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On the relative homology of q-Runge pairs

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0. Introduction

Let X be a Stein space of dimension n and $Y \subset X$ a Runge open subset, i.e. Y is a Stein open subset and the restriction map $\Gamma(X, \mathcal{O}) \to \Gamma(Y, \mathcal{O})$ has a dense image. It was shown in [4] that in this case the pair (X, Y) satisfies the following topological conditions: the relative homology group $H_i(X, Y; \mathbf{Z})$ vanishes if $i > n$ and $H_n(X, Y; \mathbf{Z})$ is torsion free (when X has isolated singularities this result was already proved in [2]). Moreover, if $Y = \emptyset$, then $H_n(X; \mathbf{Z})$ is free.

These results have been generalized in [7], [15] to q-complete spaces. Namely one has that if X is a q-complete space and $Y \subset X$ is a q-Runge open subset then the relative homology group $H_i(X, Y; \mathbf{Z})$ vanishes if $i > n+q-1$ and $H_{n+q-1}(X, Y; \mathbf{Z})$ is torsion free. If $Y = \emptyset$ then $H_{n+q-1}(X; \mathbf{Z})$ is free (note that, with our definitions, 1-complete corresponds to Stein spaces).

In fact, for $Y = \emptyset$ Hamm [11] proved a stronger result: if X is a q-complete space then X has the homotopy type of a CW complex of dimension $\leq n+q-1$.

In [3] I raised the following question.

Problem 1. Let X be a Stein space of dimension n and $Y \subset X$ a Runge open subset. Does it follow that the relative homology group $H_n(X, Y; \mathbf{Z})$ is free?

A counterexample to this problem seems to be unknown. A more general question was raised in [15].

Problem 2. Let X be a q-complete space of dimension n and $Y \subset X$ a q-Runge open subset. Does it follow that the relative homology group $H_{n+q-1}(X, Y; \mathbf{Z})$ is free?

The aim of this paper is to provide a counterexample to Problem 2. More precisely we prove the following theorem.

Theorem 1. *For all integers q and n with $2 \leq q \leq n$, and for every countable abelian torsion free group G, there exist a q-complete manifold X of dimension n*

and a q-Runge open subset $Y \subset X$ *such that* $H_{n+q-1}(X,Y;\mathbf{Z})=G$.

As an application of this result we study, in Section 3, n-Runge pairs (X,Y) with $Y\subset X$ of dimension n. It is known [5] that in this case for a given complex space of pure dimension n with no compact irreducible components, the condition on $Y\subset X$ to be n-Runge has a purely topological characterization, namely it is equivalent to each of the following assumptions:

(a) $X\backslash Y$ has no compact irreducible components;

(b) the map $H_{2n-1}(Y;\mathbf{Z})\to H_{2n-1}(X;\mathbf{Z})$ is injective;

(c) the map $H_{2n-1}(Y;\mathbf{C})\to H_{2n-1}(X;\mathbf{C})$ is injective;

(d) the map $H^{2n-1}(X;\mathbf{C})\to H^{2n-1}(Y;\mathbf{C})$ is surjective.

So it is natural to ask if it is also equivalent to

(e) the map $H^{2n-1}(X;\mathbf{Z})\to H^{2n-1}(Y;\mathbf{Z})$ is surjective.

It is easy to see that (e) implies each of the conditions (a), (b), (c) and (d). On the other hand it is proved in Corollary 1 that (e) is not equivalent to the above conditions. More precisely one has the following result.

Corollary 1. *For every integer* $n\geq 2$ *there exists an n-Runge pair* (X,Y) *of n-dimensional manifolds such that the restriction map* $H^{2n-1}(X;\mathbf{Z})\to H^{2n-1}(Y;\mathbf{Z})$ *is not surjective.*

1. Preliminaries

Let $D\subset\mathbf{C}^n$ be an open subset and $\varphi\in C^\infty(D,\mathbf{R})$. The function φ is called q-convex if its Levi form $L(\varphi)$ has at least $n-q+1$ positive (>0) eigenvalues at any point of D. Using local embeddings this definition can be easily extended to complex spaces [1].

A complex space X is called q-convex if there exists a C^∞ function $\varphi: X\to\mathbf{R}$ which is q-convex outside a compact subset $K\subset X$ and such that φ is an exhaustion on X, i.e. $\{\varphi<c\}\Subset X$ for every $c\in\mathbf{R}$. If K may be taken as the empty set then X is called q-complete. When $q=1$ this means, by Grauert's solution to the Levi problem ([8], [12]), that X is a Stein space.

Every q-complete space satisfies the following cohomological condition [1]: for every $F\in\mathrm{Coh}(X)$ one has $H^i(X,F)=0$ if $i\geq q$; q-complete spaces satisfy also the topological condition $H_i(X;\mathbf{Z})=0$ if $i>n+q-1$ and $H_{n+q-1}(X;\mathbf{Z})$ is free ([7], [15]). In fact one has the stronger result [11] that every q-complete space X of dimension n has the homotopy type of a CW complex of dimension $\leq n+q-1$.

T. Ohsawa [13] has proved the following theorem: a complex space X of dimension n is n-complete if and only if X has no compact irreducible components of dimension n (when X is smooth this was already shown in [10]).

If X is a complex space, an open subset $Y \subset X$ is said to be q-Runge if for every compact subset $K \subset Y$ there exists a q-convex exhaustion function $\varphi : X \to \mathbf{R}$ (which may depend on K) such that $K \subset \{\varphi < 0\} \Subset Y$.

It follows from this definition that X is assumed to be q-complete. When $q=1$ we get the classical definition of Runge domains, i.e. Y is Runge in X if and only if Y is Stein and the restriction map $\Gamma(X, \mathcal{O}) \to \Gamma(Y, \mathcal{O})$ has a dense image.

If $Y \subset X$ is q-Runge then the restriction map $H^{q-1}(X, F) \to H^{q-1}(Y, F)$ has a dense image for every $F \in \mathrm{Coh}(X)$ ([1]). Pairs that are q-Runge pairs also satisfy the topological condition $H_i(X, Y; \mathbf{Z}) = 0$ if $i > n+q-1$ and $H_{n+q-1}(X, Y; \mathbf{Z})$ is torsion free ([7], [15]).

For n-Runge pairs of complex spaces of dimension n one has the following characterization [5].

Let X be a complex space of pure dimension n with no compact irreducible components and $Y \subset X$ an open subset. Then the following conditions are equivalent:

(1) Y is n-Runge in X;

(2) the restriction map $H^{n-1}(X, F) \to H^{n-1}(Y, F)$ has a dense image for every $F \in \mathrm{Coh}(X)$;

(3) the restriction map $H^{n-1}(X, \Omega^n) \to H^{n-1}(Y, \Omega^n)$ has a dense image, where Ω^n denotes the canonical sheaf of X.

They are also equivalent to each of the topological conditions (a), (b), (c) and (d) stated in the introduction.

If X is a topological space we denote by $B_i(X)$ its Betti groups, i.e. $B_i(X)$ is the quotient of $H_i(X; \mathbf{Z})$ (the $\mathbf{Z}$-homology of X) by its torsion subgroup.

Let us recall the following result due to Pontrjagin [14].

Theorem 2. *Let G be a countable abelian torsion free group. Then there exists a compact connected subset $K \subset \mathbf{R}^3$ such that $B_1(\mathbf{R}^3 \setminus K) = G$.*

Remark 1. In fact it is shown in [14] that K may be taken to be a curve, but we shall not need this fact.

2. Proof of the main results

In order to prove Theorem 1 we need some lemmas.

Lemma 1. *Let X be a real orientable manifold of dimension m and $A \subset X$ a closed connected subset. Then the relative homology group $H_{m-1}(X, X \setminus A; \mathbf{Z})$ is torsion free.*

For a proof of this lemma see [6, p. 260, Corollary 3.5].

From this we deduce the following corollary.

Corollary 2. *Let $F\subset \mathbf{R}^3$ be a closed connected subset. Then $H_1(\mathbf{R}^3\setminus F;\mathbf{Z})$ is torsion free, and therefore $H_1(\mathbf{R}^3\setminus F;\mathbf{Z})=B_1(\mathbf{R}^3\setminus F)$.*

Proof. From the exact sequence for homology corresponding to the inclusion $\mathbf{R}^3\setminus F\subset \mathbf{R}^3$ we get

$$0=H_2(\mathbf{R}^3;\mathbf{Z})\longrightarrow H_2(\mathbf{R}^3,\mathbf{R}^3\setminus F;\mathbf{Z})\longrightarrow H_1(\mathbf{R}^3\setminus F;\mathbf{Z})\longrightarrow H_1(\mathbf{R}^3;\mathbf{Z})=0.$$

Therefore $H_1(\mathbf{R}^3\setminus F;\mathbf{Z})$ is isomorphic to $H_2(\mathbf{R}^3,\mathbf{R}^3\setminus F;\mathbf{Z})$ which is torsion free by Lemma 1.

Lemma 2. *Let $K\subset \mathbf{R}^3$ be a non-empty compact connected subset and $P\in K$ any point. Then there is a natural isomorphism*

$$H_2(\mathbf{R}^3\setminus\{P\},\mathbf{R}^3\setminus K;\mathbf{Z})\xrightarrow{\alpha} H_2(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z}).$$

Proof. We consider the triad $\mathbf{R}^3\setminus K\subset \mathbf{R}^3\setminus\{P\}\subset \mathbf{R}^3$ and the associated exact sequence for homology (see [9, p. 59])

$$\begin{aligned}\ldots\longrightarrow H_3(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z})&\xrightarrow{\beta} H_3(\mathbf{R}^3,\mathbf{R}^3\setminus\{P\};\mathbf{Z})\longrightarrow H_2(\mathbf{R}^3\setminus\{P\},\mathbf{R}^3\setminus K;\mathbf{Z})\\ &\xrightarrow{\alpha} H_2(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z})\longrightarrow H_2(\mathbf{R}^3,\mathbf{R}^3\setminus\{P\};\mathbf{Z})=0.\end{aligned}$$

By Alexander's duality in $\mathbf{R}^3$

$$H_3(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z})=\check{H}^0(K;\mathbf{Z})=\mathbf{Z},$$
$$H_3(\mathbf{R}^3,\mathbf{R}^3\setminus\{P\};\mathbf{Z})=\check{H}^0(\{P\};\mathbf{Z})=\mathbf{Z},$$

where $\check{H}^i$ denotes Čech cohomology. It follows that β is an isomorphism, therefore α is also an isomorphism, as desired.

Lemma 3. *For every integer $n\geq 2$ and for every countable abelian torsion free group G there exists an n-Runge pair (X,Y) with $Y\subset X$ being complex manifolds of dimension n such that $H_{2n-1}(X,Y;\mathbf{Z})=G$.*

Proof. By Theorem 2 and Corollary 2 there exists a compact connected subset $K\subset \mathbf{R}^3$ such that $H_1(\mathbf{R}^3\setminus K;\mathbf{Z})=G$. We may assume that the origin $0\in K$. We identify $\mathbf{C}^n=\mathbf{R}^{2n}$ and we consider $\mathbf{R}^3\subset \mathbf{R}^{2n}$ given by $\mathbf{R}^3=\{(x_1,x_2,\ldots,x_{2n})\in \mathbf{R}^{2n}\mid x_4=x_5=\ldots=x_{2n}=0\}$. We define $X=\mathbf{C}^n\setminus\{0\}$ and $Y=\mathbf{C}^n\setminus K$ and we shall prove that (X,Y) has the required properties. Clearly X and Y are n-complete since they are non-compact. On the other hand $X\setminus Y=K\setminus\{0\}$ has no compact connected components, therefore (X,Y) is an n-Runge pair. It remains to show that $H_{2n-1}(X,Y;\mathbf{Z})=G$. By Alexander's duality in $\mathbf{R}^{2n}\setminus\{0\}$, which contains $K\setminus\{0\}$ as a closed subset, we get $H_{2n-1}(X,Y;\mathbf{Z})=\check{H}_c^1(K\setminus\{0\},\mathbf{Z})$ where $\check{H}_c^i$ denotes the Čech cohomology groups with compact supports (see [6]). On the other hand $K\setminus\{0\}$ is a closed subset of $\mathbf{R}^3\setminus\{0\}$ and using Alexander's duality in $\mathbf{R}^3\setminus\{0\}$ we get similarly that $\check{H}_c^1(K\setminus\{0\},\mathbf{Z})=H_2(\mathbf{R}^3\setminus\{0\},\mathbf{R}^3\setminus K;\mathbf{Z})$ (we used here the fact that the Čech cohomology groups with compact supports of a closed subset $A\subset E$, with E a euclidean neighbourhood retract, depends only on A (see [6])).

By Lemma 2 we have that $H_2(\mathbf{R}^3\setminus\{0\},\mathbf{R}^3\setminus K;\mathbf{Z})=H_2(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z})$ and exactly as in Corollary 2, we get $H_2(\mathbf{R}^3,\mathbf{R}^3\setminus K;\mathbf{Z})=H_1(\mathbf{R}^3\setminus K;\mathbf{Z})$. It follows that $H_{2n-1}(X,Y;\mathbf{Z})=G$ which proves our lemma.

Lemma 4. *Let (X,Y) with $Y\subset X$ be a q-Runge pair and W be a p-complete space. Then $(X\times W, Y\times W)$ is a $(p+q-1)$-Runge pair.*

Proof. Let $K\subset Y\times W$ be a compact subset. We have to find $f: X\times W\to\mathbf{R}$ a $(p+q-1)$-convex exhaustion function such that $K\subset\{f<0\}\Subset Y\times W$. We choose compact subsets $K_1\subset Y$ and $K_2\subset W$ such that $K\subset K_1\times K_2$. Since (X,Y) is a q-Runge pair there exists a q-convex exhaustion function $\varphi: X\to\mathbf{R}$ with $K_1\subset\{\varphi<0\}\Subset Y$. Let also $\psi: W\to\mathbf{R}$ be a p-convex exhaustion function. We choose $\varepsilon>0$ sufficiently small such that $\varphi(x)+\varepsilon\exp(\psi(w))<0$ for every $x\in K_1, w\in K_2$ and we define $f(x,w)=\varphi(x)+\varepsilon\exp(\psi(w))$. Clearly f is a $(p+q-1)$-convex exhaustion function on $X\times W$ and $K\subset K_1\times K_2\subset\{f<0\}\Subset Y\times W$. Thus Lemma 4 is completely proved.

Let us recall now the Künneth formula in the relative case (see [9, p. 231]). Let X_0, W be topological spaces and $Y_0\subset X_0$ a subset. Then we have the exact sequence

$$\begin{aligned}0\longrightarrow\bigoplus_{p=0}^{m}H_p(X_0,Y_0;\mathbf{Z})\otimes H_{m-p}(W;\mathbf{Z})\longrightarrow H_m(X_0\times W,Y_0\times W;\mathbf{Z})\\ \longrightarrow\bigoplus_{p=0}^{m-1}\operatorname{Tor}(H_p(X_0,Y_0;\mathbf{Z}),H_{m+p-1}(W;\mathbf{Z}))\longrightarrow 0.\end{aligned}$$

In particular, if the homology of W is free, we have

$$H_m(X_0\times W,Y_0\times W;\mathbf{Z})\cong\bigoplus_{p=0}^{m}H_p(X_0,Y_0;\mathbf{Z})\otimes H_{m-p}(W;\mathbf{Z}).$$

We can now prove Theorem 1.

The proof of Theorem 1. If $q=n\geq 2$ then the statement of Theorem 1 follows from Lemma 3. Therefore we may assume that $2\leq q<n$. We define $W=(\mathbf{C}^{q-1}\backslash\{0\})\times(\mathbf{C}^*)^{n-q-1}$. Then W is a $(q-1)$-complete manifold of dimension $n-2$. Let (X_0,Y_0) be a 2-Runge pair, with $Y_0\subset X_0$ being 2-dimensional manifolds such that $H_3(X_0,Y_0;\mathbf{Z})=G$ (which exists by Lemma 3).

By Lemma 4 $(X_0\times W,Y_0\times W)$ is a q-Runge pair of n-dimensional manifolds. We have only to verify $H_{n+q-1}(X_0\times W,Y_0\times W;\mathbf{Z})=G$. First we study the homology of W. We have

$$(1)\qquad H_j(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})=H_j(S^{2(q-1)-1};\mathbf{Z})=\begin{cases}\mathbf{Z}, & \text{if } j=0 \text{ or } j=2q-3,\\ 0, & \text{otherwise.}\end{cases}$$

Since

$$H_j(\mathbf{C}^*;\mathbf{Z})=\begin{cases}\mathbf{Z}, & \text{if } j=0,\ 1,\\ 0, & \text{otherwise,}\end{cases}$$

it follows that

$$(2)\qquad H_j((\mathbf{C}^*)^i;\mathbf{Z})=\begin{cases}0, & \text{if } j>i,\\ \mathbf{Z}, & \text{if } j=i.\end{cases}$$

From (1) and (2) and the Künneth formula we get

$$\begin{aligned}H_{n+q-4}(W;\mathbf{Z})&=[H_0(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n+q-4}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&\quad\oplus[H_{2q-3}(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n-q-1}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&=(\mathbf{Z}\otimes 0)\oplus(\mathbf{Z}\otimes\mathbf{Z})=\mathbf{Z},\\H_{n+q-3}(W;\mathbf{Z})&=[H_0(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n+q-3}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&\quad\oplus[H_{2q-3}(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n-q}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&=(\mathbf{Z}\otimes 0)\oplus(\mathbf{Z}\otimes 0)=0,\\H_{n+q-2}(W;\mathbf{Z})&=[H_0(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n+q-2}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&\quad\oplus[H_{2q-3}(\mathbf{C}^{q-1}\backslash\{0\};\mathbf{Z})\otimes H_{n-q+1}((\mathbf{C}^*)^{n-q-1};\mathbf{Z})]\\&=(\mathbf{Z}\otimes 0)\oplus(\mathbf{Z}\otimes 0)=0.\end{aligned}$$

Since clearly the homology of W is free and $H_j(X_0,Y_0;\mathbf{Z})=0$ if $j\notin\{1,2,3\}$ it follows by Künneth's formula in the relative case that

$$\begin{aligned}H_{n+q-1}(X_0\times W,Y_0\times W;\mathbf{Z})&=\bigoplus_{i=1}^{3}H_i(X_0,Y_0;\mathbf{Z})\otimes H_{n+q-i-1}(W;\mathbf{Z})\\&=[H_1(X_0,Y_0;\mathbf{Z})\otimes 0]\oplus[H_2(X_0,Y_0;\mathbf{Z})\otimes 0]\\&\quad\oplus[H_3(X_0,Y_0;\mathbf{Z})\otimes\mathbf{Z}]\\&=G\otimes\mathbf{Z}=G.\end{aligned}$$

The proof of Theorem 1 is complete.

3. Some remarks concerning n-Runge pairs of n-dimensional manifolds

If (X, Y) is a q-Runge pair of n-dimensional manifolds (or more general of n-dimensional complex spaces) then it is known ([7], [15]) that the natural map $H_{n+q-1}(Y;\mathbf{Z})\to H_{n+q-1}(X;\mathbf{Z})$ is injective. So it is natural to ask if the restriction map $H^{n+q-1}(X;\mathbf{Z})\to H^{n+q-1}(Y;\mathbf{Z})$ is surjective. We shall prove that the answer is no, at least when $q\geq 2$. More precisely one has the following result.

Theorem 3. *For all integers q and n with $2\leq q\leq n$ there exists a q-Runge pair (X, Y) of n-dimensional manifolds such that the restriction map $H^{n+q-1}(X;\mathbf{Z})\to H^{n+q-1}(Y;\mathbf{Z})$ is not surjective.*

Proof. We consider the exact sequence for cohomology corresponding to the inclusion $Y\subset X$,

$$\begin{aligned}\ldots \longrightarrow H^{n+q-1}(X;\mathbf{Z}) \longrightarrow H^{n+q-1}(Y;\mathbf{Z}) \longrightarrow H^{n+q}(X, Y;\mathbf{Z})\\ \longrightarrow H^{n+q}(X;\mathbf{Z}) \longrightarrow \ldots .\end{aligned}$$

Since X is assumed to be q-complete it follows that $H^{n+q}(X;\mathbf{Z})=0$, therefore the surjectivity of the map $H^{n+q-1}(X;\mathbf{Z})\to H^{n+q-1}(Y;\mathbf{Z})$ is equivalent to the vanishing of the relative cohomology group $H^{n+q}(X, Y;\mathbf{Z})$. From the exact sequence ([6, p. 153]),

$$\begin{aligned}0 \longrightarrow \operatorname{Ext}(H_{n+q-1}(X, Y;\mathbf{Z});\mathbf{Z}) \longrightarrow H^{n+q}(X, Y;\mathbf{Z})\\ \longrightarrow \underset{\substack{\| \\ 0}}{\operatorname{Hom}(H_{n+q}(X, Y;\mathbf{Z});\mathbf{Z})} \longrightarrow 0\end{aligned}$$

we have only to obtain our (X, Y) with $\operatorname{Ext}(H_{n+q-1}(X, Y;\mathbf{Z});\mathbf{Z})\neq 0$.

To do this we choose any countable abelian torsion free group G such that $\operatorname{Ext}(G;\mathbf{Z})\neq 0$ (e.g. we may take $G=\mathbf{Q}$ the additive group of rational numbers). By Theorem 1 there exists a q-Runge pair (X, Y) with $Y\subset X$ being n-dimensional complex manifolds such that $H_{n+q-1}(X, Y;\mathbf{Z})=G$. As remarked above (X, Y) is the required pair satisfying the conditions of Theorem 3.

Corollary 1 is now a direct consequence of Theorem 3.

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