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An example of a real analytic strongly pseudoconvex hypersurface which is not holomorphically equivalent to any algebraic hypersurface

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1. Introduction

A submanifold in $\mathbf{C}^n$ is called algebraic if it can be defined by polynomials. As is clear, any totally real analytic submanifold is locally biholomorphic to a certain coordinates space, hence equivalent to an algebraic manifold. By the work of Moser and Webster [MW], the germ of a real analytic surface $M\subset\mathbf{C}^2$ at an isolated elliptic complex tangent $p_0\in M$ with $0<\lambda<\frac{1}{2}$ (where λ denotes the Bishop invariant), is biholomorphically equivalent to a real algebraic one. However, the corresponding statement is generally false even for CR manifolds in $\mathbf{C}^2$. An explicit example of this type has appeared in a recent survey article of Baouendi–Ebenfelt–Rothschild [BER2, Section 7]: In [BER2], they constructed a real analytic hypersurface M in $\mathbf{C}^2$ and a smooth CR map f from M into an algebraic non-Levi-flat hypersurface such that f is locally biholomorphic away from a certain subset $E\subset M$, but f is not real analytic along E, where E is a non-trivial holomorphic curve. In such an example, for each $p\in M\setminus E$, (M,p) is equivalent to the germ of a strongly pseudoconvex algebraic hypersurface, but for each $p\in E$, by the reflection principle proved in [BHR], (M,p) can not be holomorphically equivalent to the germ of any real algebraic hypersurface. The key feature in this example is the degeneracy of M along E. (See also related examples in [E] and [BHR]).

In this paper, using a different approach, we provide an explicit strongly pseudoconvex hypersurface in $\mathbf{C}^2$, that is not biholomorphically equivalent to any real algebraic manifold in the complex spaces.

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Theorem 1.1. *Let* $M=\{(z,w)\in\mathbf{C}^2:\mathrm{Im}\,w=e^{|z|^2}-1\}$, *which is apparently strongly pseudoconvex and real analytic. But for any* $p\in M$, *the germ of* M *at* p *is not holomorphically equivalent to the germ of any algebraic real hypersurface in* $\mathbf{C}^2$.

Real algebraic hypersurfaces form an important subclass of real analytic hypersurfaces in complex spaces. There are many substantial results in the CR geometry which exclusively hold in this subclass. To name a few of the works in this regard, we mention Webster's algebraic mapping theorem [We], Baouendi–Ebenfelt–Rothschild's algebraicity theorem [BER1] and an algebraic Riemann mapping theorem of the first two authors [HJ]. (See also [BER2] and [H] for a detailed list of references).

Our approach uses CR holomorphic invariant functions. For any strongly pseudoconvex real analytic hypersurface $M\subset\mathbf{C}^2$, we have a projective structure bundle $\mathcal{Y}$ associated with it [C], [F], which will further be parametrized locally by 8 complex variables: z, w, p, u, u_1^1, u^1, v_1 and t. Using the curvature functions L^{11} and P_{11}, we will derive the following Cartan-type holomorphic invariant functions on $\mathcal{Y}$ (see Lemma 4.1 for the explanation of notations):

$$L^{11}_{\omega_1|\omega_1},\quad L^{11}_{\omega_1},\quad L^{11}_{\omega^1},\quad P_{11,\omega_1}\quad\text{and}\quad P_{11,\omega^1}.$$

In case M is rigid, we will show that these *seven* invariant functions depend only on the variables z, p, u, u_1^1, u^1 and v_1. Hence, they would be generically functionally dependent. If M would be locally biholomorphically equivalent to a certain real algebraic hypersurface, there would exist a non-zero polynomial R such that

$$R(L^{11}_{\omega_1|\omega_1},L^{11},L^{11}_{\omega_1},L^{11}_{\omega^1},P_{11},P_{11,\omega_1},P_{11,\omega^1})\equiv 0.$$

Finally when M is in the specific form as in Theorem 1.1, we will get a contradiction to the existence of such an R.

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2. Preliminaries

Let $\mathcal{M}$ be a complex manifold of complex dimension 3. We say that a G-structure $\mathcal{G}$ on $\mathcal{M}$ is admissible, if it is given by holomorphic subbundles $A,B\subset T^*\mathcal{M}$ such that the fiber dimension of A (respectively, B, $A\cap B$) is 2 (respectively, 2, 1).

A basis $(\theta,\theta^1,\theta_1)$ of $T^*\mathcal{M}$ will be called a *coframe* of $\mathcal{G}$ if $\theta\in A\cap B$, $\theta^1\in A$ and $\theta_1\in B$. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two complex manifolds with admissible G-structures $\mathcal{G}_1$ and $\mathcal{G}_2$. A biholomorphic map from $\mathcal{M}_1$ onto $\mathcal{M}_2$ is called a G-isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$, if it preserves the G-structures of M_1 and M_2, respectively. The local isomorphism of G-structures can be defined in a similar way.

Assume that $M=\{(z,w)\in\mathbf{C}^2 : r(z,w,\bar z,\bar w)=0,\ d|_M r\neq 0\}$ is strictly pseudoconvex near $p\in M$. Its Segre family is then the complex three-fold $\mathcal{M}=\{(z,w,\zeta,\eta): r(z,w,\zeta,\eta)=0\}\subset\mathbf{C}^4$. And, as is well known (see [CJ1]–[CJ2]), there is a canonically associated admissible G-structure on $\mathcal{M}$ near the point $(p_0,\bar p_0)$, where A (respectively, B) is generated by dz and dw (respectively, $d\zeta$ and $d\eta$), and $A\cap B$ is generated by $\theta=(\partial r/\partial z)\,dz+(\partial r/\partial w)\,dw$. If $M\subset\mathbf{C}^2$ is biholomorphic to another real analytic hypersurface $M'\subset\mathbf{C}^2$, namely, if there is a local biholomorphism $F=(f,g)$ from an open subset of $\mathbf{C}^2$ onto an open subset of $\mathbf{C}^2$ such that its restriction $F|_M(M)\subset M'$, then it induces a local G-isomorphism from the Segre family $\mathcal{M}$ of M to the associated Segre family $\mathcal{M}'$ of M', through the map $(z,w,\zeta,\eta)\mapsto (f(z,w),g(z,w),\bar f(\zeta,\eta),\bar g(\zeta,\eta))$.

Let M be as above. Assume further $r_w(p_0)\equiv(\partial r/\partial w)|_M(p_0)\neq 0$. Define

$$S:\mathcal{M}\longrightarrow\widetilde{\mathcal{M}}\subset\mathbf{C}^2\times\mathbf{P}^1,\quad (z,w,\zeta,\eta)\longmapsto\left(z,w,\left[\frac{\partial r}{\partial z}:\frac{\partial r}{\partial w}\right](z,w,\zeta,\eta)\right). \tag{2.1}$$

Here we may regard (z,w,ζ) as a local coordinate system for $\mathcal{M}$, and use (z,w,p) as the local coordinate system for $\widetilde{\mathcal{M}}$ with $p=-r_z/r_w$. We can define a unique admissible G-structure bundle over $\widetilde{\mathcal{M}}$ to make S a G-isomorphism, by assigning its coframe along $\widetilde{\mathcal{M}}$ as

$$\theta=dw-p\,dz,\quad \theta^1=dz,\quad \theta_1=dp-p_{11}\,dz, \tag{2.2}$$

where $p_{11}(z,w,p)$ is holomorphic so that $S^*(\theta_1)$ is in the span of $d\zeta$ and $(\partial r/\partial z)\,dz+(\partial r/\partial w)\,dw$. Since $d\theta=\theta^1\wedge\theta_1 \pmod{\theta}$, the coframes, satisfying the normalization condition $d\omega=i\omega^1\wedge\omega_1 \pmod{\omega}$, are in the form of

$$\omega=u\theta,\quad \omega^1=u^1\theta+u_1^1\theta^1,\quad \omega_1=v_1\theta+v_1^1\theta_1, \tag{2.3}$$

where u, u_1^1, u^1 and v_1 are holomorphic functions with $u=iu_1^1v_1^1\neq 0$.

In what follows, we will perform calculations on $\widetilde{\mathcal{M}}$ and its associated bundles. This will greatly simplify the later computation.

Over $\mathcal{M}$, there exists a holomorphic principal bundle $\mathcal{Y}$, called the projective structure bundle. A result of Chern [C] asserts that there is a uniquely determined holomorphic Cartan connection, called *Hachtroudi connection*, which is defined on $\mathcal{Y}$. The Hachtroudi connection is given by the holomorphic 1-forms ω, ω^1,

ω_1, ϕ, ϕ_1^1, ϕ^1, ϕ_1 and ψ defined on $\mathcal{Y}$, which satisfy the structure equations

$$
(2.4)\qquad \begin{cases} d\omega = i\omega^1\wedge\omega_1+\omega\wedge\phi, \\ d\omega^1 = \omega^1\wedge\phi_1^1+\omega\wedge\phi^1, \\ d\omega_1 = \phi_1^1\wedge\omega_1+\omega_1\wedge\phi+\omega\wedge\phi_1, \\ d\phi = i\omega^1\wedge\phi_1+i\phi^1\wedge\omega_1+\omega\wedge\psi, \\ \Phi_1^1 := d\phi_1^1-i\omega_1\wedge\phi^1+2i\phi_1\wedge\omega^1+\frac{1}{2}\psi\wedge\omega=0, \\ \Phi^1 := d\phi^1-\phi\wedge\phi^1-\phi^1\wedge\phi_1^1+\frac{1}{2}\psi\wedge\omega^1=L^{11}\omega\wedge\omega_1, \\ \Phi_1 := d\phi_1-\phi_1^1\wedge\phi_1+\frac{1}{2}\psi\wedge\omega_1=P_{11}\omega\wedge\omega^1, \\ \Psi := d\psi-\phi\wedge\psi-2i\phi^1\wedge\phi_1=H_1\omega\wedge\omega^1+K^1\omega\wedge\omega_1. \end{cases}
$$

Here the functions L^{11}, P_{11}, H_1 and K^1 are called *CR curvature functions*. If we let $(\sigma_1,\dots,\sigma_8)=(\omega,\omega^1,\dots,\psi)$, it is known that $\mathcal{M}$ is locally G-isomorphic to $\mathcal{M}'$ coming from M' if and only if there is a local biholomorphic map $\mathcal{F}$ from $\mathcal{Y}$ onto $\mathcal{Y}'$ such that $\mathcal{F}^*\sigma_j'=\sigma_j$ for all $1\le j\le 8$. In what follows, we denote the push-forward of $\mathcal{Y}$ to $\widetilde{\mathcal{Y}}$ by $\widetilde{\mathcal{Y}}$.

3. CR curvature functions

Theorem 3.1. *Let $\widetilde{\mathcal{M}}$ be as given by (2.1) and let $\widetilde{\mathcal{Y}}$ the corresponding holomorphic principle bundle associated to $\widetilde{\mathcal{M}}$. Keep the notation which we have set up in (2.2) and (2.3). Then besides the three holomorphic 1-forms in (2.3), there exist five more holomorphic 1-forms ϕ, ϕ_1^1, ϕ^1, ϕ_1 and ψ, defined over $\widetilde{\mathcal{Y}}$, with complex variables z, w, p, u, u_1^1, u^1, v_1 and t, where $u, u_1^1\neq 0$ and $(z,w)\approx p_0$. These holomorphic forms are linearly independent, satisfy the structure equations (2.4), and are explicitly given by the formulas*

$$
\phi=-\frac{du}{u}+t\omega+\frac{iv_1}{u}\omega^1-\frac{iu^1}{u}\omega_1,
$$

$$
\phi_1^1=-\frac{du_1^1}{u_1^1}+\frac{iu^1}{u}\omega_1+\left(\frac{2iv_1}{u}+\frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\omega^1+\left(\frac{t}{2}-\frac{3iu^1v_1}{2u^2}-\frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p}+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega,
$$

$$
\begin{aligned}
\phi^1=&-\frac{du^1}{u}-\frac{u^1}{u}\phi_1^1+\left(\frac{t}{2}+\frac{3iu^1v_1}{2u^2}+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega^1 \\
&+\left(\frac{tu^1}{2u}-\frac{i(u^1)^2v_1}{2u^3}-\frac{u^1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2}+\frac{u_1^1}{6u^2}\frac{\partial^3 p_{11}}{\partial p^3}\right)\omega,
\end{aligned}
$$

$$\phi_1 = -\frac{dv_1}{u} - \frac{v_1}{u}\phi + \frac{v_1}{u}\phi_1^1 + \frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega^1 + \left(\frac{t}{2} - \frac{3iu^1v_1}{2u^2} - \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega_1$$
$$+\left[\frac{v_1 t}{2u} + \frac{iu^1(v_1)^2}{2u^3} - \frac{iu^1}{u(u_1^1)^2}\frac{\partial p_{11}}{\partial w} + \frac{v_1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{2i}{3uu_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w}\right.$$
$$\left. - \frac{i}{6uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial p^2 \partial z} + p_{11}\frac{\partial^3 p_{11}}{\partial p^3} + p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\right]\omega,$$
$$\psi = -dt + t\phi + \frac{iv_1}{u}\phi^1 - \frac{iu^1}{u}\phi_1 + \left(\frac{iu^1 t}{2u} - \frac{(u^1)^2 v_1}{2u^3} + \frac{iu^1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} - \frac{1}{6uv_1^1}\frac{\partial^3 p_{11}}{\partial p^3}\right)\omega_1$$
$$+\left[-\frac{iv_1 t}{2u} - \frac{u^1(v_1)^2}{2u^3} + \frac{u^1}{u(u_1^1)^2}\frac{\partial p_{11}}{\partial w} + \frac{iv_1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} - \frac{2}{3uu_1^1}\frac{\partial^2 p_{11}}{\partial p \partial w}\right.$$
$$\left. + \frac{1}{6uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z} + p_{11}\frac{\partial^3 p_{11}}{\partial p^3} + p\frac{\partial^3 p_{11}}{\partial p^2 \partial w}\right)\right]\omega^1$$
$$+\left[-\frac{t^2}{2} + \frac{(u^1)^2(v_1)^2}{2u^4} - \frac{iu^1 v_1}{2u^3}\frac{\partial^2 p_{11}}{\partial p^2} - \frac{(u^1)^2}{u^2(u_1^1)^2}\frac{\partial p_{11}}{\partial w} + \frac{4u^1}{3u^2 u_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w}\right.$$
$$- \frac{u^1}{3u^2u_1^1}\frac{\partial^3 p_{11}}{\partial p^2 \partial z} + \left(\frac{iu_1^1 v_1}{3u^3} - \frac{u^1 p_{11}}{3u^2 u_1^1}\right)\frac{\partial^3 p_{11}}{\partial p^3} - \frac{u^1 p}{3u^2 u_1^1}\frac{\partial^3 p_{11}}{\partial p^2 \partial w} - \frac{1}{8u^2}\left(\frac{\partial^2 p_{11}}{\partial p^2}\right)^2$$
$$\left. + \frac{1}{3u^2}\frac{\partial p_{11}}{\partial p}\frac{\partial^3 p_{11}}{\partial p^3} - \frac{1}{2u^2}\frac{\partial^3 p_{11}}{\partial w \partial p^2} + \frac{1}{3u^2}\frac{\partial^4 p_{11}}{\partial z \partial p^3} + \frac{p_{11}}{3u^2}\frac{\partial^4 p_1}{\partial p^4} + \frac{p}{3u^2}\frac{\partial^4 p_{11}}{\partial w \partial p^3}\right]\omega.$$

Moreover, the CR curvature functions are given by

$$L^{11} = -\frac{i(u_1^1)^2}{6u^3}\frac{\partial^4 p_{11}}{\partial p^4},$$
$$P_{11} = \frac{i}{u(u_1^1)^2}\left[\frac{\partial^2 p_{11}}{\partial w^2} - \frac{1}{2}\frac{\partial p_{11}}{\partial w}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{2}{3}\frac{\partial p_{11}}{\partial p}\frac{\partial^2 p_{11}}{\partial p \partial w} + \frac{p_{11}}{6}\frac{\partial^3 p_{11}}{\partial p^2 \partial w}\right.$$
$$- \frac{1}{6}\frac{\partial p_{11}}{\partial p}\left(\frac{\partial^3 p_{11}}{\partial p^2 \partial z} + p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right) - \frac{2}{3}\left(\frac{\partial^3 p_{11}}{\partial z \partial w \partial p} + p_{11}\frac{\partial^3 p_{11}}{\partial p^2 \partial w} + p\frac{\partial^3 p_{11}}{\partial p \partial w^2}\right)$$
$$+ \frac{1}{6}\left(\frac{\partial^4 p_{11}}{\partial p^2 \partial z^2} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3 \partial z} + p\frac{\partial^4 p_{11}}{\partial p^2 \partial z \partial w}\right) + \frac{1}{6}\frac{\partial^3 p_{11}}{\partial p^3}\left(\frac{\partial p_{11}}{\partial z} + p\frac{\partial p_{11}}{\partial w}\right)$$
$$+ \frac{p_{11}}{6}\left(\frac{\partial^4 p_{11}}{\partial z \partial p^3} + p_{11}\frac{\partial^4 p_{11}}{\partial p^4} + p\frac{\partial^4 p_{11}}{\partial p^3 \partial w}\right)$$
$$\left. + \frac{p}{6}\left(\frac{\partial^4 p_{11}}{\partial z \partial p^2 \partial w} + p_{11}\frac{\partial^4 p_{11}}{\partial p^3 \partial w} + p\frac{\partial^4 p_{11}}{\partial p^2 \partial w^2}\right)\right],$$
$$K^1 = \frac{2}{u_1^1}\left(\frac{\partial L^{11}}{\partial z} + p_{11}\frac{\partial L^{11}}{\partial p} + p\frac{\partial L^{11}}{\partial w}\right) + 2iv_1\frac{\partial L^{11}}{\partial u} + 2u_1^1\left(\frac{2iv_1}{u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\frac{\partial L^{11}}{\partial u},$$
$$H_1 = \frac{2iu_1^1}{u}\frac{\partial P_{11}}{\partial p} - 2iu^1\frac{\partial P_{11}}{\partial u} + \frac{2iu^1u_1^1}{u}\frac{\partial P_{11}}{\partial u_1^1}.$$

Proof. These forms are constructed by applying a similar, but complicated, process for getting (2.4) as in [CM], [C]. Here, for the convenience of the reader, we verify, in certain details, that the connection forms constructed satisfy the structure equations in (2.4). This is good enough to complete the proof of the theorem, by the uniqueness property of connection forms.

For simplicity, we write

$$\begin{cases} \phi^1 = -\dfrac{du^1}{u} - \dfrac{u^1}{u}\phi_1^1 + A^1\omega^1 + A\omega, \\ \phi_1 = -\dfrac{dv_1}{u} - \dfrac{v_1}{u}\phi + \dfrac{v_1}{u}\phi_1^1 + \dfrac{i}{(u_1^1)^2}\dfrac{\partial p_{11}}{\partial w}\omega^1 + B_1\omega_1 + B\omega, \\ \psi = -dt + t\phi + \dfrac{iv_1}{u}\phi^1 - \dfrac{iu^1}{u}\phi_1 + C_1\omega_1 + C^1\omega^1 + C\omega, \end{cases}$$

where A^1, A, B_1, B, C, C^1 and C_1 denote the corresponding coefficients in the formulas.

It is easy to see that the first three equations in (2.4) hold. Let us first verify the fourth identity in (2.4). In fact

$$\begin{aligned} d\phi &= d\left(-\frac{du}{u} + t\omega + \frac{iv_1}{u}\omega^1 - \frac{iu^1}{u}\omega_1\right) \\ &= dt\wedge\omega + t\,d\omega + i\frac{u\,dv_1 - v_1\,du}{u^2}\wedge\omega^1 + \frac{iv_1}{u}d\omega^1 - i\frac{u\,du^1 - u^1\,du}{u^2}\wedge\omega_1 - \frac{iu^1}{u}d\omega_1 \\ &= \left(C^1 - iB + \frac{iv_1 t}{u}\right)\omega^1\wedge\omega + + \left(C_1 + iA - \frac{iu^1 t}{u}\right)\omega_1\wedge\omega \\ &\quad + (it - iB_1 - iA^1)\omega^1\wedge\omega_1 + i\omega^1\wedge\phi_1 + i\phi^1\wedge\omega_1 + \omega\wedge\psi \\ &= i\omega^1\wedge\phi_1 + i\phi^1\wedge\omega_1 + \omega\wedge\psi. \end{aligned}$$

Next we verify the fifth identity in (2.4),

$$\begin{aligned} d\phi_1^1 &= d\Bigg[-\frac{du_1^1}{u_1^1} + \frac{iu^1}{u}\omega_1 + \left(\frac{2iv_1}{u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\omega^1 \\ &\quad + \left(\frac{t}{2} - \frac{3iu^1 v_1}{2u^2} - \frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega\Bigg] \\ &= \mathrm{id}\left(\frac{u^1}{u}\right)\wedge\omega_1 + \frac{iu^1}{u}d\omega_1 + d\left(\frac{2iv_1}{u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\wedge\omega^1 \\ &\quad + \left(\frac{2iv_1}{u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)d\omega^1 + d\left(\frac{t}{2} - \frac{3iu^1 v_1}{2u^2} - \frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge\omega \\ &\quad + \left(\frac{t}{2} - \frac{3iu^1 v_1}{2u^2} - \frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge d\omega \end{aligned}$$

$$
\begin{aligned}
&= \left(iA^1 - 2iB_1 + \frac{9u^1v_1}{2u^2} - \frac{3i}{4u}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{it}{2}\right)\omega^1\wedge\omega_1 \\
&\quad + \left(iA + \frac{3iu^1}{2u}B_1 - \frac{1}{2}C_1 - \frac{iu^1t}{u} - \frac{3(u^1)^2v_1}{u^3} + \frac{3iu^1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} - \frac{1}{4uv_1^1}\frac{\partial^3 p_{11}}{\partial p^3}\right)\omega\wedge\omega_1 \\
&\quad + \left[2iB - \frac{1}{2}C^1 + \left(\frac{3iv_1}{2u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)A^1 - \frac{3u^1}{2u(u_1^1)^2}\frac{\partial p_{11}}{\partial w} + \frac{3u^1(v_1)^2}{u^3} - \frac{2iv_1t}{u}\right. \\
&\qquad - \frac{\partial p_{11}}{\partial p}\left(\frac{3iu^1v_1}{2u^2u_1^1} + \frac{t}{2u_1^1}\right) - \frac{1}{4uu_1^1}\frac{\partial p_{11}}{\partial p}\frac{\partial^2 p_{11}}{\partial p^2} - \frac{3iv_1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{1}{uu_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w} \\
&\qquad \left. - \frac{1}{4uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial z\partial p^2} + p_{11}\frac{\partial^3 p_{11}}{\partial p^3} + p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\right]\omega\wedge\omega^1 + i\omega_1\wedge\phi^1 - 2i\phi_1\wedge\omega^1 - \frac{1}{2}\psi\wedge\omega \\
&= i\omega_1\wedge\phi^1 - 2i\phi_1\wedge\omega^1 - \tfrac{1}{2}\psi\wedge\omega.
\end{aligned}
$$

We check the sixth identity $\Phi^1 = L^{11}\omega\wedge\omega_1$ in (2.4) as follows,

$$
\begin{aligned}
\Phi^1 &= d\phi^1 - \phi\wedge\phi^1 - \phi^1\wedge\phi_1^1 + \tfrac{1}{2}\psi\wedge\omega^1 \\
&= d\left[-\frac{du^1}{u} - \frac{u^1}{u}\phi_1^1 + \left(\frac{tu^1}{2u} - \frac{i(u^1)^2v_1}{2u^3} - \frac{u^1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2} + \frac{u_1^1}{6u^2}\frac{\partial^3 p_{11}}{\partial p^3}\right)\omega\right. \\
&\quad \left. + \left(\frac{t}{2} + \frac{3iu^1v_1}{2u^2} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega^1\right] - \phi\wedge\phi^1 - \phi^1\wedge\phi_1^1 + \frac{1}{2}\psi\wedge\omega^1 \\
&= \left(-\phi^1 - \frac{u^1}{u}\phi_1^1 + A\omega + A^1\omega^1\right) \\
&\quad \wedge\left[\left(\phi - t\omega - \frac{iv_1}{u}\omega^1 + \frac{iu^1}{u}\omega_1\right) - \phi_1^1 + \frac{t}{2}\omega - \frac{iu^1v_1}{u^2}\omega - \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\omega + \frac{3iv_1}{2u}\omega^1\right] \\
&\quad + \left(-\psi + t\phi + \frac{iv_1}{u}\phi^1 - \frac{iu^1}{u}\phi_1 + C_1\omega_1 + C^1\omega^1 + C\omega\right)\wedge\left(\frac{u^1}{2u}\omega + \frac{1}{2}\omega^1\right) \\
&\quad + \left(-\phi_1 - \frac{v_1}{u}\phi + \frac{v_1}{u}\phi_1^1 + \frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega^1 + B_1\omega_1 + B\omega\right)\wedge\left(-\frac{i(u^1)^2}{2u^2}\omega + \frac{3iu^1}{2u}\omega^1\right) \\
&\quad + \left(-\phi + t\omega + \frac{iv_1}{u}\omega^1 - \frac{iu^1}{u}\omega_1\right)\wedge\left(\frac{u^1}{u}\phi_1^1 - \frac{tu^1}{2u}\omega + \frac{3i(u^1)^2v_1}{2u^3}\omega + \frac{u^1}{2u^2}\frac{\partial^2 p_{11}}{\partial p^2}\omega\right. \\
&\qquad \left. - \frac{u_1^1}{3u^2}\frac{\partial^3 p_{11}}{\partial p^3}\omega - \frac{3iu^1v_1}{u^2}\omega^1 - \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\omega^1\right) \\
&\quad + \left[-\phi_1^1 + \frac{iu^1}{u}\omega_1 + \left(\frac{2iv_1}{u} + \frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\omega^1\right. \\
&\qquad \left. + \left(\frac{t}{2} - \frac{3iu^1v_1}{2u^2} - \frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega\right]\wedge\frac{u_1^1}{6u^2}\frac{\partial^3 p_{11}}{\partial p^3}\omega \\
&\quad - \frac{u^1}{u}\left(i\omega_1\wedge\phi^1 - 2i\phi_1\wedge\omega^1 - \frac{1}{2}\psi\wedge\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{tu^1}{2u}-\frac{i(u^1)^2v_1}{2u^3}-\frac{u^1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2}+\frac{u_1^1}{6u^2}\frac{\partial^3 p_{11}}{\partial p^3}\right)(i\omega^1\wedge\omega_1+\omega\wedge\phi)\\
&+\left(\frac{t}{2}+\frac{3iu^1v_1}{2u^2}+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)(\omega^1\wedge\phi_1^1+\omega\wedge\phi^1)-\phi\wedge\phi^1-\phi^1\wedge\phi_1^1+\frac{1}{2}\psi\wedge\omega^1\\
&-\frac{u^1}{4u^2}d\left(\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge\omega+\frac{u_1^1}{6u^2}d\left(\frac{\partial^3 p_{11}}{\partial p^3}\right)\wedge\omega+\frac{1}{4u}d\left(\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge\omega^1\\
=\,&L^{11}\omega\wedge\omega_1.
\end{aligned}
$$

We continue to check the seventh identity $\Phi_1=P_{11}\omega\wedge\omega^1$ in (2.4),

$$
\begin{aligned}
\Phi_1 =\,& d\phi_1-\phi_1^1\wedge\phi_1+\tfrac{1}{2}\psi\wedge\omega_1\\
=\,& d\left(-\frac{dv_1}{u}-\frac{v_1}{u}\phi+\frac{v_1}{u}\phi_1^1+\frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega^1+B_1\omega_1+B\omega\right)-\phi_1^1\wedge\phi_1+\frac{1}{2}\psi\wedge\omega_1\\
=\,& \left(-\phi_1-\frac{v_1}{u}\phi+\frac{v_1}{u}\phi_1^1+\frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega^1+B_1\omega_1+B\omega\right)\\
&\wedge\left(\phi-t\omega-\frac{iv_1}{u}\omega^1+\frac{iu^1}{u}\omega_1-\phi+\phi_1^1-\frac{3iu^1}{2u}\omega_1+\frac{t}{2}\omega+\frac{iu^1v_1}{u^2}\omega+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\omega\right)\\
&+\left(-\phi+t\omega+\frac{iv_1}{u}\omega^1-\frac{iu^1}{u}\omega_1\right)\\
&\wedge\left[\frac{v_1}{u}\phi-\frac{v_1}{u}\phi_1^1+\frac{3iu^1v_1}{u^2}\omega_1+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\omega_1-\frac{v_1t}{2u}\omega-\frac{3iu^1(v_1)^2}{2u^3}\omega+\frac{iu^1}{u(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega\right.\\
&\left.-\frac{v_1}{2u^2}\frac{\partial^2 p_{11}}{\partial p^2}\omega-\frac{2i}{3uu_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w}\omega+\frac{i}{6uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z}+p_{11}\frac{\partial^3 p_{11}}{\partial p^3}+p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\omega\right]\\
&+\left[-\phi_1^1+\frac{iu^1}{u}\omega_1+\left(\frac{2iv_1}{u}+\frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\omega^1\right.\\
&\left.\quad+\left(\frac{t}{2}-\frac{3iu^1v_1}{2u^2}-\frac{u^1}{uu_1^1}\frac{\partial p_{11}}{\partial p}+\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega\right]\\
&\wedge\left[-\frac{2i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega^1+\frac{2iu^1}{u(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega-\frac{2i}{3uu_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w}\omega\right.\\
&\left.\quad+\frac{i}{6uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z}+p_{11}\frac{\partial^3 p_{11}}{\partial p^3}+p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\omega\right]\\
&+\left(-\psi+t\phi+\frac{iv_1}{u}\phi^1-\frac{iu^1}{u}\phi_1+C_1\omega_1+C^1\omega^1+C\omega\right)\wedge\left(\frac{1}{2}\omega_1+\frac{v_1}{2u}\omega\right)\\
&+\left(-\phi^1-\frac{u^1}{u}\phi_1^1+A\omega+A^1\omega^1\right)\wedge\left(-\frac{3iv_1}{2u}\omega_1+\frac{i(v_1)^2}{2u^2}\omega-\frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}\omega\right)\\
&-\frac{v_1}{u}\left(i\omega^1\wedge\phi_1+i\phi^1\wedge\omega_1+\omega\wedge\psi\right)+\frac{v_1}{u}\left(i\omega_1\wedge\phi^1-2i\phi_1\wedge\omega^1-\frac{1}{2}\psi\wedge\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{(u_1^1)^2}\frac{\partial p_{11}}{\partial w}(\omega^1\wedge\phi_1^1+\omega\wedge\phi^1)\\
&+\left(\frac{t}{2}-\frac{3iu^1v_1}{2u^2}-\frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge(\phi_1^1\wedge\omega_1+\omega_1\wedge\phi+\omega\wedge\phi_1)\\
&+\left[\frac{v_1t}{2u}+\frac{iu^1(v_1)^2}{2u^3}-\frac{iu^1}{u(u_1^1)^2}\frac{\partial p_{11}}{\partial w}+\frac{v_1}{4u^2}\frac{\partial^2 p_{11}}{\partial p^2}+\frac{2i}{3uu_1^1}\frac{\partial^2 p_{11}}{\partial p\partial w}\right.\\
&\left.\quad-\frac{i}{6uu_1^1}\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z}+p_{11}\frac{\partial^3 p_{11}}{\partial p^3}+p\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\right](i\omega^1\wedge\omega_1+\omega\wedge\phi)\\
&-\phi_1^1\wedge\phi_1+\frac{1}{2}\psi\wedge\omega_1+\frac{i}{(u_1^1)^2}d\left(\frac{\partial p_{11}}{\partial w}\right)\wedge\omega^1\\
&-\frac{1}{4u}d\left(\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge\omega_1-\frac{iu^1}{u(u_1^1)^2}d\left(\frac{\partial p_{11}}{\partial w}\right)\wedge\omega+\frac{v_1}{4u^2}d\left(\frac{\partial^2 p_{11}}{\partial p^2}\right)\wedge\omega\\
&+\frac{2i}{3uu_1^1}d\left(\frac{\partial^2 p_{11}}{\partial p\partial w}\right)\wedge\omega-\frac{i}{6uu_1^1}d\left(\frac{\partial^3 p_{11}}{\partial p^2\partial z}\right)\wedge\omega-\frac{i}{6uu_1^1}\frac{\partial^3 p_{11}}{\partial p^3}dp_{11}\wedge\omega\\
&-\frac{ip_{11}}{6uu_1^1}d\left(\frac{\partial^3 p_{11}}{\partial p^3}\right)\wedge\omega-\frac{i}{6uu_1^1}\frac{\partial^3 p_{11}}{\partial p^2\partial w}dp\wedge\omega-\frac{ip}{6uu_1^1}d\left(\frac{\partial^3 p_{11}}{\partial p^2\partial w}\right)\wedge\omega\\
=&\,P_{11}\omega\wedge\omega^1.
\end{aligned}
$$

Finally, we verify the last equation in (2.4). Taking differential on the fifth equation of (2.4), $\Phi_1^1=0$, we obtain

$$d\psi=2i\phi^1\wedge\phi_1+\phi\wedge\psi+\varrho\omega\wedge\omega^1+\varkappa\omega\wedge\omega_1+\chi\wedge\omega.$$

Taking differential on the sixth equation of (2.4): $\Phi^1=L^{11}\omega\wedge\omega_1$, we get $\chi=0$, and

$$
\begin{aligned}
\varkappa&=2(L^{11})_{\omega_1}\\
&=\frac{2}{u_1^1}\left(\frac{\partial L^{11}}{\partial z}+p_{11}\frac{\partial P_{11}}{\partial p}+p\frac{\partial L^{11}}{\partial w}\right)+2iv_1\frac{\partial L^{11}}{\partial u}+2u_1^1\left(\frac{2iv_1}{u}+\frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\frac{\partial L^{11}}{\partial u}\\
&=K_1.
\end{aligned}
$$

Taking the exterior differentiation on the seventh equation in (2.4), we obtain

$$\varrho=2(dP_{11})_{\omega_1}=\frac{2iu_1^1}{u}\frac{\partial P_{11}}{\partial p}-2iu^1\frac{\partial P_{11}}{\partial u}+\frac{2iu^1u_1^1}{u}\frac{\partial P_{11}}{\partial u_1^1}=H_1.$$

So the last equation in (2.4) holds. □

Lemma 3.2. *Let $h(z,p,u,u^1_1)$ be any differentiable function on $\widetilde{\mathcal{Y}}$. Then it holds that*

$$dh = -u\frac{\partial h}{\partial u}\phi - u^1_1\frac{\partial h}{\partial u^1_1}\phi^1_1 + \left[\frac{iu^1_1}{u}\frac{\partial h}{\partial p} - iu^1\frac{\partial h}{\partial u} + \frac{iu^1u^1_1}{u}\frac{\partial h}{\partial u^1_1}\right]\omega_1$$
$$+\left[\frac{1}{u^1_1}\left(\frac{\partial h}{\partial z} + p_{11}\frac{\partial h}{\partial p}\right) + iv_1\frac{\partial h}{\partial u} + u^1_1\left(\frac{2iv_1}{u} + \frac{1}{u^1_1}\frac{\partial p_{11}}{\partial p}\right)\frac{\partial h}{\partial u^1_1}\right]\omega^1$$
$$+\left[\left(-\frac{iu^1_1v_1}{u^2} - \frac{u^1p_{11}}{uu^1_1}\right)\frac{\partial h}{\partial p} - \frac{u^1}{uu^1_1}\frac{\partial h}{\partial z} + ut\frac{\partial h}{\partial u}\right.$$
$$\left. + u^1_1\left(\frac{t}{2} - \frac{3iu^1v_1}{2u^2} - \frac{u^1}{uu^1_1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\frac{\partial h}{\partial u^1_1}\right]\omega.$$

Proof. First, from (2.3) we can easily derive

$$dz = \frac{1}{u^1_1}\omega^1 - \frac{u^1}{u^1_1u}\omega,$$
$$dp = \frac{iu^1_1}{u}\omega_1 + \frac{p_{11}}{u^1_1}\omega^1 + \left(-\frac{v_1}{uv^1_1} - \frac{u^1p_{11}}{uu^1_1}\right)\omega.$$

Applying the formulas in Theorem 3.1, we then also see that

$$du = u\left(-\phi + t\omega + \frac{iv_1}{u}\omega^1 - \frac{iu^1}{u}\omega_1\right),$$
$$du^1_1 = u^1_1\left[-\phi^1_1 + \frac{iu^1}{u}\omega_1 + \left(\frac{2iv_1}{u} + \frac{1}{u^1_1}\frac{\partial p_{11}}{\partial p}\right)\omega^1\right.$$
$$\left. + \left(\frac{t}{2} - \frac{3iu^1v_1}{2u^2} - \frac{u^1}{uu^1_1}\frac{\partial p_{11}}{\partial p} + \frac{1}{4u}\frac{\partial^2 p_{11}}{\partial p^2}\right)\omega\right].$$

Now, to conclude the proof of the lemma, it suffices to substitute the above to

$$dh = \frac{\partial h}{\partial z}dz + \frac{\partial h}{\partial p}dp + \frac{\partial h}{\partial u}du + \frac{\partial h}{\partial u^1_1}du^1_1. \quad \square$$

4. Calculation of invariant functions

For any differentiable function h on the projective bundle $\widetilde{\mathcal{Y}}$, there is a unique representation for its differential dh into the covariant differentials in terms of the connection forms in Theorem 3.1,

$$dh = h_\omega\omega + h_{\omega_1}\omega_1 + h_{\omega^1}\omega^1 + h_\phi\phi + h_{\phi^1_1}\phi^1_1 + h_{\phi^1}\phi^1 + h_{\phi_1}\phi_1 + h_\psi\psi.$$

Using the CR curvature functions L^{11} and P_{11} defined in Theorem 3.1, we can get five more invariant holomorphic functions from their first and second covariant differentials,

$$L^{11}_{\omega^1},\quad L^{11}_{\omega_1}|_{\omega_1},\quad P_{11,\omega_1},\quad P_{11}\quad\text{and}\quad P_{11,\omega^1}.$$

Notice that when r is Nash algebraic, all these holomorphic invariant functions become algebraic too, by the way they were constructed from the defining function.

Lemma 4.1. *Let M be defined as in Theorem* 1.1. *Then we have the following formulas for the above invariant functions*:

$$\begin{aligned}
L^{11} &= \frac{(u_1^1)^2 A_1(z,\zeta)}{u^3 e^{3z\zeta}(1+z\zeta)^7},\\
P_{11} &= \frac{A_2(z,\zeta)}{u(u_1^1)^2 e^{z\zeta}(1+z\zeta)^7},\\
L^{11}_{\omega_1} &= \frac{(u_1^1)^3 B_3(z,\zeta)}{u^4 e^{4z\zeta}(1+z\zeta)^9}+\frac{u^1(u_1^1)^2 A_3(z,\zeta)}{u^4 e^{3z\zeta}(1+z\zeta)^7},\\
L^{11}_{\omega^1} &= \frac{u_1^1 B_4(z,\zeta)}{u^3 e^{3z\zeta}(1+z\zeta)^9}+\frac{v_1(u_1^1)^2 A_4(z,\zeta)}{u^4 e^{3z\zeta}(1+z\zeta)^7},\\
L^{11}_{\omega_1|\omega_1} &= \frac{(u_1^1)^4 C_5(z,\zeta)}{u^5 e^{5z\zeta}(1+z\zeta)^{11}}+\frac{u^1(u_1^1)^3 B_5(z,\zeta)}{u^5 e^{4z\zeta}(1+z\zeta)^9}+\frac{(u^1)^2(u_1^1)^2 A_5(z,\zeta)}{u^5 e^{3z\zeta}(1+z\zeta)^7},\\
P_{11,\omega_1} &= \frac{B_6(z,\zeta)}{u^2 u_1^1 e^{2z\zeta}(1+z\zeta)^9}+\frac{u^1 A_6(z,\zeta)}{u^2(u_1^1)^2 e^{z\zeta}(1+z\zeta)^7},\\
P_{11,\omega^1} &= \frac{B_7(z,\zeta)}{u(u_1^1)^3 e^{z\zeta}(1+z\zeta)^9}+\frac{v_1 A_7(z,\zeta)}{u^2(u_1^1)^2 e^{z\zeta}(1+z\zeta)^7}.
\end{aligned}$$

Here, A_j, B_j and C_j are polynomials in (z,ζ) with $A_j(\chi,\chi)=D_j\chi^2+o(\chi^2)$ and $D_j\neq 0$.

Proof. When M is as in Theorem 1.1, then $r(z,w,\zeta,\eta)=2i(e^{z\zeta}-1)+\eta-w$. A simple calculation shows that

$$p=2i\zeta e^{z\zeta},\quad p_{11}=2i\zeta^2 e^{z\zeta}. \tag{4.1}$$

In particular, we get $p_{11}(z,\zeta)=\zeta p(z,\xi)$; and for any integer $k\geq 1$, we have

$$\frac{\partial^k p_{11}}{\partial p^k}=k\frac{\partial^{k-1}\zeta}{\partial p^{k-1}}+p\frac{\partial^k\zeta}{\partial p^k}.$$

Applying the differential operator $\partial/\partial p$ to (4.1), we find

$$\begin{aligned}
\frac{\partial\zeta}{\partial z} &= -\frac{\zeta^2}{1+z\zeta},\\
\frac{\partial\zeta}{\partial p} &= \frac{1}{2ie^{z\zeta}(1+z\zeta)},
\end{aligned}$$

$$\frac{\partial^2\zeta}{\partial p^2}=\frac{2z+z^2\zeta}{4e^{2z\zeta}(1+z\zeta)^3},$$
$$\frac{\partial^3\zeta}{\partial p^3}=\frac{-9z^2-8z^3\zeta-2z^4\zeta^2}{8ie^{3z\zeta}(1+z\zeta)^5},$$
$$\frac{\partial^4\zeta}{\partial p^4}=-\frac{64z^3+79z^4\zeta+36z^5\zeta^2+6z^6\zeta^3}{16e^{4z\zeta}(1+z\zeta)^7}.$$

Hence

$$\frac{\partial p_{11}}{\partial p}=\zeta+p\frac{\partial\zeta}{\partial p}=\frac{2\zeta+z\zeta^2}{1+z\zeta},$$
$$\frac{\partial p_{11}}{\partial z}=-\frac{2i\zeta^3e^{z\zeta}}{1+z\zeta},$$
$$\frac{\partial^2 p_{11}}{\partial p^2}=\frac{2+2z\zeta+z^2\zeta^2}{2ie^{z\zeta}(1+z\zeta)^3},$$
$$\frac{\partial^2 p_{11}}{\partial z\partial p}=-\frac{3\zeta^2+3z\zeta^3+z^2\zeta^4}{(1+z\zeta)^3},$$
$$\frac{\partial^2 p_{11}}{\partial z^2}=2ie^{z\zeta}\frac{3\zeta^4+2z\zeta^5}{(1+z\zeta)^3},$$
$$\frac{\partial^3 p_{11}}{\partial p^3}=\frac{6z+6z^2\zeta+4z^3\zeta^2+z^4\zeta^3}{4e^{2z\zeta}(1+z\zeta)^5},$$
$$\frac{\partial^3 p_{11}}{\partial p^2\partial z}=-\frac{6\zeta+6z\zeta^2+4z^2\zeta^3+z^3\zeta^4}{2ie^{z\zeta}(1+z\zeta)^5},$$
$$\frac{\partial^3 p_{11}}{\partial p\partial z^2}=\frac{12\zeta^3+16z\zeta^4+9z^2\zeta^5+2z^3\zeta^6}{(1+z\zeta)^5},$$
$$\frac{\partial^4 p_{11}}{\partial p^4}=-\frac{36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4}{8ie^{3z\zeta}(1+z\zeta)^7},$$
$$\frac{\partial^4 p_{11}}{\partial^3 p\partial z}=\frac{6-18z\zeta-18z^2\zeta^2-14z^3\zeta^3-6z^4\zeta^4-z^5\zeta^5}{4e^{2z\zeta}(1+z\zeta)^7},$$
$$\frac{\partial^4 p_{11}}{\partial p^2\partial z^2}=\frac{36\zeta^2+40z\zeta^3+29z^2\zeta^4+12z^3\zeta^5+2z^4\zeta^6}{2ie^{z\zeta}(1+z\zeta)^7}.$$

Now applying Theorem 3.1 and the above data, we get

$$L^{11}=\frac{(u_1^1)^2A_1(z,\zeta)}{u^3e^{3z\zeta}(1+z\zeta)^7},$$

with $A_1(z,\xi)=36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4$.

Notice that p_{11} depends only on (z,ζ). Applying Theorem 3.1, we get

$$P_{11}=\frac{i}{u(u_1^1)^2}\left[-\frac{1}{6}\frac{\partial p_{11}}{\partial p}\frac{\partial^3 p_{11}}{\partial p^2\partial z}+\frac{1}{6}\frac{\partial^4 p_{11}}{\partial p^2\partial z^2}+\frac{p_{11}}{3}\frac{\partial^4 p_{11}}{\partial p^3\partial z}+\frac{1}{6}\frac{\partial^3 p_{11}}{\partial p^3}\frac{\partial p_{11}}{\partial z}+\frac{p_{11}^2}{6}\frac{\partial^4 p_{11}}{\partial p^4}\right].$$

Hence, applying the data we just obtained, we can get

$$P_{11}=\frac{A_2(z,\zeta)}{u(u_1^1)^2e^{z\zeta}(1+z\zeta)^7},$$

with $A_2(z,\zeta)=36\zeta^2+112z\zeta^3+145z^2\zeta^4+110z^3\zeta^5+55z^4\zeta^6+16z^5\zeta^7+2z^6\zeta^8$. Applying the just obtained formulas for L^{11} and P_{11}, we have

$$\frac{\partial L^{11}}{\partial z}=\frac{(u_1^1)^2(72z-96z^2\zeta-120z^3\zeta^2-96z^4\zeta^3-53z^5\zeta^4-16z^6\zeta^5-2z^7\zeta^6)}{48u^3e^{3z\zeta}(1+z\zeta)^9},$$

$$\frac{\partial L^{11}}{\partial p}=\frac{(u_1^1)^2(-320z^3-410z^4\zeta-316z^5\zeta^2-163z^6\zeta^3-48z^7\zeta^4-6z^8\zeta^5)}{96iu^3e^{4z\zeta}(1+z\zeta)^9},$$

$$\frac{\partial L^{11}}{\partial u}=-\frac{(u_1^1)^2(36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4)}{16u^4e^{3z\zeta}(1+z\zeta)^7},$$

$$\frac{\partial L^{11}}{\partial u_1^1}=\frac{u_1^1(36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4)}{24u^3e^{3z\zeta}(1+z\zeta)^7},$$

$$\frac{1}{u_1^1}\left(\frac{\partial L^{11}}{\partial z}+p_{11}\frac{\partial L^{11}}{\partial p}\right)$$
$$=\frac{u_1^1(72z-96z^2\zeta-440z^3\zeta^2-506z^4\zeta^3-369z^5\zeta^4-179z^6\zeta^5-50z^7\zeta^6-6z^8\zeta^7)}{48u^3e^{3z\zeta}(1+z\zeta)^9},$$

$$\frac{\partial P_{11}}{\partial z}$$
$$=\frac{-248\zeta^3-826z\zeta^4-1166z^2\zeta^5-985z^3\zeta^6-580z^4\zeta^7-233z^5\zeta^8-56z^6\zeta^9-6z^7\zeta^{10}}{12u(u_1^1)^2e^{z\zeta}(1+z\zeta)^9},$$

$$\frac{\partial P_{11}}{\partial p}$$
$$=\frac{72\zeta+120z\zeta^2-16z^2\zeta^3-142z^3\zeta^4-145z^4\zeta^5-108z^5\zeta^6-55z^6\zeta^7-16z^7\zeta^8-2z^8\zeta^9}{24iu(u_1^1)^2e^{2z\zeta}(1+z\zeta)^9},$$

$$\frac{\partial P_{11}}{\partial u}=-\frac{36\zeta^2+112z\zeta^3+145z^2\zeta^4+110z^3\zeta^5+55z^4\zeta^6+16z^5\zeta^7+2z^6\zeta^8}{12u^2(u_1^1)^2e^{z\zeta}(1+z\zeta)^7},$$

$$\frac{\partial P_{11}}{\partial u_1^1}=-\frac{36\zeta^2+112z\zeta^3+145z^2\zeta^4+110z^3\zeta^5+55z^4\zeta^6+16z^5\zeta^7+2z^6\zeta^8}{6u(u_1^1)^3e^{z\zeta}(1+z\zeta)^7},$$

$$\frac{1}{u_1^1}\left(\frac{\partial P_{11}}{\partial z}+p_{11}\frac{\partial P_{11}}{\partial p}\right)$$
$$=\frac{1}{12u(u_1^1)^3e^{z\zeta}(1+z\zeta)^9}(-176\zeta^3-706z\zeta^4-1182z^2\zeta^5$$
$$-1127z^3\zeta^6-725z^4\zeta^7-341z^5\zeta^8-111z^6\zeta^9-22z^7\zeta^{10}-2z^8\zeta^{11}).$$

We next use Lemma 3.2 to compute some covariant derivatives of L^{11} and P_{11}. We have

$$L^{11}_{\omega_1}=\frac{iu_1^1}{u}\frac{\partial L^{11}}{\partial p}-iu^1\frac{\partial L^{11}}{\partial u}+\frac{iu^1u_1^1}{u}\frac{\partial L^{11}}{\partial u_1^1},$$
$$L^{11}_{\omega^1}=\frac{1}{u_1^1}\left(\frac{\partial L^{11}}{\partial z}+p_{11}\frac{\partial L^{11}}{\partial p}\right)+iv_1\frac{\partial L^{11}}{\partial u}+u_1^1\left(\frac{2iv_1}{u}+\frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\frac{\partial L^{11}}{\partial u_1^1},$$
$$L^{11}_{\omega_1|\omega_1}=\frac{iu_1^1}{u}\frac{\partial L^{11}_{\omega_1}}{\partial p}-iu^1\frac{\partial L^{11}_{\omega_1}}{\partial u}+\frac{iu^1u_1^1}{u}\frac{\partial L^{11}_{\omega_1}}{\partial u_1^1},$$
$$P_{11,\omega_1}=\frac{iu_1^1}{u}\frac{\partial P_{11}}{\partial p}-iu^1\frac{\partial P_{11}}{\partial u}+\frac{iu^1u_1^1}{u}\frac{\partial P_{11}}{\partial u_1^1},$$
$$P_{11,\omega^1}=\frac{1}{u_1^1}\left(\frac{\partial P_{11}}{\partial z}+p_{11}\frac{\partial P_{11}}{\partial p}\right)+iv_1\frac{\partial P_{11}}{\partial u}+u_1^1\left(\frac{2iv_1}{u}+\frac{1}{u_1^1}\frac{\partial p_{11}}{\partial p}\right)\frac{\partial P_{11}}{\partial u_1^1}.$$

From the above calculation and applying these formulas, it is clear that A_j, B_j and C_j are polynomials in (z,ζ) with $A_j(\chi,\chi)=D_j\chi^2+o(\chi^2)$ and $D_j\neq 0$. Indeed, a tedious but routine calculation yields

$$A_3=\tfrac{5}{48}i(36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4),$$
$$B_3=(-320z^3-410z^4\zeta-316z^5\zeta^2-163z^6\zeta^3-48z^7\zeta^4-6z^8\zeta^5),$$
$$A_4=\tfrac{1}{48}i(36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+2z^6\zeta^4),$$
$$B_4=\tfrac{1}{48}(72z+48z^2\zeta-64z^3\zeta^2-78z^4\zeta^3-67z^5\zeta^4-41z^6\zeta^5-14z^7\zeta^6-2z^8\zeta^7),$$
$$A_5=-\tfrac{5}{8}(36z^2+40z^3\zeta+29z^4\zeta^2+12z^5\zeta^3+z^6\zeta^4),$$
$$B_5=-\tfrac{1}{8}i(320z^3+410z^4\zeta+316z^5\zeta^2+163z^6\zeta^3+48z^7\zeta^4+6z^8\zeta^6),$$
$$C_5=\tfrac{1}{192}(3750z^4+5568z^5+4627z^6\zeta^2+2702z^7\zeta^3+1054z^8\zeta^4+240z^9\zeta^5+24z^{10}\zeta^6),$$
$$A_6=-\tfrac{1}{12}i(36\zeta^2+112z\zeta^3+145z^2\zeta^4+110z^3\zeta^5+55z^4\zeta^6+16z^5\zeta^7+2z^6\zeta^8),$$
$$B_6=\tfrac{1}{24}(72\zeta+120z\zeta^2-16z^2\zeta^3-142z^3\zeta^4-145z^4\zeta^5$$
$$-108z^5\zeta^6-55z^6\zeta^7-16z^7\zeta^8-2z^8\zeta^9),$$
$$A_7=-\tfrac{5}{12}i(36\zeta^2+112z\zeta^3+145z^2\zeta^4+110z^3\zeta^5+55z^4\zeta^6+16z^5\zeta^7+2z^6\zeta^8),$$
$$B_7=-\tfrac{1}{12}(320\zeta^3+1370z\zeta^4+2506z^2\zeta^5+2661z^3\zeta^6+1895z^4\zeta^7$$
$$+955z^5\zeta^8+325z^6\zeta^9+66z^7\zeta^{10}+6z^8\zeta^{11}).\quad\square$$

Remark 4.2. One of the main features of Lemma 4.1 is that the invariant functions there depend only on z, p, u, u_1^1, u_1 and v_1. Indeed, the same computation also shows that even if M is a general rigid strongly pseudoconvex hypersurface, namely $M=\{(z,w):\operatorname{Im} w=\varrho(z,\bar z)\}$, the same property holds.

5. Proof of Theorem 1.1

We now give the proof of Theorem 1.1. To proceed, we start with a general fact. Let $F(z)=(f_1,\dots,f_m)(z)$ be a holomorphic map from a domain D in $\mathbf{C}^n$ into $\mathbf{C}^m$. Assume that the generic rank k of F is strictly smaller than m. Assume, for simplicity, that $\{f_{m-k},\dots,f_m\}$ is generically functionally independent. Then there is a complex variety E such that for each $a\in D\setminus E$ and $l<m-k$ one can find a unique holomorphic function $\Lambda_{a,l}$ in the variables $(Y_1,\dots,Y_k)$, defined near $(f_{m-k}(a),\dots,f_m(a))$, such that $f_l(z)\equiv\Lambda_{a,l}(f_{m-k}(z),\dots,f_m(z))$ for $z\approx a$. In particular, when F is Nash algebraic, then so is $\Lambda_{a,l}$.

Proof of Theorem 1.1. Seeking a contradiction, suppose for some point $a\in M$, that (M,a) is equivalent to the germ of a certain algebraic hypersurface. By Lemma 4.1, we have seven holomorphic invariant functions $L^{11}_{\omega_1|\omega_1}$, L^{11}, $L^{11}_{\omega_1}$, $L^{11}_{\omega^1}$, P_{11}, P_{11,ω_1} and P_{11,ω^1}. Since ζ depends only on z and p, we see that these seven invariant functions are only depending on the six variables z, p, u, u_1^1, u_1 and v_1, by the formulas in Lemma 4.1. Let k be the generic rank of the map

$$Y=(Y_1,\dots,Y_7)=(L^{11}_{\omega_1|\omega_1},L^{11},L^{11}_{\omega_1},L^{11}_{\omega^1},P_{11},P_{11,\omega_1},P_{11,\omega^1}),$$

then $k\le 6$.

Assume without loss of generality that $\{Y_{7-k},\dots,Y_7\}$ is the maximally independent set. (Indeed, by a tedious calculation, it can be shown that $k=6$ and the last six invariant functions are generically independent.) Then, for a generic point $\tilde A\in\widetilde{\mathcal{Y}}$ whose projection is sufficiently close to a, there is a unique holomorphic function $\Lambda_{\tilde A}$ such that $Y_1\equiv\Lambda_a(Y_1,\dots,Y_k)$ near $\tilde A$. Notice that Λ_a is also intrinsically defined.

Since we assumed that (M,a) is CR isomorphic to some real algebraic hypersurface $\widetilde M\subset\mathbf{C}^2$, it implies that $\Lambda_{\tilde A}$ can also be derived in the same manner from an algebraic hypersurface and thus must be algebraic as observed in the beginning of Section 4 and Section 5. Hence, there exists a non-constant polynomial R such that

$$R(L^{11}_{\omega_1|\omega_1},L^{11},L^{11}_{\omega_1},L^{11}_{\omega^1},P_{11},P_{11,\omega_1},P_{11,\omega^1})\equiv 0. \tag{5.1}$$

We next show that such an R is identically constant, obtaining a contradiction. It should be noticed that a priori, (5.1) is only known to hold on a certain open subset. However, by the holomorphic continuation and using the formulas in Lemma 4.1, it is clear that (5.1) holds for any $u, u_1^1, (1+z\zeta)\neq 0$.

The formula (5.1) can be written as

$$(5.2)\quad \sum_{\substack{m+n+\alpha+\beta\\+\mu+s+\tau\leq N}} C_{mn\alpha\beta\mu s\tau}(L^{11}_{\omega^1})^m(P_{11,\omega^1})^n(L^{11}_{\omega_1})^\alpha(P_{11,\omega_1})^\beta(L^{11}_{\omega_1|\omega_1})^\mu(L^{11})^s(P_{11})^\tau\equiv 0.$$

Suppose that i_1 is the biggest integer such that $C_{mn\alpha\beta\mu s\tau}\neq 0$ for some $m+n=i_1$, and non-negative integers α, β, μ, s and τ.

We remark that the left-hand side of (5.2) is a polynomial in u_1 and v_1, by the formulas in Lemma 4.1. Since the only terms containing v_1 are $L^{11}_{\omega^1}$ and P_{11,ω^1}, and by considering the highest v_1-power terms in (5.2), we can conclude

$$(5.3)\quad \sum_{\substack{\alpha+\beta+\mu+s+\tau\leq N-i_1\\ m+n=i_1}} C_{mn\alpha\beta\mu s\tau}\left(\frac{v_1(u_1^1)^2A_4}{u^4e^{3z\zeta}(1+z\zeta)^7}\right)^m\left(\frac{v_1A_7}{u^2(u_1^1)^2e^{z\zeta}(1+z\zeta)^7}\right)^n$$
$$\times(L^{11}_{\omega_1})^\alpha(P_{11,\omega_1})^\beta(L^{11}_{\omega_1|\omega_1})^\mu(L^{11})^s(P_{11})^\tau\equiv 0.$$

Suppose that i_2 is the biggest integer such that $C_{mn\alpha\beta\mu s\tau}\neq 0$ for some $m+n=i_2$, $\alpha+\beta+\mu=i_2$, and non-negative integers s and τ. And suppose that μ_0 is the biggest possible integer such that $C_{mn\alpha\beta\mu_0 s\tau}\neq 0$ for some $m+n=i_1$, $\alpha+\beta+\mu_0=i_2$, and non-negative integers s and τ.

Since the only terms containing u^1 are $L^{11}_{\omega_1}$, P_{11,ω_1} and $L^{11}_{\omega_1|\omega_1}$, we similarly get, by considering the highest power u^1-terms in (5.3),

$$(5.4)\quad \sum_{\substack{s+\tau\leq N-i_1-i_2\\ m+n=i_1\\ \alpha+\beta+\mu_0=i_2}} C_{mn\alpha\beta\mu_0 s\tau}\left(\frac{v_1(u_1^1)^2A_4}{u^4e^{3z\zeta}(1+z\zeta)^7}\right)^m\left(\frac{v_1A_7}{u^2(u_1^1)^2e^{z\zeta}(1+z\zeta)^7}\right)^n$$
$$\times\left(\frac{u^1(u_1^1)^2A_3}{u^4e^{3z\zeta}(1+z\zeta)^7}\right)^\alpha\left(\frac{u^1A_6}{u^2(u_1^1)^2e^{3z\zeta}(1+z\zeta)^7}\right)^\beta\left(\frac{(u^1)^2(u_1^1)^2A_5}{u^5e^{3z\zeta}(1+z\zeta)^7}\right)^{\mu_0}(L^{11})^s(P_{11})^\tau\equiv 0.$$

Using the formulas for L^{11} and P_{11}, we get from (5.4),

$$(5.5)\quad \sum_{\substack{s+\tau\leq N-i_1-i_2\\ m+n=i_1\\ \alpha+\beta+\mu_0=i_2}} C_{mn\alpha\beta\mu_0 s\tau}\frac{(v_1)^{i_1}(u^1)^{i_2}}{u^{4m+2n+4\alpha+2\beta+5\mu_0+3s-\tau}}$$
$$\times\frac{(u_1^1)^{2m-2n+2\alpha-2\beta+2\mu_0+2s-2\tau}A_4^mA_7^nA_3^\alpha A_6^\beta A_5^{\mu_0}A_1^sA_2^\tau}{e^{(3m+n+3\alpha+3\beta+3\mu_0+3s+\tau)z\zeta}(1+z\zeta)^{7(i_1+i_2+s+\tau)}}\equiv 0.$$

Observe that in the summation in (5.5), i_1, i_2 and μ_0 are fixed. Deleting the common factors $1/(1+z\zeta)^{7(i_1+i_2)}$, $(u_1^1)^{-2i_1-2i_2+4\mu_0}$, $u^{-2i_1-2i_2-3\mu_0}$, $e^{-(i_1+3\mu_0)}$, $A_5^{\mu_0}$, etc., we can simplify it as follows:

$$\sum_{\substack{m+n=i_1\\ \alpha+\beta+\mu_0=i_2}} C_{mn\alpha\beta\mu_0 s\tau}\frac{(u_1^1)^{4m+4u+2s-2\tau}A_4^m A_7^n A_3^\alpha A_6^\beta A_1^s A_2^\tau}{u^{2m+2\alpha+3s-\tau}e^{(2m+3s+\tau)z\zeta}(1+z\zeta)^{7(s+\tau)}}\equiv 0. \tag{5.6}$$

Rewrite

$$\frac{(u_1^1)^{4m+4\alpha+2s-2\tau}}{u^{2m+2\alpha+3s+\tau}e^{(2m+3s+\tau)z\zeta}(1+z\zeta)^{s+\tau}}=\frac{(u_1^1)^{\lambda_1}}{u^{\lambda_2}e^{\lambda_3 z\zeta}(1+z\zeta)^{s+\tau}}$$

with

$$\begin{cases}\lambda_1=4m+4\alpha+2s-2\tau,\\ \lambda_2=2m+2\alpha+3s+\tau,\\ \lambda_3=2m+3s+\tau.\end{cases}$$

We have

$$s=\tfrac{1}{2}\lambda_2-\tfrac{1}{4}\lambda_1-\tau,\quad m=-\tfrac{1}{2}\tau+\tfrac{1}{2}\lambda_3-\tfrac{3}{2}s,\quad \alpha=\tfrac{1}{2}\lambda_2-\tfrac{1}{2}\lambda_3. \tag{5.7}$$

Notice that the basic property of the exponential function also indicates that

$$\sum D_{k_1k_2k_3}(z,\zeta)\frac{(u_1^1)^{k_1}}{u^{k_2}e^{k_3 z\zeta}}\equiv 0$$

if and only if $D_{k_1k_2k_3}(z,\zeta)\equiv 0$ for all k_1, k_2 and k_3, where k_j are running over a finite set of integer numbers and $D_{k_1k_2k_3}(z,\zeta)$ are rational functions.

Hence by (5.6), we arrive at

$$\sum_{\substack{\tau\le i_3-s\\ n=i_1-m\\ \beta=-\alpha-\mu_0+i_2}} C_{mn\alpha\beta\mu_0 s\tau}\frac{A_4^m A_7^n A_3^\alpha A_6^\beta A_1^s A_2^\tau}{(1+z\zeta)^{7(\tau+s)}}\equiv 0, \tag{5.8}$$

where m, u and s are determined in (5.7) and the summation is taken over τ.

Let τ_0 be the smallest integer τ in (5.8) such that the corresponding coefficient $C_{m_0n_0\alpha_0\beta_0s_0\tau_0}$ is non-zero. Let $z=\zeta=\chi$ in (5.8). Applying Lemma 4.1, we conclude that (5.8) can be expressed as

$$KC_{m_0n_0\alpha_0\beta_0s_0\tau_0}\chi^{2(i_1+i_2-\mu_0+s_0+\tau_0)}+o(\chi^{2i_1+i_2-\mu_0+\tau_0+s_0})\equiv 0,$$

for some non-zero constant K. This is apparently a contradiction. The proof of Theorem 1.1 is complete. □

Remark 5.1. By Remark 4.2 and the discussion in this section, it is clear that a general rigid strongly pseudoconvex hypersurface M cannot be equivalent to any algebraic one if there is no non-zero polynomial R such that

$$R(L^{11}_{\omega_1|\omega_1}, L^{11}, L^{11}_{\omega_1}, L^{11}_{\omega^1}, P_{11}, P_{11,\omega_1}, P_{11,\omega^1}) \equiv 0.$$

More generally, let M be a real analytic strongly pseudoconvex hypersurface. Let $\mathcal{Y}$ be its structure bundle with $E:=\{R_\alpha\}_\alpha$ being the complete set of its Cartan–Chern–Moser curvature functions. Write $\{R_1, \dots, R_N\}$ for a maximal subset whose elements are generically functionally independent. Then, when M is equivalent to an algebraic strongly pseudoconvex hypersurface, for any $R \in E$ there is a non-constant polynomial P_R in $N+1$-variables such that $P_R(R, R_1, \dots, R_N) \equiv 0$ along $\mathcal{Y}$. It is not clear to us if the converse of this statement holds or not. (See Section 1.C in [H] for some related questions.)

References

[BER1] Algebraicity of holomorphic mappings between real algebraic sets in $\mathbf{C}^n$, *Acta Math.* **177** (1996), 225–273.

[BER2] Baouendi, M. S., Ebenfelt, P. and Rothschild, L. P., Local geometric properties of real submanifolds in complex space—a survey, *Bull. Amer. Math. Soc* **37** (2000), 309–336.

[BHR] Baouendi, M. S., Huang, X. and Rothschild, L. P., Regularity of CR mappings between algebraic hypersurfaces, *Invent. Math.* **125** (1996), 13–36.

[C] Chern, S. S., On the projective structure of a real hypersurface in $\mathbf{C}_{n+1}$, *Math. Scand.* **36** (1975), 581–600.

[CJ1] Chern, S. S. and Ji, S., Projective geometry and Riemann's mapping problem, *Math. Ann.* **302** (1995), 581–600.

[CJ2] Chern, S. S. and Ji, S., On the Riemann mapping theorem, *Ann. of Math.* **144** (1996), 421–439.

[CM] Chern, S. S. and Moser, J. K., Real hypersurfaces in complex manifolds, *Acta Math.* **133** (1974), 219–271.

[E] Ebenfelt, P., On the unique continuation problem for CR mappings into nonminimal hypersurfaces, *J. Geom. Anal.* **6** (1996), 385–405.

[F] Faran, J., Segre families and real hypersurfaces, *Invent. Math.* **60** (1980), 135–172.

[H] Huang, X., On some problems in several complex variables and CR geometry, in *Proceedings of ICCM* (Yau, S. T., ed.), pp. 231–244, International Press, Cambridge, Mass., 2000.

[HJ] HUANG, X. and JI, S., Global holomorphic extension of a local map and a Riemann mapping theorem for algebraic domains, *Math. Res. Lett.* **5** (1998), 247–260.

[MW] MOSER, J. and WEBSTER, S. M., Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations, *Acta Math.* **150** (1983), 255–296.

[W] WEBSTER, S. M., On the mapping problem for algebraic real hypersurfaces, *Invent. Math.* **43** (1977), 53–68.

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