; this implies that $\tilde{\zeta}=(1,0)$. After integrating with respect to z_2, we obtain

$$J_2 \lesssim (1-|\zeta|^2)^2 \int_{|1-z_1|<\gamma h} \frac{\sqrt{1-|z_1|^2}}{|1-rz_1|^4}\, d\lambda(z_1)$$
$$\lesssim (1-|\zeta|^2)^2 \int_{|1-z_1|<\gamma h} \frac{d\lambda(z_1)}{|1-rz_1|^{7/2}} = (1-|\zeta|^2)^2 L.$$

We make the change of variables $w=1/r-z_1$ in L, and obtain

$$L = \frac{1}{r^{7/2}} \int_C \frac{d\lambda(w)}{|w|^{7/2}},$$

where

$$C := \left\{ w\in\mathbf{C} \,\middle|\, \left|\frac{1}{r}-w\right|<1 \right\} \cap \left\{ w\in\mathbf{C} \,\middle|\, \left|1-\frac{1}{r}+w\right|<\gamma h \right\}.$$

We majorize if we integrate on the corona

$$C' := \left\{ w \in \mathbf{C} \,\middle|\, \frac{1}{r} - 1 < |w| < \frac{1}{r} - 1 + \gamma h \right\} \supset C.$$

Hence

$$L \leq \frac{1}{r^{7/2}} \int_{C'} \frac{d\lambda(w)}{|w|^{7/2}} \lesssim \left(\frac{1}{r} - 1\right)^{-3/2} - \left(\frac{1}{r} - 1 + \gamma h\right)^{-3/2} \leq (1-r)^{-3/2};$$

putting in J_2 we get $J_2 \lesssim \sqrt{1-|\zeta|^2}$, recalling that $r = |\zeta|$.

Putting this in the integral of I_1, we get

$$I_1' \leq \int_{Q_1} \sqrt{1-|\zeta|^2}\, |\omega(\zeta)|\, dm(\zeta) \lesssim h^2,$$

since $\sqrt{1-|\zeta|^2}\, |\omega(\zeta)|$ is Carleson.

Now look at J_1. The kernels are majorized on $|\Phi| < \delta$ by

$$|A_j| \lesssim \frac{|1-\zeta\cdot\bar{z}|}{(1-|z|^2)^3} \frac{1}{|\Phi(z)|^2},$$
$$|B_j| \lesssim \frac{\sqrt{1-|\zeta|^2}}{\sqrt{|1-\zeta\cdot\bar{z}|}\,(1-|z|^2)^2} \frac{1}{|\Phi(z)|^3},$$

hence we have, with $p=2$ for the kernels A_j and $p=3$ for the B_j,

$$|K(z,\zeta)| \lesssim \frac{1}{(1-|\zeta|^2)^{5/2}} \frac{1}{|\Phi(z)|^p},$$

since on $|\Phi| < \delta$, $|1-\zeta\cdot\bar{z}| \simeq 1-|\zeta|^2$ and $1-|z|^2 \simeq 1-|\zeta|^2$, and we still have to divide by $\sqrt{1-|z|^2}$. We get

$$J_1 := \int_{|\Phi|<\delta} |K(z,\zeta)|\, dm(z).$$

Using Lemma 6.1 with p, we get

$$J_1 \lesssim \frac{1}{(1-|\zeta|^2)^{5/2}} (1-|\zeta|^2)^3 = \sqrt{1-|\zeta|^2}.$$

Putting this in the integral of I_1, we get

$$I_1'' \lesssim \int_{Q_1} J_1 |\omega(\zeta)|\, dm(\zeta),$$

but $\sqrt{1-|\zeta|^2}\,|\omega(\zeta)|\,dm(\zeta)$ is Carleson, so

$$I_1''\lesssim h^2$$

and $I_1=I_1'+I_1''\lesssim h^2$.

Now let us look at I_n. If $\zeta\notin Q_n$ and $z\in Q$ then $|\Phi(z)|\geq\delta$ and $|1-\bar\zeta z|\geq 2^n h$; the kernels are

$$|K|\lesssim\frac{(1-|\zeta|^2)^2}{(1-|z|^2)^{1/2}|1-\zeta\cdot\bar z|^4},$$

and the volume of Q is of the order h^3, hence

$$\begin{aligned}J_n &:= (1-|\zeta|^2)^2\int_{Q(a,h)}\frac{1}{(1-|z|^2)^{1/2}|1-\bar\zeta z|^4}\,dm(z)\\ &\leq (1-|\zeta|^2)^2\frac{1}{2^{4n}h^4}\int_{Q(a,h)}\frac{1}{(1-|z|^2)^{1/2}}\,dm(z)\\ &\leq \frac{(1-|\zeta|^2)^2}{2^{4n}h^4}\int_{1-h}^{1}\frac{dt}{\sqrt{1-t}}h^2\leq 2\frac{(1-|\zeta|^2)^2}{2^{4n}h^{3/2}}.\end{aligned}$$

Putting this in I_n, we get

$$I_n\leq\frac{2}{2^{4n}h^{3/2}}\int_{Q_{n+1}}(1-|\zeta|^2)^{3/2}\sqrt{1-|\zeta|^2}\,|\omega(\zeta)|\,dm(\zeta)\lesssim 2^{1-1/2n}h^2,$$

as $\zeta\in Q_{n+1}\Rightarrow 1-|\zeta|^2\leq 2^{n+1}h$ and $\sqrt{1-|\zeta|^2}\,|\omega(\zeta)|$ is Carleson.

Now we see that the sum is convergent and we obtain the first case.

The case of bounded measures ($\alpha=0$). We have to show that the coefficients of R and the coefficient of $R\wedge\bar\partial|z|^2/\sqrt{1-|z|^2}$ are bounded measures provided that it is the case for $\sqrt{1-|z|^2}\,\omega$, and the treatment, exactly as above, will be left to the reader.

The case of $0<\alpha<1$. This is obtained by interpolation between the two previous cases, since we know that Carleson measures of order α are obtained by (Banach space) interpolation between bounded measures and Carleson measures [5]. □

6.2. Proof of Theorem 1.5

Now let $\lambda=\{\lambda_j\in\mathbf{C}\}_{j\in\mathbf{N}}$ be such that $\sum_{j=0}^{\infty}|\lambda_j|^p(1-|a_j|^2)^2=:\|\lambda\|_p^p<+\infty$. Since

$$\mu:=\sqrt{1-|z|^2}\sum_{j=0}^{\infty}(1-|a_j|^2)^{3/2}\delta_{a_j}=\sum_{j=0}^{\infty}(1-|a_j|^2)^2\delta_{a_j}$$

is Carleson, $\lambda=\{\lambda_j\}_{j\in\mathbf{N}}$ is in $L^p(\mu)$, hence $|\omega_3|\leq\lambda\cdot\mu$ is in the class $W^{\alpha}_{(0,2)}(\mathbf{B})$, [5].

To conclude, we have that the $(0,1)$ forms μ_j are still in the same Carleson class, hence we can solve them in $L^p(\partial\mathbf{B})$, again using results of [5]. □

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