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On the Poincaré inequality for vector fields

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Abstract. We prove the Poincaré inequality for vector fields on the balls of the control distance by integrating along subunit paths. Our method requires that the balls are representable by means of suitable "controllable almost exponential maps".

1. Introduction

In this paper we are concerned with the following Poincaré-type inequality,

$$\int_{B\times B}|u(x)-u(y)|\,dx\,dy\le cr|B|\int_{\lambda B}|Xu(x)|\,dx,\quad u\in C^1(\overline{\lambda B}),\tag{1.1}$$

where $B=B(x_0,r):=\{x\in\mathbf{R}^n:d(x_0,x)<r\}$ is a ball of the *control distance* d generated by a family $X=(X_1,\dots,X_m)$ of locally Lipschitz continuous vector fields $X_j:\mathbf{R}^n\to\mathbf{R}^n$. By $|B|$ we denote the Lebesgue measure of B, whereas λB stands for the homothetic ball $B(x_0,\lambda r)$. Moreover, $|Xu|$ denotes the euclidean norm of the X-gradient of u, i.e. $Xu=(X_1u,\dots,X_mu)$, $X_ju=\langle X_j,\nabla u\rangle$, $j=1,\dots,m$.

The most commonly used definition of control distance is based on the notion of subunit curve. An absolutely continuous path $\gamma:[0,T]\to\mathbf{R}^n$ is *X-subunit* if it satisfies $\dot\gamma(t)=\sum_{j=1}^m a_j(t)X_j(\gamma(t))$, with $\sum_{j=1}^m a_j(t)^2\le1$, for almost every $t\in[0,T]$. Assuming that for each $x,y\in\mathbf{R}^n$ there exists at least one X-subunit path that connects x and y one defines

$$d(x,y)=\inf\{T>0:\text{there is }\gamma:[0,T]\to\mathbf{R}^n\text{ subunit, with }\gamma(0)=x\text{ and }\gamma(T)=y\}.$$

Then $(x,y)\mapsto d(x,y)$ is a distance on $\mathbf{R}^n$ which is called the *control* (or *Carnot–Carathéodory*) *distance* related to X. We shall always assume that the d-topology

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is equivalent to the euclidean one, a condition that is satisfied in many important contexts.

As is well known, inequalities like (1.1) play a crucial role when studying eigenvalue problems for $\sum_j X_j^* X_j$ with Neumann boundary conditions. Over the last six years it has been shown that seemingly unexpected important results follow naturally from (1.1) making this inequality even more relevant. Indeed, let us assume that the following *dilation condition*

$$|\delta B| \leq c\delta^Q |B| \tag{1.2}$$

is satisfied, for any B and for every $\delta \geq 1$, where c and Q are suitable constants independent of B and δ. Then (1.1) is essentially equivalent to the following Poincaré–Sobolev inequality

$$\left(\fint_B |u-u_B|^q \right)^{1/q} \leq cr \left(\fint_{\lambda B} |Xu|^p \right)^{1/p}, \quad u \in C^1(\overline{\lambda B}), \tag{1.3}$$

where $u_B = \fint_B u = (1/|B|) \int_B u$, r is the radius of the ball B, and $1/p - 1/q = 1/Q$. Significant references are Saloff-Coste [S], Biroli and Mosco [BM], Hajłasz and Koskela [HK], Maheux and Saloff-Coste [MS], Franchi, Lu and Wheeden [FLW], Garofalo and Nhieu [GN]. Franchi, Lu and Wheeden actually prove that (1.1) and (1.2) imply a stronger estimate than (1.3), viz. an integral representation formula for $u - u_B$ in terms of Xu. Garofalo and Nhieu, assuming (1.2) and a slightly weaker inequality than (1.1), developed a general geometric theory of first order Sobolev spaces related to X.

It is also remarkable that the term λB in the right-hand side of (1.3) can be replaced by B, if (1.2) and (1.3) hold for any B. This important fact comes from a clever remark by Jerison [J].

In the euclidean setting, corresponding to the case in which $Xu = \nabla u$, i.e. $X_j = \partial/\partial x_j$, $j = 1, \dots, n$, inequality (1.1) is well known and can be proved starting from the identity

$$u(x) - u(y) = \int_0^{|x-y|} \frac{d}{dt} u\left(y + t \frac{x-y}{|x-y|} \right) dt. \tag{1.4}$$

It is noteworthy that the right-hand side of (1.4) is the integral of $\sum (\partial u / \partial x_j)\, dx_j$ along the "shortest" X-subunit path connecting x and y.

The idea of estimating the difference between the value of u at different points by integration along subunit curves was also used in [FL1]. In that paper, seemingly for the first time, the Poincaré inequality was proved in a non-euclidean setting. The class of vector fields studied in [FL1] however, is rather particular: the family X, indeed, was required to be diagonal, i.e. $X_j = \lambda_j \partial/\partial x_j$, $j = 1, \dots, n$, and the λ_j's were required to satisfy some strong conditions.

A much wider class of vector fields for which (1.1) holds was found by Jerison [J], who proved the Poincaré inequality for any family $X = (X_1, \dots, X_m)$ of smooth vector fields satisfying the Hörmander condition

$$\text{Rank}(\mathcal{L}(X_1, \dots, X_m))(x) = n \quad \text{for all } x \in \mathbf{R}^n. \tag{1.5}$$

To the authors' knowledge, the problem of finding other and different conditions assuring that (1.1) is satisfied is still widely open. The aim of this paper is to give a contribution in this direction. We show that the Poincaré inequality (1.1) holds if the ball B is *representable* by means of *X-controllable almost exponential maps* (see the next section for the precise statement). This hypothesis enables us to prove (1.1) starting from the idea of estimating $u(x) - u(y)$ by integration along an almost d-shortest X-subunit path connecting x and y. It should be immediately noticed that our condition, unlike (1.5) and that of [FL1] is not directly readable on the fields, it requires to know a "good representation" of the d-balls. However, for the vector fields studied in [FL1], this representation is already available, it was proved in [FL2]. For Hörmander's vector fields our conditions can be verified by slightly improving a well-known representation theorem of d-balls due to Nagel, Stein and Wainger [NSW].

This paper is organized as follows. In Section 2, after introducing our notion of X-controllable almost exponential map, we prove the main theorem of the paper. In Section 3 we apply our result to a diagonal case, thus giving a new proof of the Poincaré inequality of [FL1]. In Section 4 we first show the previously mentioned improvement of the Nagel–Stein–Wainger's representation theorem. Then, by using our main theorem, we provide a new proof of Jerison's Poincaré inequality. We want to stress that the result of Section 4 answers a question raised by Jerison in [J] about the possibility to demonstrate the Poincaré inequality starting from an improvement of Nagel–Stein–Wainger's result. In the last section of the paper we show an application of our theorem to a pair of non-smooth vector fields in $\mathbf{R}^3$. Such a pair, that we call "of step two", naturally arises in studying non Levi-flat real surfaces in $\mathbf{C}^2$. We directly refer to Section 5 for a few more comments on this application.

Before closing this introduction we would like to quote two other papers which are related to our work. In [FL3] a "non-invariant" Poincaré inequality is proved assuming that a sufficiently rich family of subunit curves sweeps out from any point of $\mathbf{R}^n$. Although this hypothesis may be difficult to verify and may only lead to a weak form of the Poincaré inequality, our main theorem was partially inspired by [FL3]. Finally, by a technique similar to the one used here, Varopoulos [V] gave an easy proof of the Poincaré inequality for vector fields which are left invariant on a homogeneous group.

2. Almost exponential maps, controllability and the Poincaré inequality

Let us consider, in $\mathbf{R}^n$, an open set Ω and an open neighborhood Q of the origin. We will say that a map $E:\Omega\times Q\to\mathbf{R}^n$ is an *almost exponential map* of type $a>0$ if $E(x,0)=x$ for every $x\in\Omega$, $E(x,\cdot)$ is C^1 and one-to-one on Q and the jacobian determinant $D(x,h):=|\det\partial E(x,h)/\partial h|$ satisfies the estimate

$$0<\frac{1}{a}D(x,0)\le D(x,h)\le aD(x,0)\quad\text{for all } x\in\Omega \text{ and } h\in Q. \tag{2.1}$$

A map $E:\Omega\times Q\to\mathbf{R}^n$ will be said to be *X-controllable* with a *hitting time* $T>0$ if there exists a function $\gamma:\Omega\times Q\times[0,T]\to\mathbf{R}^n$ satisfying the following conditions.

(C1) For any $(x,h)\in\Omega\times Q$, $t\mapsto\gamma(x,h,t)$ is an X-subunit path connecting x and $E(x,h)$, i.e. $\gamma(x,h,0)=x$, $\gamma(x,h,T(x,h))=E(x,h)$ for a suitable $T(x,h)\le T$.

(C2) For any $(h,t)\in Q\times[0,T]$, $x\mapsto\gamma(x,h,t)$ is a one-to-one map having continuous first derivatives and jacobian determinant uniformly bounded away from zero, i.e.

$$b:=\inf_{\Omega\times Q\times[0,T]}\left|\frac{\partial\gamma}{\partial x}\right|>0. \tag{2.2}$$

We will call any function γ satisfying (C1) and (C2) a *control function* of E.

Roughly speaking, our main observation is that the Poincaré inequality relies on the representability of the d-balls by means of X-controllable almost exponential maps.

The following theorem makes this assertion precise.

Theorem 2.1. *Let $B=B(x_0,r)$ be a fixed d-ball. Assume that there exists an open set $\Omega\subseteq B$, an almost exponential map $E:\Omega\times Q\to\mathbf{R}^n$ and two positive constants α and β satisfying the following conditions*:

(i) $|B|\le\alpha|\Omega|$ *and* $B\subseteq E(x,Q)$ *for every* $x\in\Omega$;

(ii) *E is X-controllable with a hitting time* $T\le\alpha r$;

(iii) $|(\alpha+1)B|\le\beta|B|$.

Then there exists $c>0$ such that

$$\int_{B\times B}|u(x)-u(y)|\,dx\,dy\le cr|B|\int_{(\alpha+1)B}|Xu(z)|\,dz,\quad u\in C^1(\overline{(\alpha+1)B}). \tag{2.3}$$

The constant c depends only on α and β, on the type of E (the constant a in (2.1)), and on the constant b in (2.2) (related to a control function γ of E).

Proof. We first prove some simple consequences of the hypotheses (i), (ii) and (iii). Let $\gamma:\Omega\times Q\times[0,T]\to\mathbf{R}^n$ be an X-control function of E such that $T\le\alpha r$ (hypothesis (ii)). Since $t\mapsto\gamma(x,h,t)$ is a subunit curve and $\gamma(x,h,0)=x$,

$$d(x,\gamma(x,h,t))\le t\le T\le\alpha r,\quad x\in\Omega. \tag{2.4}$$

Then, keeping in mind that $E(x,h)=\gamma(x,h,T(x,h))$ for a suitable $T(x,h)\le T$, $E(x,Q)\subseteq B(x,\alpha r)\subseteq B(x_0,(\alpha+1)r)$. From these inclusions and by also using (iii), we obtain

$$|E(x,Q)|\le\beta|B|\quad\text{for all } x\in\Omega. \tag{2.5}$$

On the other hand, by (2.1),

$$|E(x,Q)|=\int_Q D(x,h)\,dh\ge\frac{1}{a}|D(x,0)|\,|Q|.$$

Then we have $D(x,0)\le\beta a|B|/|Q|$ which, together with (2.1), implies

$$D(x,h)\le\beta a^2\frac{|B|}{|Q|}\quad\text{for all } x\in\Omega \text{ and } h\in Q. \tag{2.6}$$

We are now in a position to prove (2.3). We denote by c any positive constant only depending on α, β, a and b. For any function $u\in C^1(\overline{(\alpha+1)B})$ we have, using (i), (2.6), (C1) and that γ is subunit,

$$\begin{aligned}\int_{B\times B}|u(x)-u(y)|\,dx\,dy&\le\int_{B\times B}|u(x)-u_\Omega|\,dx\,dy+\int_{B\times B}|u(y)-u_\Omega|\,dx\,dy\\&=2|B|\int_B|u(y)-u_\Omega|\,dy\\&\le\frac{2|B|}{|\Omega|}\int_B\int_\Omega|u(x)-u(y)|\,dx\,dy\\&\le 2\alpha\int_\Omega\int_{E(x,Q)}|u(x)-u(y)|\,dy\,dx \qquad (2.7)\end{aligned}$$

$$
\begin{aligned}
&= 2\alpha \int_\Omega \int_Q D(x,h)|u(x)-u(E(x,h))|\,dh\,dx \\
&\le c\frac{|B|}{|Q|}\int_\Omega\int_Q |u(x)-u(E(x,h))|\,dh\,dx \\
&= c\frac{|B|}{|Q|}\int_\Omega\int_Q \left|\int_0^{T(x,h)} \frac{d}{dt}u(\gamma(x,h,t))\,dt\right| dh\,dx \\
&\le c\frac{|B|}{|Q|}\int_\Omega\int_Q\int_0^T |\langle\nabla u(\gamma(x,h,t)),\dot\gamma(x,h,t)\rangle|\,dt\,dh\,dx \\
&\le c\frac{|B|}{|Q|}\int_Q\int_0^T\int_\Omega |Xu(\gamma(x,h,t))|\,dx\,dt\,dh.
\end{aligned}
$$

We then use (C2) to estimate the last integral by means of the change of variable $z=\gamma(x,h,t)$. Keeping in mind that $|\det\partial\gamma/\partial x|\ge b$ and noticing that $\gamma(x,h,t)\in B(x_0,(\alpha+1)r)$ for every $(x,h,t)\in\Omega\times Q\times[0,T]$ (see (2.4)), we have

$$
\int_\Omega |Xu(\gamma(x,h,t))|\,dx \le \frac{1}{b}\int_{B(x_0,(\alpha+1)r)} |Xu(z)|\,dz.
$$

This estimate, together with (2.7), gives

$$
\begin{aligned}
\int_{B\times B}|u(x)-u(y)|\,dx\,dy &\le c\frac{|B|}{|Q|}T\int_Q\int_{(\alpha+1)B}|Xu(z)|\,dz\,dh \\
&\le cr|B|\int_{(\alpha+1)B}|Xu(z)|\,dz,
\end{aligned}
$$

using that $T\le\alpha r$. This completes the proof. □

Remark 2.2. Under the same hypotheses as Theorem 2.1 we can directly prove the L^p version $(1\le p<\infty)$ of inequality (2.3),

$$
\int_{B\times B}|u(x)-u(y)|^p\,dx\,dy\le cr^p|B|\int_{(\alpha+1)B}|Xu(z)|^p\,dz,\quad u\in C^1(\overline{(\alpha+1)B}).
$$

We only have to modify, in an obvious way, the proof of Theorem 2.1.

3. A diagonal case

In this section we will apply Theorem 2.1 to the case in which $X=(X_1,\dots,X_n)$, $X_j=\lambda_j(x)\partial/\partial x_j$, $j=1,\dots,n$, and the λ_j's are continuous functions on $\mathbf{R}^n$ with continuous first derivatives outside $\Pi=\{(x_1,\dots,x_n):x_1x_2\dots x_n=0\}$. Moreover, $\lambda_1=1$, $\lambda_j=\lambda_j(x_1,\dots,x_{j-1})$ and

$$0<x_k\frac{\partial\lambda_j}{\partial x_k}\le\varrho\lambda_j \quad \text{in } \mathbf{R}^n\setminus\Pi, \tag{3.1}$$

if $1\le k<j\le n$. Under these hypotheses, the following results were proved in [FL2].

(D1) There exists at least one X-subunit path connecting any pair of points x and y in $\mathbf{R}^n$.

(D2) The X-control distance d is continuous with respect to the euclidean topology and the d-balls satisfy the dilation condition (1.2) with c and Q only depending on the constant ϱ in (3.1).

(D3) There exist n continuous functions $\Lambda_1,\dots,\Lambda_n:\mathbf{R}^n\times]0,+\infty[\to\mathbf{R}$, with continuous first derivatives in $\mathbf{R}^n\setminus\Pi$, such that

$$B(x,r)\subset\{x+h:h=(h_1,\dots,h_n),\ |h_j|\le r\Lambda_j(x,r)\}$$

for every $x\in\mathbf{R}^n$ and $r>0$ (in the notation of [FL2], $\Lambda_j(x,r)=F_j(x,r)/r$).

(D4) There exists $\alpha>0$ such that, defining

$$\Omega=\Omega_B:=\{x\in B(x_0,r):\Lambda_j(x_0,r)\le\alpha\lambda_j(x)\} \tag{3.2}$$

for any d-ball $B=B(x_0,r)$, then

$$|B|\le\alpha|\Omega|. \tag{3.3}$$

The last assertion is not explicitly stated in [FL2], but it can be deduced from (3.1) by elementary computation. With (D1)–(D4) in hand it is easy to show that the hypotheses of Theorem 2.1 are satisfied. Indeed, let $B=B(x_0,r)$ be a fixed d-ball. Define $Q:=\{h=(h_1,\dots,h_n):|h_j|\le 2r\Lambda_j(x_0,r),\ j=1,\dots,n\}$ and $E:\Omega\times Q\to\mathbf{R}^n$, $E(x,h)=x+h$. Obviously E is an almost exponential map and, thanks to (3.2),

$$B\subseteq\{x_0+h:|h_j|\le r\Lambda_j(x_0,r)\}\subseteq\{x+h:|h_j|\le 2r\Lambda_j(x_0,r)\}=E(x,Q) \tag{3.4}$$

for every $x\in B$. Thus, also keeping in mind (3.3), the condition (i) of Theorem 2.1 holds. Let us now check (ii) assuming $n=3$ for sake of simplicity (the general case can be handled in the same way). For every $(x,h)\in\Omega\times Q$ we define the path $\gamma(x,h,\cdot)$ as a sum of three paths, i.e.

$$\gamma(x,h,\cdot)=\gamma_3(x,h,\cdot)+\gamma_2(x,h,\cdot)+\gamma_1(x,h,\cdot)$$

where

$$\begin{aligned}
\gamma_3(x,h,t) &= (x_1, x_2, x_3+t\tilde{\Lambda}_3), && 0\le t\le |h_3/\tilde{\Lambda}_3|,\\
\gamma_2(x,h,t) &= (x_1, x_2+t\tilde{\Lambda}_2, x_3+h_3), && 0\le t\le |h_2/\tilde{\Lambda}_2|,\\
\gamma_1(x,h,t) &= (x_1+t\tilde{\Lambda}_1, x_2+h_2, x_3+h_3), && 0\le t\le |h_1/\tilde{\Lambda}_1|,
\end{aligned}$$

and $\tilde{\Lambda}_i=(\operatorname{sgn}(h_i)/\alpha)\Lambda_i(x_0,r)$, $i=1,\ 2,\ 3$. Then, if we define

$$T(x,h):=|h_3/\tilde{\Lambda}_3|+|h_2/\tilde{\Lambda}_2|+|h_1/\tilde{\Lambda}_1|,$$

$\gamma(x,h,0)=\gamma_3(x,h,0)=x$ and $\gamma(x,h,T(x,h))=\gamma_1(x,h,|h_1|/\tilde{\Lambda}_1)=x+h$. Using the definition of Q the hitting time $T=\sup_{x,h}T(x,h)$ can be estimated by $6\alpha r$.

Moreover, noticing that

$$\dot{\gamma}_i=\tilde{\Lambda}_i e_i=\frac{\tilde{\Lambda}_i}{\lambda_i(\gamma_i)}X_i(\gamma_i),$$

we can assert that $\gamma(x,h,\cdot)$ is X-subunit since $\Lambda_i(x_0,r)\le\alpha\lambda_i(x)$ for any $x\in\Omega$ (see (3.2)). Finally, for every fixed $h\in Q$ and $t\in[0,T]$, $x\mapsto\gamma(x,h,t)$ is a smooth function having jacobian determinant $\det\partial\gamma/\partial x\equiv 1$. Thus, also keeping in mind (D2), all the hypotheses of Theorem 2.1 are satisfied and the Poincaré inequality (1.1) holds, in this case, for every d-ball B.

We would like to close this section by quoting a paper by Franchi [F] containing a Poincaré inequality for diagonal vector fields more general than ours. However, the representation theorem for the d-balls proved in [F] could be seemingly used to prove that the hypotheses of Theorem 2.1 are satisfied in that more general case, too.

4. Hörmander vector fields and Jerison's Poincaré inequality

In this section $X=(X_1,\dots,X_m)$ will be a family of smooth vector fields on $\mathbf{R}^n$ satisfying the Hörmander condition (1.5). It is well known that the control distance d related to X is locally Hölder continuous with respect to the euclidean distance (see [FP]; see also [VSC], for a simpler proof). By using a representation theorem of the d-balls that slightly generalizes and improves a well-known result by Nagel, Stein and Wainger, we will show that the hypotheses of our Theorem 2.1 are satisfied for every d-ball with small enough radius. This will provide a new proof of Jerison's Poincaré inequality.

We begin by introducing some notation and definitions. If $I=(i_1,\dots,i_p)$ is a multi-index such that $1\le i_j\le m$, we set

$$X_I=[X_{i_1},[X_{i_2}\dots[X_{i_{p-1}},X_{i_p}]\dots]],$$

and $|I|=p$. We shall say that X_I is a commutator of length $|I|$ and we assume that each of the vector fields X_j is a commutator of length 1.

If $S_1,\dots,S_l\in\{X_1,\dots,X_m\}$, for any $\sigma\in\mathbf{R}$ we set

$$\begin{aligned}C_1(\sigma,S_1)&=e^{\sigma S_1},\\ C_2(\sigma;S_1,S_2)&=e^{-\sigma S_2}e^{-\sigma S_1}e^{\sigma S_2}e^{\sigma S_1},\\ C_l(\sigma;S_1,\dots,S_l)&=C_{l-1}(\sigma;S_2,\dots,S_l)^{-1}e^{-\sigma S_1}C_{l-1}(\sigma;S_2,,\dots,S_l)e^{\sigma S_1},\quad l\ge 2,\end{aligned}$$

and

$$\exp^*(\sigma S)=\begin{cases}C_l(\sigma^{1/l};S_1,\dots,S_l), & \text{if } \sigma>0,\\ C_l(|\sigma|^{1/l};S_1,\dots,S_l)^{-1}, & \text{if } \sigma<0,\end{cases}\tag{4.1}$$

where S denotes the commutator $[S_1,[S_2\dots[S_{l-1},S_l]\dots]]$. By the Campbell–Hausdorff formula (see [NSW, Lemma 2.21]) we have

$$\exp^*(\sigma S)(x)=x+\sigma S(x)+O(|\sigma|^{1+1/l}),\quad \text{as } \sigma\to 0,\tag{4.2}$$

which in particular means that $\exp^*(\sigma S)(x)$ is a perturbation of the exponential map $\exp(\sigma S)(x)$.

Let now K be an arbitrary compact subset of $\mathbf{R}^n$. By the Hörmander condition (1.5) there exists a positive integer $\nu=\nu(K)$ such that

$$\operatorname{span}\{X_I(x):|I|\le\nu\}=\mathbf{R}^n\quad \text{for all } x\in K.$$

Denote by $Y_1,\dots,Y_q$ an enumeration of the commutators of length $\le\nu$. The length of Y_j will be denoted by l_j. Given an n-tuple $\eta=(Y_{j_1},\dots,Y_{j_n})$, we finally let

$$\begin{aligned}l(\eta)&=l_{j_1}+\dots+l_{j_n},\\ \|h\|_\eta&=\max_{k=1,\dots,n}|h_k|^{1/l_{j_k}},\qquad \text{if } h=(h_1,\dots,h_n),\\ Q_\eta(r)&=\{h\in\mathbf{R}^n:\|h\|_\eta<r\},\quad \text{if } r>0\end{aligned}\tag{4.3}$$

and

$$E_\eta(x,h)=\exp^*(h_1Y_{j_1})\dots\exp^*(h_nY_{j_n})(x).\tag{4.4}$$

By using (4.2) it is easy to see that $E_\eta(x,h)=x+\sum_{k=1}^n Y_{j_k}(x)h_k+o(h)$, as $h\to 0$. Then

$$D_{E_\eta}(x,0)=|\det[Y_{j_1}(x),\dots,Y_{j_n}(x)]|.$$

We are now ready to state our version of Nagel–Stein–Wainger's representation theorem for the d-balls.

Theorem 4.1. *Let $K\subset\mathbf{R}^n$ be a compact set. Then there exists three positive constants r_0, c_1 and c_2 depending on K, with $c_2<c_1<1$, such that, given an n-tuple η, a point $x\in K$ and a positive $r\le r_0$ satisfying the inequality*

$$D_{E_\eta}(x,0)r^{l(\eta)}\ge\tfrac{1}{2}\max_\zeta D_{E_\zeta}(x,0)r^{l(\zeta)}, \tag{4.5}$$

the following assertions hold:

(a) *if $h\in Q_\eta(c_1r)$ then $\frac{1}{4}D_{E_\eta}(x,0)\le D_{E_\eta}(x,h)\le 4D_{E_\eta}(x,0)$;*

(b) *$B(x,c_2r)\subset E_\eta(x,Q_\eta(c_1r))$;*

(c) *the function $E_\eta(x,\cdot\,)$ is one-to-one on the set $Q_\eta(c_1r)$.*

The proof of this theorem is very similar to that of Theorem 7 in [NSW] and relies on a careful estimate of the derivatives of the map $h\mapsto E_\eta(x,h)$ obtained by means of the Campbell–Hausdorff formula. We refer to [M] for a detailed proof of this result. Theorem 4.1 contains all we need to show that condition (i) of Theorem 2.1 is satisfied. Indeed, let $B=B(x_0,r)$ be a d-ball centered at $x_0\in K$ and of radius $r<\frac{1}{2}c_2r_0$. For any n-tuple $\eta=(Y_{j_1},\dots,Y_{j_n})$ we define

$$\Omega_\eta=\left\{x\in B:D_{E_\eta}(x,0)\left(\frac{2r}{c_2}\right)^{l(\eta)}>\frac{1}{2}\max_\zeta D_{E_\zeta}(x,0)\left(\frac{2r}{c_2}\right)^{l(\zeta)}\right\}.$$

At least one of the sets Ω_η satisfies

$$|\Omega_\eta|\ge\frac{1}{N}|B|, \tag{4.6}$$

where N is the total number of n-tuples available. Let us choose one such η and denote by Q the box

$$Q:=\left\{h\in\mathbf{R}^n:\|h\|_\eta<\frac{2c_1}{c_2}r\right\}. \tag{4.7}$$

Then, by (a) and (c) of Theorem 4.1, the function

$$E_\eta:\Omega_\eta\times Q\longrightarrow\mathbf{R}^n \tag{4.8}$$

is an almost exponential map. Moreover, $|B|\le N|\Omega_\eta|$ (see (4.6)) and $B\subseteq E(x,Q)$ for every $x\in\Omega_\eta$, since $B\subseteq B(x,2r)$ and $B(x,2r)\subseteq E_\eta(x,Q)$, by the definition of Ω_η and the assertion (b) of Theorem 4.1. Thus hypothesis (i) of Theorem 2.1 is satisfied. We next prove that the map E_η in (4.8) is X-controllable with a hitting time $T\le cr$. We would like to stress that the X-controllability of the map E_η seems to be much easier than that of the maps Φ studied in [NSW]. This is the only reason for which we use the E_η's rather than the Φ's.

Our claim will be a straight consequence of the following lemma.

Lemma 4.2. *Let $E_i:\Omega_i\times Q\to\mathbf{R}^n$, $i=1$, 2, be two maps with continuous first derivatives such that $E_1(x,h)\subseteq\Omega_2$ for any $(x,h)\in\Omega_1\times Q$. Let us assume that E_i is X-controllable with a control function $\gamma_i:\Omega_i\times Q\times[0,T_i]\to\mathbf{R}^n$. Let also assume that $\gamma_i(x,h,\cdot)$ reaches $E_i(x,h)$ in a time $T_i(h)$ independent of x, i.e.*

$$\gamma_i(x,h,T_i(h))=E_i(x,h)\quad \textit{for all } (x,h)\in\Omega_i\times Q.$$

Then

$$E:\Omega_1\times Q\longrightarrow\mathbf{R}^n,\quad E(x,h)=E_2(E_1(x,h),h)$$

is X-controllable with a hitting time $T=T_1+T_2$.

Proof. Let us define $\gamma:\Omega_1\times Q\times[0,T_1+T_2]\to\mathbf{R}^n$ by

$$\gamma(x,h,t)=\begin{cases}\gamma_1(x,h,t), & \text{if } 0\le t\le T_1(h),\\ \gamma_2(E_1(x,h),h,t-T_1(h)), & \text{if } T_1(h)\le t\le T_1(h)+T_2(h),\\ E_2(E_1(x,h),h), & \text{if } T_1(h)+T_2(h)\le t\le T_1+T_2.\end{cases}$$

It is easy to see that γ is a control function for E (note that $x\mapsto E_1(x,h)$ is one-to-one since $x\mapsto\gamma_1(x,h,T_1(h))$ is one-to-one). □

The explicit form of the map $E_\eta(x,h)$ in (4.4) is

$$E_\eta(x,h)=\prod_{j=1}^{M}\exp(|h_{k_j}|^{1/l_{k_j}}\sigma_j X_{r_j})(x),$$

where $M=M(\eta)$ is a suitable integer, $k_j\in\{1,\dots,n\}$, $r_j\in\{1,\dots,m\}$, $\sigma_j=\pm1$ (r_j and σ_j also depend on the sign of h_{k_j}, see (4.1)). Elementary standard results of ordinary differential equation theory show that any map of the form $(x,h)\mapsto\exp(|h_{k_j}|^{1/l_{k_j}}X_{r_j})(x)$ is controllable with hitting time $|h_{k_j}|^{1/l_{k_j}}$. Thus, applying Lemma 4.2, M times, we conclude that the map E_η is X-controllable with a hitting time

$$T\le c\sup_{h\in Q}\|h\|_\eta=\alpha r,$$

where Q is the box defined in (4.7) and α only depends on η, c_1 and c_2. Thus, also the hypothesis (ii) of Theorem 2.1 is satisfied.

Finally we remark that the dilation condition (iii) directly follows from the doubling condition for the d-balls proved in [NSW, Theorem 1]. Then the Poincaré inequality (1.1) holds on the ball B.

5. A pair of non-smooth vector fields of step two

We consider, in $\mathbf{R}^3$, the pair of vector fields $X=(X_1,X_2)$, where $X_j=\partial_{x_j}+a_j\partial_{x_3}$ and a_j is a C^1-function with bounded first derivatives, $j=1$, 2. We assume that

$$p:=X_1a_2-X_2a_1>0 \tag{5.1}$$

at any point of $\mathbf{R}^3$. Since $[X_1,X_2]=p\partial_{x_3}$, (5.1) implies that the vector fields X_1, X_2, and $[X_1,X_2]$ are linearly independent at any point.

Then (5.1) formally implies the Hörmander condition for (X_1,X_2). Nevertheless, due to the "minimal" regularity assumptions on the coefficients a_j, neither Jerison's inequality, nor Nagel–Stein–Wainger's representation theorem can be directly applied to the pair (X_1,X_2). On the contrary, our method naturally works also in this case. The explicit statement of our result is the following.

Theorem 5.1. *Let $X_j=\partial_{x_j}+a_j\partial_{x_3}$, $j=1$, 2, be a pair of C^1 vector fields in $\mathbf{R}^3$. Assume that* (5.1) *holds. Then, for any compact set $K\subset\mathbf{R}^3$ there are positive constants λ, c and R_0 such that*

$$\int_{B\times B}|u(x)-u(y)|\,dx\,dy\le cr|B|\int_{\lambda B}|Xu(x)|\,dx,\quad u\in C^1(\lambda B), \tag{5.2}$$

for any ball $B=B(x_0,r)$ such that $x_0\in K$ and $r<R_0$.

Proof. Let us consider the map $\exp^*(\sigma[X_1,X_2])$, obtained by setting $l=2$ and $S=[X_1,X_2]$ in (4.1). By means of elementary computations we obtain

$$\exp^*(\sigma[X_1,X_2])(x)=(x_1,x_2,x_3+P(x,\sigma)), \tag{5.3}$$

where

$$P(x,\sigma)=p(x)\sigma+R(x,\sigma) \tag{5.4}$$

and

$$\sup_{x\in K}\left(\left|\frac{\partial}{\partial\sigma}R(x,\sigma)\right|+|R(x,\sigma)|\right)\to 0,\quad \text{as } \sigma\to 0, \tag{5.5}$$

for any fixed compact set $K\subset\mathbf{R}^3$.

We now define

$$E(x,h)=\exp^*(h_3[X_1,X_2])\exp(h_2X_2)\exp(h_1X_1)(x),$$

and $Q(r)=\{h\in\mathbf{R}^3:\|h\|<r\}$, where $\|h\|=\max\{|h_1|,\ |h_2|,\ |h_3|^{1/2}\}$. Keeping in mind (5.3) it is easy to see that

$$E(x,h)=\bigg(x_1+h_1, x_2+h_2,$$

$$x_3+\int_0^{h_1} a_1(e^{\tau X_1}(x))\,d\tau+\int_0^{h_2} a_2(e^{\tau X_2}e^{h_1X_1}(x))\,d\tau+P(e^{h_2X_2}e^{h_1X_1}(x),h_3)\bigg). \tag{5.6}$$

Let $K\subset\mathbf{R}^3$ be a compact set. By using (5.1), (5.4), (5.5) and (5.6) we can find three positive constants r_0, δ and μ such that

$$E(x,Q(r_0))\supseteq\{y\in\mathbf{R}^3:|x-y|<\delta\}\quad\text{for any } x\in K \tag{5.7}$$

and

$$|E(x,h)-x|\geq\mu|h|\quad\text{for any } x\in K,\ h\in Q(r_0). \tag{5.8}$$

Since $t\mapsto\exp(tX_j)(x)$ is a subunit path, from the definition of E we have

$$d(x,E(x,h))\leq|h_1|+|h_2|+4|h_3|^{1/2}\leq 6\|h\|. \tag{5.9}$$

Putting together (5.7), (5.8) and (5.9) we get

$$d(x,y)\leq c|x-y|^{1/2}\quad\text{for all } x\in K \text{ and } y\in\mathbf{R}^3 \text{ with } |x-y|\leq\delta,$$

where c only depends on μ and r_0. As a consequence the distance d is well defined and locally Hölder continuous with respect to the euclidean distance.

We next prove the following inclusions:

$$E\big(x,Q\big(\tfrac{1}{6}r\big)\big)\subseteq B(x,r)\subseteq E(x,Q(\theta r)) \tag{5.10}$$

for any $x\in K$ and $0<r\leq r_0$. Here θ denotes a positive constant only depending on K and r_0. The first inclusion in (5.10) is a trivial consequence of (5.9). To prove the second one we argue as follows. Given a point $z\in B(x,r)$, there exists an absolutely continuous path γ on $[0,1]$ satisfying $\gamma(0)=x$, $\gamma(1)=z$, $\dot\gamma(t)=\sum_{j=1}^2 c_j(t)X_j(\gamma(t))$ and

$$|c_j(t)|\leq r\quad\text{for a.e. } t\in[0,1],\ j=1,\ 2. \tag{5.11}$$

The second inclusion in (5.10) will be proved if we show the existence of a point $h\in Q(\theta r)$ such that $E(x,h)=z$. Using (5.6) we see that this equation is satisfied if and only if

$$h_j=\int_0^1 c_j(t)\,dt,\quad j=1,\ 2, \tag{5.12}$$

and

$$(5.13)\qquad \begin{aligned} P(e^{h_2X_2}e^{h_1X_1}(x),h_3) = {} & \sum_{j=1}^{2}\int_0^1 c_j(t)[a_j(\gamma(t))-a_j(x)]\,dt \\ & + \int_0^{h_1}[a_1(x)-a_1(e^{tX_1}(x))]\,dt \\ & + \int_0^{h_2}[a_2(x)-a_2(e^{tX_2}e^{h_1X_1}(x))]\,dt. \end{aligned}$$

The equations (5.12) uniquely determine h_1 and h_2 in such a way that, due to (5.11), $|h_1|,|h_2|\le r$. Moreover, if r is small enough (we may suppose this condition being satisfied if $r\le r_0$), there exists a unique $h_3=h_3(x,h_1,h_2)$ satisfying (5.13). By using (5.4) and the Lipschitz continuity of a_1 and a_2, we obtain $|h_3|\le\theta r^2$ for a suitable $\theta=\theta(K,r_0)>0$. Thus (5.10) holds.

In view of the second inclusion in (5.10) we infer that the set

$$K_0:=\overline{\bigcup_{x\in K}B(x,r_0)}$$

is a compact subset of $\mathbf{R}^3$. Following the argument described before we get

$$(5.14)\qquad B(x,r)\subset E(x,Q(\theta' r))$$

for any $x\in K_0$, $r\le\varrho_0$ (θ' and ϱ_0 are suitable constants and we may assume that $\theta'\ge\theta$ and $\varrho_0\le\frac{1}{2}r_0$). Moreover, we may also assume the existence of a positive constant $a>0$ such that, for any $x\in K_0$, the map $h\mapsto E(x,h)$ is a one-to-one map on the box $Q(2\theta'\varrho_0)$ and

$$(5.15)\qquad \frac{1}{a}\le\left|\det\frac{\partial E}{\partial h}(x,h)\right|\le a \quad \text{for all } x\in K_0 \text{ and } h\in Q(2\theta'\varrho_0).$$

We are now ready to prove the Poincaré inequality (5.2) by using Theorem 2.1. Let $B=B(x_0,r)$ be a d-ball centered at a point $x_0\in K$ and having radius $r\le\varrho_0$. Define now

$$\Omega:=B \quad\text{and}\quad Q:=Q(2\theta' r).$$

By the triangle inequality and (5.14) we get

$$B\subset B(x,2r)\subset E(x,Q)$$

for any $x\in\Omega$. In view of (5.15) the map $E:\Omega\times Q\to\mathbf{R}^n$ is an almost exponential map X-controllable with a hitting time $T\le 12\theta' r$. We explicitly remark that the constant $\alpha{:=}12\theta'$, together with the *type* of E and the constant related to the *control function* (i.e. the constants in (2.1) and (2.2), respectively) are independent on $r \le \varrho_0$.

Finally, by using the inclusions (5.10) and the estimates (5.15), we easily prove the existence of a positive constant β, independent of r, such that $|(\alpha+1)B| \le \beta|B|$ if $0<r<R_0$, where $R_0 \le \varrho_0$ is a suitable constant only depending on θ', a and ϱ_0. Then all the hypotheses of Theorem 2.1 are satisfied and the Poincaré inequality (5.2) holds with $\lambda=\alpha+1$. $\square$

We close the section by giving a motivation for our interest in the vector fields X_1 and X_2 satisfying (5.1). Let f be a real C^2 function defined on an open set $O \subseteq \mathbf{R}^3$. Define the functions

$$a_1 = \frac{f_{x_1} - f_{x_2} f_{x_3}}{1+f_{x_3}^2}, \quad a_2 = -\frac{f_{x_2}+f_{x_1} f_{x_3}}{1+f_{x_3}^2},$$

where $f_{x_i}=\partial f/\partial x_i$, $i=1, 2, 3$. Then the C^1-vector fields $X_j=\partial_{x_j}+a_j\partial_{x_3}$, $j=1, 2$, satisfy (5.1) if and only if the *Levi curvature* of the graph of f is *strictly positive* at any point (see [C, p. 519]). This last property plays an important role in the analysis of the classical and viscosity solutions to the prescribed Levi-curvature equation (cf. [C] and [CLM]).

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