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Normality and shared values

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Abstract. Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ and let a and b be distinct values. If for every $f\in\mathcal{F}$, f and f' share a and b on Δ, then $\mathcal{F}$ is normal on Δ.

I. Introduction

Let D be a domain in $\mathbf{C}$. Define for f meromorphic on D and $a\in\mathbf{C}$

$$\overline{E}_f(a)=f^{-1}(\{a\})\cap D=\{z\in D: f(z)=a\}.$$

Two functions f and g on D are said to share the value a if $\overline{E}_f(a)=\overline{E}_g(a)$.

A meromorphic function f on $\mathbf{C}$ is called a normal function if there exists a positive number M such that

$$f^{\#}(z)\le M.$$

Here, as usual, $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$ denotes the spherical derivative.

W. Schwick seems to have been the first to draw a connection between normality criteria and shared values. He proved the following theorem [12].

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2 and a_3 for every $f\in\mathcal{F}$, then $\mathcal{F}$ is normal on Δ.*

In the present paper, we prove the following result.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, and let a and b be distinct complex numbers and c a nonzero complex number. If for every $f\in\mathcal{F}$,*

$$\overline{E}_f(0)=\overline{E}_{f'}(a),\quad \overline{E}_f(c)=\overline{E}_{f'}(b),$$

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then $\mathcal{F}$ is normal on Δ.

The special case $a=0$, $b=c=1$ was proved in [10].

As an immediate consequence, we have the following result.

Theorem 2. *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ, and let a and b be distinct complex numbers. If f and f' share a and b for every $f\in\mathcal{F}$, then $\mathcal{F}$ is normal on Δ.*

Earlier, Mues and Steinmetz had proved the following theorem [8].

Theorem B. *Let f be a meromorphic function on $\mathbf{C}$ and a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, and a_3, then $f(z)=ce^z$.*

We prove the following result.

Theorem 3. *Let f be a meromorphic function on $\mathbf{C}$ and a and b be distinct complex numbers. If f and f' share a and b, then f is a normal function.*

Example. Let $f(z)=\tan z$. Then $f'(z)=1+\tan^2 z$, so f and f' share the values $\frac{1}{2}\left(1\pm i\sqrt{3}\right)$. More generally, if f is a solution of the differential equation

$$w'=aw^2+(b+1)w+c, \quad a,b,c\in\mathbf{C},$$

and the quadratic $y=ax^2+bx+c$ has two distinct roots, then f and f' share the values $\left(-b\pm\sqrt{b^2-4ac}\right)/2a$.

II. Lemmas

Lemma 1. ([11]) *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ all of whose zeros have multiplicity at least k, and suppose there exists $A\geq 1$ such that $|f^{(k)}(z)|\leq A$ whenever $f(z)=0$, $f\in\mathcal{F}$. Then if $\mathcal{F}$ is not normal, there exist, for each $0\leq\alpha\leq k$,*

(a) *a number r, $0<r<1$;*
(b) *points z_n, $|z_n|<r$;*
(c) *functions $f_n\in\mathcal{F}$; and*
(d) *positive numbers $\varrho_n\to 0$;*

such that

$$\frac{f_n(z_n+\varrho_n\zeta)}{\varrho_n^{\alpha}}=g_n(\zeta)\to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a meromorphic function on $\mathbf{C}$ such that $g^{\#}(\zeta)\leq g^{\#}(0)=kA+1$.

Remark. In fact, Lemma 1 holds also for $-1<\alpha<0$, [9]. For $-1<\alpha<k$, the hypothesis on $f^{(k)}(z)$ can be dropped, and $kA+1$ can be replaced by an arbitrary positive constant [2].

In the sequel, we shall make use of the standard notation of value distribution theory, see [6] and [13].

Lemma 2. (Milloux, [6, Theorem 3.2, cf. Theorem 2.2]) *Let f be a meromorphic function of finite order. Then*

$$T(r,f)<\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-b}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f),$$

where $b\neq 0,\infty$ and $S(r,f)=O(\log r)$. If f is a rational function, then $S(r,f)=O(1)$.

Lemma 3. ([3]; cf. [7]) *A normal meromorphic function has order at most 2. A normal entire function (Yosida function) is of exponential type.*

Lemma 4. (Frank and Weissenborn [4], [13, Lemma 4.6]) *Let f be a transcendental meromorphic function of finite order. Then for every positive number ε, we have*

$$k\overline{N}(r,f)<(1+\varepsilon)N\left(r,\frac{1}{f^{(k+1)}}\right)+(1+\varepsilon)N_1(r,f)+S(r,f),$$

where $N_1(r,f)=N(r,f)-\overline{N}(r,f)$ and $S(r,f)=O(\log r)$.

Lemma 5. *Let f be a meromorphic function of finite order and a and b be distinct nonzero numbers. Suppose that all poles of f are multiple, $\overline{E}_f(0)=\overline{E}_{f'}(a)$, and $f'(z)\neq b$. If there exists a nonzero number d such that*

$$F(z)=\frac{f(z)f''(z)}{(f'(z)-a)(f'(z)-b)}\equiv d, \tag{2.1}$$

then

(i) $f(z)=b(z-c)+A/n(z-c)^n$, $d=1+1/n$;

(ii) $(n+1)b=a$;

here $A(\neq 0)$ and c are complex numbers and $n(\geq 2)$ is a positive integer.

Proof. Clearly, $f''(z)\not\equiv 0$. We claim that f must satisfy the following conditions:

(1) all poles of f have the same multiplicity n $(2\leq n\leq+\infty)$;

(2) the principal part of each pole has only one term;

(3) f has at most finitely many poles;

(4) all zeros of $f'-a$ have the same multiplicity τ; and

(5) $\overline{E}_{f''}(0)\subset\overline{E}_{f'}(a)=\overline{E}_f(0)$.

(1) Let z_0 be a pole of f of multiplicity n, so that

$$f(z)=\frac{a_{-n}}{(z-z_0)^n}+\frac{a_{-n+1}}{(z-z_0)^{n-1}}+\dots, \quad a_{-n}\neq 0. \tag{2.2}$$

A simple calculation gives

$$d=F(z_0)=1+\frac{1}{n}.$$

Since d is constant, n is independent of z_0.

(2) From (2.2), we have

$$f'(z)=\frac{-na_{-n}}{(z-z_0)^{n+1}}+\frac{(-n+1)a_{-n+1}}{(z-z_0)^n}+\dots,$$
$$f''(z)=\frac{n(n+1)a_{-n}}{(z-z_0)^{n+2}}+\frac{n(n-1)a_{-n+1}}{(z-z_0)^{n+1}}+\dots.$$

If $\{i:1\le i\le n-1,\ a_{-i}\neq 0\}\neq\emptyset$, put $j=\max\{i:1\le i\le n-1,\ a_{-i}\neq 0\}$. Now

$$f(z)f''(z)=d(f'(z)-a)(f'(z)-b)=\left(1+\frac{1}{n}\right)(f'(z)-a)(f'(z)-b).$$

Comparing coefficients of $1/(z-z_0)^{n+j+2}$ on both sides, we have

$$n(n+1)+j(j+1)=2dnj=2(n+1)j,$$

whence $(n-j)^2+(n-j)=0$. This contradicts $1\le j\le n-1$.

(3) Suppose f is a transcendental meromorphic function. From (1) and (2), there exists a transcendental meromorphic function g such that

$$f(z)=g^{(n-1)}(z), \tag{2.3}$$

where all poles of $g(z)$ are simple. Utilizing Lemma 4, for $\varepsilon=\frac{1}{2}$, we have

$$n\overline{N}(r,g)<\frac{3}{2}N\left(r,\frac{1}{g^{(n+1)}}\right)+O(\log r).$$

It follows from (2.3) that

$$n\overline{N}(r,f)<\frac{3}{2}N\left(r,\frac{1}{f''}\right)+O(\log r). \tag{2.4}$$

From Lemma 2 and $f'(z)\neq b$, we have

$$T(r,f)<\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+O(\log r). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$T(r,f) < \left(1-\frac{2n}{3}\right)\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+O(\log r).$$

Now f and g have finite order and $N(r,1/f)\le T(r,f)+O(1)$, so

$$\left(\frac{2n}{3}-1\right)\overline{N}(r,f)=O(\log r).$$

Since $n\ge 2$, it follows that $\overline{N}(r,f)=O(\log r)$, so $f(z)$ has at most finitely many poles.

(4) Let z_0 be a zero of $f'-a$. Then $f(z_0)=0$. Write

$$f(z)=a(z-z_0)+a_{\tau+1}(z-z_0)^{\tau+1}+\dots \;. \tag{2.6}$$

As $f''(z)\not\equiv 0$, $1\le\tau<+\infty$. It follows from (2.1) and (2.6) that

$$d=F(z_0)=\frac{a\tau}{a-b},$$

so τ is independent of z_0.

(5) Suppose z_0 is a zero of f''. If $f'(z_0)-a\neq 0$, then $F(z_0)=0\neq d$, a contradiction.

Having established the properties claimed for f, we turn now to the proof of (i) and (ii).

Suppose then that f is a transcendental meromorphic function. Since $T(r,f)=T(r,1/f)+O(1)$, $\overline{N}(r,f)=O(\log r)$, it follows from (2.5) that

$$\begin{aligned} T(r,f) &\le T\left(r,\frac{1}{f}\right)-m\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+\overline{N}(r,f)+O(\log r)\\ &\le T(r,f)-m\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+O(\log r), \end{aligned} \tag{2.7}$$

so that

$$m\left(r,\frac{1}{f}\right)=O(\log r). \tag{2.8}$$

Put $Q(z)=(f'(z)-a)/f(z)$. Since $\overline{E}_f(0)=\overline{E}_{f'}(a)$, we have

$$T(r,Q)=N(r,Q)+m(r,Q)\le\overline{N}(r,f)+m\left(r,\frac{f'}{f}\right)+m\left(r,\frac{a}{f}\right). \tag{2.9}$$

From (2.8) and (2.9), we have $T(r,Q)=O(\log r)$. Thus Q is a rational function, whose poles are those of f. As $f'(z)\neq b$, we may assume that

$$f'(z)=b+\frac{e^{P_1(z)}}{P_2(z)}, \tag{2.10}$$

where P_1 and P_2 are polynomials, $\deg P_1\geq 1$. Since

$$f(z)=\frac{f'(z)-a}{Q(z)}=\frac{f'(z)-b}{Q(z)}+\frac{b-a}{Q(z)}=\frac{e^{P_1(z)}}{Q(z)P_2(z)}+\frac{b-a}{Q(z)},$$

we have from (2.10)

$$\left(\frac{b-a}{Q(z)}\right)'+\left(\frac{e^{P_1(z)}}{Q(z)P_2(z)}\right)'=b+\frac{e^{P_1(z)}}{P_2(z)}.$$

Since $\deg P_1\geq 1$, we must have

$$\left(\frac{b-a}{Q(z)}\right)'=b.$$

Thus there exists a constant c, such that

$$Q(z)=\frac{b-a}{b(z-c)}.$$

Clearly, $-n=\operatorname{Res}_{z=c}(f'(z)/f(z))=\operatorname{Res}_{z=c}Q(z)=(b-a)/b$; so f satisfies the differential equation

$$w'+\frac{n}{z-c}w=a. \tag{2.11}$$

But all solutions of (2.11) are rational functions, which contradicts the assumption on f.

Hence f must be rational. If $\tau=1$, it follows from (2.1) and (5) that $f''(z)\neq 0$. Since all poles of f have the same (finite) multiplicity n and $f'(z)\neq b$, we have

$$f'(z)=b-\frac{A}{P(z)^{n+1}}, \tag{2.12}$$

where $A(\neq 0)$ is a constant and P is a polynomial all of whose zeros are simple. Then

$$f''(z)=(n+1)A\frac{P'(z)}{P(z)^{n+2}}.$$

Since P and P' have no common zeros and $f''(z)\neq 0$, we must have $P'(z)\neq 0$, i.e., P is a linear polynomial. We may assume that

$$P(z)=(z-c). \tag{2.13}$$

Then

$$f(z)=b(z-c)+\frac{A}{n(z-c)^n}+D. \tag{2.14}$$

Since

$$f(z)f''(z)=\left(1+\frac{1}{n}\right)(f'(z)-a)(f'(z)-b),$$

it follows from (2.12), (2.13) and (2.14) that

$$nb(z-c)^{n+1}+nD(z-c)^n+A=(a-b)(z-c)^{n+1}+A.$$

Thus $a=(n+1)b$ and $D=0$, i.e.,

$$f(z)=b(z-c)+\frac{A}{n(z-c)^n},\quad a=(n+1)b.$$

If $\tau\geq 2$, it follows from (5) that

$$\overline{E}_{f''}(0)=\overline{E}_{f'}(a)=\overline{E}_f(0). \tag{2.15}$$

Again utilizing Lemma 2, we obtain

$$T(r,f)<\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+S(r,f),$$

where $S(r,f)=O(1)$. As

$$N\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{1}{f''}\right),\quad \overline{N}(r,f)=\frac{1}{n}N(r,f),\quad n\geq 2,$$

we have

$$\left(1-\frac{1}{n}\right)T(r,f)=O(1),$$

which contradicts $n\geq 2$.

This completes the proof of Lemma 5.

Lemma 6. *Let f be a nonconstant meromorphic function of finite order, all of whose poles are multiple, and let a and b be distinct nonzero numbers. If $\overline{E}_f(0)=\overline{E}_{f'}(a)$, $f'(z)\neq b$, and $f''(z)\not\equiv 0$, then*

$$f(z)=b(z-c)+\frac{A}{n(z-c)^n},\qquad n\geq 2,$$

and

$$a=(n+1)b.$$

Proof. Following the notation of [13, p. 105], let $N_{1)}(r,1/(f'-a))$ be the counting function for simple zeros of $f'-a$ and let

$$N_{(2}\left(r,\frac{1}{f'-a}\right)=N\left(r,\frac{1}{f'-a}\right)-N_{1)}\left(r,\frac{1}{f'-a}\right).$$

Clearly,

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f'-a}\right)&=N_{1)}\left(r,\frac{1}{f'-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f'-a}\right),\\ \overline{N}_{(2}\left(r,\frac{1}{f'-a}\right)&-N\left(r,\frac{1}{f''}\right)\leq 0.\end{aligned}\tag{2.16}$$

Since $\overline{E}_f(0)=\overline{E}_{f'}(a)$, $a\neq 0$, and $f'(z)\neq b$, we have by Lemma 2,

$$\begin{aligned}T(r,f)&\leq\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+S(r,f)\\ &=\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'-a}\right)-N\left(r,\frac{1}{f''}\right)+S(r,f).\end{aligned}\tag{2.17}$$

It follows from (2.16) and (2.17) that

$$T(r,f)\leq\overline{N}(r,f)+N_{1)}\left(r,\frac{1}{f'-a}\right)+S(r,f).\tag{2.18}$$

Set

$$F(z)=\frac{f(z)f''(z)}{(f'(z)-a)(f'(z)-b)}.$$

Then F is an entire function. If F is identically constant, Lemma 5 gives the desired result. Suppose, therefore, that F is not constant. Then

$$m(r,F)=m\left(r,\frac{ff''}{(f'-a)(f'-b)}\right)\leq m(r,f)+m\left(r,\frac{f''}{(f'-a)(f-b)}\right).$$

Using

$$\frac{f''}{(f'-a)(f'-b)}=\frac{1}{2(b-a)}\left(\frac{f''}{f'-a}-\frac{f''}{f'-b}\right)$$

and the lemma on the logarithmic derivative ([6, Lemma 2.3] or [13, Lemma 1.3]), we have

$$m(r,F)\le m(r,f)+S(r,f), \tag{2.19}$$

where again $S(r,f)=O(\log r)$ and $S(r,f)=O(1)$ in case f is a rational function.

Assume now that z_0 is a simple zero of $f'-a$. As $\overline{E}_f(0)=\overline{E}_{f'}(a)$, $f(z_0)=0$, so that writing $f(z)=a(z-z_0)+a_2(z-z_0)^2+\dots$, we have $f'(z)=a+2a_2(z-z_0)+\dots$ and $f''(z)=2a_2+\dots$, $a_2\ne 0$. It follows that $F(z_0)=a/(a-b)$, so that

$$N_{1)}\left(r,\frac{1}{f'-a}\right)\le N\left(r,\frac{1}{F-a/(a-b)}\right)\le T(r,F)+O(1). \tag{2.20}$$

Since F is an entire function and all poles of f are multiple, we have from (2.18), (2.19) and (2.20),

$$T(r,f)\le \tfrac{1}{2}N(r,f)+m(r,f)+S(r,f)$$

i.e.,

$$N(r,f)=S(r,f)=O(\log r). \tag{2.21}$$

Thus f has only finitely many poles. By (2.17),

$$\begin{aligned}T(r,f)&\le \overline{N}(r,f)+N\left(r,\frac{1}{f}\right)-N\left(r,\frac{1}{f''}\right)+S(r,f)\\ &\le T\left(r,\frac{1}{f}\right)-m\left(r,\frac{1}{f}\right)+S(r,f)\le T(r,f)-m\left(r,\frac{1}{f}\right)+S(r,f).\end{aligned}$$

Thus

$$m\left(r,\frac{1}{f}\right)=S(r,f)=O(\log r). \tag{2.22}$$

From (2.21), (2.22) and $\overline{E}_f(0)=\overline{E}_{f'}(a)$, we have, as in Lemma 5, that $f(z)$ is a rational function.

Thus

$$\overline{N}(r,f)=S(r,f)=O(1), \tag{2.23}$$

i.e., f is a polynomial. Since $f'(z)\ne b$, f' is a constant. This contradicts $f''(z)\not\equiv 0$.

III. Proofs of the theorems

Proof of Theorem 1. Assume $|a|\le|b|$. (Otherwise, we consider the family $\mathcal{F}_1=\{f-c: f\in\mathcal{F}\}$.) Suppose that $\mathcal{F}$ is not normal on Δ. Then, by Lemma 1, we have $f_n\in\mathcal{F}$, $z_n\in\Delta$, and $\varrho_n\to 0+$ such that

$$g_n(\zeta)=\frac{f_n(z_n+\varrho_n\zeta)}{\varrho_n}\to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function satisfying $g^{\#}(\zeta)\le g^{\#}(0)=(|a|+1)+1=|a|+2$.

We claim that $\overline{E}_g(0)=\overline{E}_{g'}(a)$, $g'(\zeta)\ne b$.

Indeed, suppose $g(\zeta_0)=0$. Since g is not constant, there exist ζ_n, $\zeta_n\to\zeta_0$, such that

$$g_n(\zeta_n)=\frac{f_n(z_n+\varrho_n\zeta_n)}{\varrho_n}=0 \quad (n \text{ large enough}).$$

Since $\overline{E}_{f_n}(0)=\overline{E}_{f_n'}(a)$, we have $g_n'(\zeta_n)=f_n'(z_n+\varrho_n\zeta_n)=a$. It follows that $g'(\zeta_0)=\lim_{n\to\infty}g_n'(\zeta_n)=a$. Thus $\overline{E}_g(0)\subset\overline{E}_{g'}(a)$.

Suppose now that ζ_0 is a point such that $g'(\zeta_0)=a$. If $g'(\zeta)\equiv a$, then $g^{\#}(\zeta)\le|a|$, which contradicts $g^{\#}(0)=|a|+2$. Thus, $g'(\zeta)\not\equiv a$, so there exist ζ_n, $\zeta_n\to\zeta_0$, such that

$$g_n'(\zeta_n)=f_n'(z_n+\varrho_n\zeta_n)=a;$$

and hence

$$g_n(\zeta_n)=\frac{f_n(z_n+\varrho_n\zeta_n)}{\varrho_n}=0.$$

Thus $g(\zeta_0)=\lim_{n\to\infty}g_n(\zeta_n)=0$. It follows that $\overline{E}_{g'}(a)\subset\overline{E}_g(0)$, so that $\overline{E}_g(0)=\overline{E}_{g'}(a)$.

Finally, suppose that there exists ζ_0 satisfying $g'(\zeta_0)=b$. One sees easily that $g'(\zeta)\not\equiv b$, so the previous reasoning shows that there exist $\zeta_n\to\zeta_0$, such that

$$g_n'(\zeta_n)=f_n'(z_n+\varrho_n\zeta_n)=b$$

and

$$g_n(\zeta_n)=\frac{f_n(z_n+\varrho_n\zeta_n)}{\varrho_n}=\frac{c}{\varrho_n}.$$

It follows that

$$g(\zeta_0)=\lim_{n\to\infty}g_n(\zeta_n)=\infty,$$

which contradicts $g'(\zeta_0)=b$. Thus $g'(\zeta)\neq b$.

If $ab=0$, then $a=0$ since $|a|\leq|b|$. Thus $\overline{E}_g(0)=\overline{E}_{g'}(0)$, so all zeros of g are multiple. By Lemma 3, the order of g is at most 2. It follows from [1, Theorem 3] that g has only a finite number of zeros. Hence, by Hayman's inequality ([5], [13, Theorem 4.5]), g must be a rational function. Since $g'(\zeta)\neq b$, it follows that

$$g'(\zeta)=b+o(1), \quad g(\zeta)=b\zeta+O(1), \quad \zeta\to\infty.$$

But $g(\zeta)/g'(\zeta)$ is a polynomial, which must be linear; and this contradicts $\overline{E}_g(0)=\overline{E}_{g'}(0)$.

Suppose, therefore, that $ab\neq 0$. Let ζ_0 be a pole of $g(\zeta)$. Since $g(\zeta)\not\equiv\infty$, there exists a closed disc $K=\{\zeta:|\zeta-\zeta_0|\leq\delta\}$ on which $1/g$ and $1/g_n$ are holomorphic (for n sufficiently large) and $1/g_n\to 1/g$ uniformly. Since $1/g_n(\zeta)-\varrho_n/c\to 1/g(\zeta)$ uniformly on K and $1/g$ is nonconstant, there exist ζ_n, $\zeta_n\to\zeta_0$, such that (for n large enough)

$$\frac{1}{g_n(\zeta_n)}-\frac{\varrho_n}{c}=0,$$

i.e.,

$$g_n(\zeta_n)-\frac{c}{\varrho_n}=\frac{f_n(z_n+\varrho_n\zeta_n)-c}{\varrho_n}=0. \tag{3.1}$$

Thus $f_n(z_n+\varrho_n\zeta_n)=c$, so that

$$g_n'(\zeta_n)=f_n'(z_n+\varrho_n\zeta_n)=b. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\left(\frac{1}{g(\zeta)}\right)'\Bigg|_{\zeta=\zeta_0}=-\frac{g'(\zeta_0)}{g^2(\zeta_0)}=\lim_{n\to\infty}-\frac{g_n'(\zeta_n)}{g_n^2(\zeta_n)}=0,$$

so that ζ_0 is a multiple pole of $g(\zeta)$. Thus all poles of g are multiple.

By Lemma 6, either $a=(n+1)b$, where n is a positive integer, or $g''(\zeta)\equiv 0$. If $a=(n+1)b$, then $|a|>|b|$, which contradicts $|a|\leq|b|$. If $g''(\zeta)\equiv 0$, then $g(\zeta)=a(\zeta-\zeta_0)$, which contradicts $\overline{E}_g(0)=\overline{E}_{g'}(a)$. This completes the proof.

Proof of Theorem 2. By Theorem 1, $\mathcal{F}_1=\{f-a:f\in\mathcal{F}\}$ is normal; hence, so is $\mathcal{F}$.

Proof of Theorem 3. Suppose f is not a normal function. Then there exist $z_n\to\infty$ such that $\lim_{n\to\infty}f^{\#}(z_n)=\infty$. Write $f_n(z)=f(z+z_n)$ and set $\mathcal{F}=\{f_n\}$. Then by Marty's criterion, $\mathcal{F}$ is not normal on the unit disc. On the other hand, since $\overline{E}_{f_n}(a)=\overline{E}_{f_n'}(a)$ and $\overline{E}_{f_n}(b)=\overline{E}_{f_n'}(b)$, Theorem 2 implies that $\mathcal{F}$ *is* normal. The contradiction proves the theorem.

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