Ark. Mat., **38** (2000), 93–96

On Gröchenig, Heil, and Walnut's proof of the local three squares theorem

Kristian Seip

It is plain that the positive (and nontrivial) part of Theorem 5.1 of the preceding paper by K. Gröchenig, C. Heil, and D. Walnut [1] is equivalent to the following theorem. (We keep the notation from [1] with the following slight exception: μ_k denotes the characteristic function of the interval $[-r_k, r_k]$, and ν_k is the characteristic function of the d-dimensional cube $[-r_k, r_k]^d$.)

Theorem. *Suppose* $0<r_1<r_2$ *and* $r_1/r_2 \notin \mathbf{Q}$, *and set* $R=r_1+r_2$. *Then the set of functions of the form* $(g_1\mu_1)*\mu_2+\mu_1*(g_2\mu_2)$, *with* $g_1, g_2 \in L^2(-R,R)$, *is dense in* $L^2(-R,R)$.

This observation underlies the proof of Theorem 5.1 of [1]. Below I will give a more direct proof of the theorem just stated. A d-dimensional extension, equivalent to the local three "squares" theorem in dimension $d>1$ (cf. Theorem 6.1 of [1]), will be obtained as a corollary of this theorem.

Proof. It is enough to consider linear combinations of $(g_1\mu_1)*\mu_2$, with $g_1(t)=e^{i\pi kt/r_1}$, and $\mu_1*(g_2\mu_2)$, with $g_2(t)=e^{i\pi kt/r_2}$, k denoting an arbitrary integer. Taking Fourier transforms, we see that the question is whether the linear span of the functions

$$\frac{G(t)}{t(t-\lambda_{j,k})},$$

with $\lambda_{j,k}=k/2r_j$ and $G(t)=\sin(2\pi r_1 t)\sin(2\pi r_2 t)$, is dense in the Paley–Wiener space PW_{2R}. This is answered in two steps.

First we prove that $G(t)/t$ belongs to the closed span of these functions: Choosing $a_k>0$ such that $\sum_k a_k=1$ and $\sum_k a_k^2=\varepsilon$, we obtain

$$\int_{\mathbf{R}}\left|\frac{G(t)}{t}-\sum_k \frac{a_k\lambda_{1,k}G(t)}{t(\lambda_{1,k}-t)}\right|^2 dt=\int_{\mathbf{R}}\left|\sum_k \frac{a_k G(t)}{\lambda_{1,k}-t}\right|^2 dt \le 2r_1\pi^2\varepsilon,$$

by the orthogonality of the functions $\sin(2\pi r_1 t)/(\lambda_{1,k}-t)$. The claim follows since $\varepsilon>0$ is arbitrary.

In the second step, we check that the functions $G(t)/t$, $G(t)/t^2$ and $G(t)/(t-\lambda_{j,k})$ constitute a complete sequence in PW_{2R}. By step one, this will prove the theorem, since each of these functions is a linear combination of functions from the original sequence along with $G(t)/t$. This completeness can be proved in different ways. By performing a suitable block summation of a corresponding Lagrange-type interpolation formula, one may in fact obtain a considerably stronger result about the approximation of functions in PW_{2R} by linear combinations of functions from this sequence. Such an approximation result and thus in particular the completeness were proved by Levin [2]; see also [3, pp. 151–153], where this type of interpolation is investigated in the case that the zeros of the corresponding function G are simple and uniformly separated.

To make this note self-contained, I give an argument for the completeness. It is convenient to switch to the equivalent norm and inner product for PW_{2R} obtained by integrating along the horizontal line $\operatorname{Im} z=-1$. We shall prove that a function F which is orthogonal to each of the given functions, must vanish identically. We see that

$$\int_{-\infty}^{\infty} F(x-i)\overline{\left(\frac{G(x-i)}{x-i-\lambda}\right)}\,dx=\int_{-\infty}^{\infty} F(x-i)\frac{\overline{G(\overline{x+i})}}{x-(\lambda-i)}\,dx.$$

For $\operatorname{Im}\lambda>-1$, the integrand times $e^{i2\pi Rx}$ belongs to H^1 of the upper half-plane. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} F(x-i)\overline{\left(\frac{G(x-i)}{x-i-\lambda}\right)}\,dx&=\int_{-\infty}^{\infty} F(x-i)\frac{\overline{G(x-i)}}{x-(\lambda-i)}\left(1-e^{i2\pi R(x-(\lambda-i))}\right)dx\\&=e^{-i2\pi R(\lambda-i)}\\&\quad\times\int_{-\infty}^{\infty} F(x-i)\overline{G(\overline{x+i})}e^{-i2\pi Rx}\frac{\sin 2\pi R(x-(\lambda-i))}{x-(\lambda-i)}\,dx\end{aligned}$$

for $\operatorname{Im}\lambda>-1$. As a function of λ, the right-hand side belongs to PW_{2R}. The orthogonality of F to each function in our sequence implies that this function is of the form $H(z)G(z)$ for some entire function H. This is impossible unless $H\equiv 0$, whence $F(x-i)\overline{G(\overline{x+i})}e^{-i2\pi Rx}$ must be orthogonal to PW_{2R}. We let f denote the inverse Fourier transform of F and use the Paley–Wiener theorem to see that then a linear combination of $f(t)$, $f(t-2r_1)$, $f(t-2r_2)$, $f(t-2R)$ is orthogonal to $L^2(-R,R)$ considered as a subspace of $L^2(\mathbf{R})$. This can only occur when f is zero almost everywhere. $\square$

We shall deduce two corollaries of this theorem, of which the second is equivalent to the positive part of Theorem 6.1 of [1]. To simplify the book-keeping, we set

$$\Pi^*_m h_m = h_1 * h_2 * \ldots * h_n$$

when $h_1, h_2, \ldots, h_n$ are functions defined on $\mathbf{R}$.

Corollary 1. *Suppose* $0<r_1<r_2<\ldots<r_m$ $(m\geq 2)$ *and* $r_j/r_k\notin\mathbf{Q}$ *for* $j\neq k$, *and set* $R=\sum_{j=1}^m r_j$. *Then the set of functions of the form* $\sum_{j=1}^m (g_j\mu_j)*(\Pi^*_{k\neq j}\mu_k)$, *with* $g_j\in L^2[-R,R]$, *is dense in* $L^2(-R,R)$.

Proof. The proof is by induction on m. We note that when $m=2$, the statement is identical to that of the theorem. We set $m=n+1$ and assume the corollary has been established for $m=n$.

In what follows we let $\hat{\mu}_k$ denote the characteristic function of the interval $[-R+r_k, R-r_k]$. The induction hypothesis implies that any function $\mu_k*(\hat{h}_k\hat{\mu}_k)$, $\hat{h}_k\in L^2(-R,R)$, may be approximated by functions of the form $\sum_{j\neq k}(g_j\mu_j)*(\Pi^*_{l\neq j}\mu_l)$. In particular, we may choose

$$\hat{h}_l\hat{\mu}_l = (h_j\mu_j)*(h_k\mu_k)*\Pi^*_{p\neq j,k,l}\mu_p,$$

assuming of course that $j,k\neq l$. In other words, any function $(h_j\mu_j)*(h_k\mu_k)*\Pi^*_{p\neq j,k}\mu_p$ may be approximated by functions of the form stated in the corollary. Using the induction hypothesis a second time (freezing j and letting k vary), we find that any function $(h_j\mu_j)*(\hat{h}_j\hat{\mu}_j)$ may be approximated by those same functions. Now remains only the simple exercise to check that the linear combinations of functions of the form $(h_j\mu_j)*(\hat{h}_j\hat{\mu}_j)$, $h_j, \hat{h}_j\in L^2(-R,R)$, constitute a dense subset of $L^2(-R,R)$. □

Corollary 2. *Suppose* $0<r_1<r_2<\ldots<r_{d+1}$ *and* $r_j/r_k\notin\mathbf{Q}$ *when* $j\neq k$, *and set* $R=\sum_{j=1}^{d+1} r_j$. *Then the set of functions of the form* $\sum_{j=1}^{d+1}\nu_j*(\hat{g}_j\hat{\nu}_j)$, *with* $\hat{\nu}_j$ *denoting the characteristic function of* $[-R+r_j, R-r_j]^d$ *and* $\hat{g}_j\in L^2[-R,R]^d$, *is dense in* $L^2[-R,R]^d$.

Proof. Corollary 2 is an immediate consequence of Corollary 1 (with $m=d+1$), since any product $f_1(x_1)f_2(x_2)\ldots f_d(x_d)$, with each f_j a function as in Corollary 1, is a function as stated in Corollary 2. □

References

1. Gröchenig, K., Heil, C. and Walnut, D., Nonperiodic sampling and the local three squares theorem, *Ark. Mat.* **38** (2000), 77–92.
2. Levin, B. Ya., On bases of exponential functions in $L^2(-\pi,\pi)$, *Zap. Mekh.-Mat. Fak. i Khar'kov Mat. Obshch.* **27** (1961), 39–48 (Russian).

3. Levin, B. Ya., *Lectures on Entire Functions,* Transl. Math. Monographs **150**, Amer. Math. Soc., Providence, R. I., 1996.

Received July 2, 1999

Kristian Seip
Department of Mathematical Sciences
Norwegian University of Science and Technology
NO-7491 Trondheim
Norway
email: seip@math.ntnu.no