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Invariant metric estimates for $\bar{\partial}$ on some pseudoconvex domains

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1. Introduction

Let $\Omega \Subset \mathbf{C}^n$ be a smoothly bounded, pseudoconvex domain of finite type (see [D'A] for the definition of finite type). Call Ω *simple* if it satisfies at least one of the following conditions: (i) Ω is strongly pseudoconvex, (ii) Ω is convex, (iii) $n=2$, or (iv) Ω is decoupled (see [M5] for the definition of decoupled).

The main result in this paper is the following result.

Theorem 1.1. *If Ω is simple, there exists a constant C such that if α is a $\bar{\partial}$-closed $(n,1)$-form on Ω, then there exists an $(n,0)$-form u solving $\bar{\partial}u=\alpha$ with the estimate*

$$\|u\|_I \le C\|\alpha\|_I, \tag{1.1}$$

where $\|\cdot\|_I$ denotes the norm associated to any one of the metrics of Carathéodory, Bergman, or Kobayashi on the domain Ω.

The result follows in two steps. First, the three invariant metrics, acting on a given tangent vector, are shown to be comparable to each other and to a fairly explicit pseudometric which directly encodes the geometry of $b\Omega$. If Ω is convex, this result was obtained by Chen [Ch] in his thesis; as this result has not been published elsewhere, we give a proof of Chen's theorem in Section 2. For the other types of simple domains, the estimates on the metrics are due to several authors and we collect the references to those works also in Section 2. Second, the estimate on the solution u is proved by exploiting a variant of the usual L^2 inequalities for $\bar{\partial}$ due to Hörmander. This variant has its origins in the work of Donnelly–Fefferman [DF] and

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Ohsawa–Takegoshi [OT] though recently more transparent proofs of the inequality have been given by Berndtsson [B], McNeal [M5] and Siu [S]. Estimate (1.1) is, as usual, shown by duality. The point of the new inequality is that it allows (non-bounded) weight factors to be introduced, which have large hessians, without having to change the volume form in the integrals due to the weight function's behavior at $b\Omega$.

Since the three metrics vanish at $b\Omega$, when acting on forms, in a measured way, (1.1) is a non-isotropic, base-level, regularity result about a solution to the $\bar\partial$-equations. For the Carathéodory and Kobayashi metrics, as far as we know, this kind of result has not been formulated previously in the literature. However for the Bergman metric, our theorem is a special case ($p=n$, $q=1$) of the cohomology vanishing theorems obtained by Donnelly–Fefferman [DF] and Donnelly [Do]. In our view, though, the proof of (1.1) given here for the Bergman metric is simpler than the proofs in [DF] and [Do]; in connection with this see also the paper [BC]. Furthermore, the condition (3.2) we use in order to exploit inequality (3.1) is less restrictive for certain kinds of potential functions than a related condition of Gromov [G] (which is used in [Do]) and may be useful in extending the vanishing theorems to wider classes of domains.

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2. Metrics and volume forms

We first recall the definitions of the infinitesimal invariant metrics.

If $U_1\subset\mathbf{C}^{n_1}$ and $U_2\subset\mathbf{C}^{n_2}$ are open sets, let $H(U_1,U_2)$ denote the holomorphic mappings from U_1 to U_2. Let B^n denote the unit ball in $\mathbf{C}^n$. Let $X\in\mathbf{C}^n$ and view it as a $(1,0)$ tangent vector. If $\Omega\subset\mathbf{C}^n$ is a domain and $z\in\Omega$, the Carathéodory length of X at z is defined as

$$M_C(z;X)=\sup\{|Xf(z)|:f\in H(\Omega,B^n),\ f(z)=0\}. \tag{2.1}$$

Here X acts naturally on f as a derivation.

Let $e_1=(1,0,\dots,0)\in\mathbf{C}^n$. The Kobayashi length of X at z is given by

$$M_K(z;X)=\inf\{r>0:\text{there is } f\in H(B^n,\Omega) \text{ with } f(0)=z \text{ and } f'(0)\cdot e_1=X/r\}. \tag{2.2}$$

We mention that the use of B^n in both definitions is not universal and that other choices of model domains exist in the literature, e.g., a polydisc.

To define the Bergman metric, we consider two L^2 extreme value problems:

$$D(z,z)=\sup\{|f(z)|:f\in H(\Omega,\mathbf{C}) \text{ and } \|f\|_2\leq 1\}, \tag{2.3}$$

$$N(z;X)=\sup\{|Xf(z)|:f\in H(\Omega,\mathbf{C}),\ f(z)=0, \text{ and } \|f\|_2\leq 1\}. \tag{2.4}$$

Here $\|f\|_2$ denotes the euclidean L^2 norm of f on Ω. The square $D(z,z)^2$ gives the value of the Bergman kernel function, associated with Ω, at the diagonal point (z,z) in $\Omega\times\Omega$. The Bergman length of X at z is defined as

$$M_B(z;X)=\frac{N(z;X)}{D(z,z)}. \tag{2.5}$$

We mention that there is an alternative, equivalent way to express the Bergman metric. Let

$$b_{lk}(z)=\frac{\partial^2}{\partial z_l\partial\bar{z}_k}\log D^2(z,z), \tag{2.6}$$

where $(z_1,\dots,z_n)$ are the standard coordinates on $\mathbf{C}^n$. If $X=\sum_{l=1}^n a_l\partial/\partial z_l$, then

$$M_B(z;X)=\left(\sum_{l,k=1}^n b_{lk}(z)a_l\bar{a}_k\right)^{1/2}. \tag{2.7}$$

This representation of the Bergman metric shows that it is a Hermitian (indeed, Kähler) metric on $T^{1,0}(\Omega)$.

It is very difficult to calculate the exact value of the above metrics, except in a few special cases of domains with high degrees of symmetry. For our purposes, though, it will be sufficient to know that the three metrics are comparable to each other, in a certain sense, as z approaches $b\Omega$ and, also, that each metric can be approximated by a reasonably explicit pseudometric defined using the geometry of $b\Omega$. For the various kinds of domains which make up our class of simple domains, these approximate size results have been obtained by several authors (see references below) using somewhat different techniques and notation. The estimates, however, may all be cast in a unified way, which we proceed to indicate. As the basic ideas and notation are easiest to state in the convex case, we concentrate on this situation first. For notational convenience, we write $A\lesssim B$ if there exists a constant C so that $A\leq CB$ and use the symbols $\gtrsim$ and $\approx$ similarly. In practice C will be independent of certain parameters on which A and B depend; usually, the parameters are the point $q\in\Omega$ and the positive real number ε.

Let $\Omega\Subset\mathbf{C}^n$ be a smoothly bounded, convex domain of finite type, and r a (smooth, convex) defining function for Ω: $\Omega=\{z:r(z)<0\}$ and $dr\neq 0$ when $r=0$.

We will assume, without loss of generality, that the sets $\{z:r(z)<\eta\}$ for $-c<\eta<c$, $c>0$, are also convex.

Let $p\in b\Omega$ and $\varepsilon>0$ be given. We recall some local measurements related to the order of vanishing of r near p introduced in [M4]. For $q\in U\cap\Omega$, U a neighborhood of p, and $\lambda\in S^{2n-1}$, let $\sigma(q,\lambda,\varepsilon)$ denote the distance from q to the set $\{z:r(z)=\varepsilon\}$ along the complex line generated by λ. Set $\tau_1(q,\varepsilon)=\varepsilon$ and let $L_1\in S^{2n-1}$ be a vector so that $\sigma(q,L_1,\varepsilon)=\tau_1(q,\varepsilon)$. We distinguish the (essentially) maximal distances from q to $\{z:r(z)=\varepsilon\}$ along independent directions and, for convenience, do so in an orthogonal fashion. Define

$$\tau_2(q,\varepsilon)=\max\{\sigma(q,\lambda,\varepsilon):\lambda\perp L_1\},$$

and let L_2 be a vector such that $\sigma(q,L_2,\varepsilon)=\tau_2(q,\varepsilon)$. Inductively, define

$$\tau_k(q,\varepsilon)=\max\{\sigma(q,\lambda,\varepsilon):\lambda\perp\operatorname{span}(L_1,\dots,L_{k-1})\}$$

for $k=3,\dots,n$. Note that

$$\tau_1(q,\varepsilon)<\tau_n(q,\varepsilon)\leq\dots\leq\tau_2(q,\varepsilon). \tag{2.8}$$

The following result is proved in [M4].

Proposition 2.1. *For every $q\in\Omega\cap U$ and every $\varepsilon>0$ sufficiently close to 0, there exist coordinates $(z_1,\dots,z_n)$ centered at q and points $p_1,\dots,p_n\in\{z:r(z)=\varepsilon+r(q)\}$ such that, in the coordinates $(z_1,\dots,z_n)$, the defining function r satisfies*

(i) $$\frac{\tau_1(q,\varepsilon)}{\tau_i(q,\varepsilon)}\lesssim\left|\frac{\partial r}{\partial z_i}(p_i)\right|\lesssim\frac{\tau_1(q,\varepsilon)}{\tau_i(q,\varepsilon)}\quad \textit{for } 1\leq i\leq n,$$

(ii) $$\left|\frac{\partial r}{\partial z_i}(p_j)\right|\lesssim\frac{\tau_1(q,\varepsilon)}{\tau_i(q,\varepsilon)},\quad \textit{if } i<j,$$

(iii) $$\left|\frac{\partial r}{\partial z_i}(p_j)\right|=0,\quad \textit{if } i>j.$$

Also, if we define the polydisc

$$P_\varepsilon(q)=\{z\in U:|z_1|<\tau_1(q,\varepsilon),\dots,|z_n|<\tau_n(q,\varepsilon)\},$$

then there exists a constant $C>0$, independent of $q\in\Omega\cap U$, such that $CP_\varepsilon(q)\subset\{z\in U:r(z)<\varepsilon+r(q)\}$.

The coordinates given by Proposition 2.1 simplify all the subsequent constructions and estimates. The first important construction is of a family of plurisubharmonic functions which are, in a sense, maximal with respect to the polydiscs $P_\varepsilon(q)$.

Let $p\in b\Omega$ be fixed. For $\varepsilon\geq 0$, let $C_{q,\varepsilon}\Subset\mathbf{C}^n$ be a smoothly bounded, convex domain such that

$$\{z\in U: r(z)<r(q)+\varepsilon\}\subset C_{q,\varepsilon},$$
$$\{z\in U: r(z)=r(q)+\varepsilon\}\subset bC_{q,\varepsilon},$$

for some neighborhood U of p. Without loss of generality, we may assume that $\operatorname{dist}(C_{q,\varepsilon},\{z:r(z)=r(q)+\varepsilon+a\})=\operatorname{dist}(\{z:r(z)=r(q)+\varepsilon\},\{z:r(z)=r(q)+\varepsilon+a\})$, for small $a>0$. We will suppress the index ε when $\varepsilon=0$. The following result is proved in [M4], see also [M3].

Proposition 2.2. *Let Ω be a smoothly bounded, convex domain of finite type and the notation be as above. Let $q\in U\cap\Omega$ and let $\varepsilon>0$ be small. There exists a constant $c>0$, independent of q, and a function $\phi_{q,\varepsilon}\in C^\infty(\overline{C}_q)$ so that*

(i) $|\phi_{q,\varepsilon}(z)|\leq 1$, $z\in\overline{C}_q$;

(ii) *$\phi_{q,\varepsilon}$ is plurisubharmonic on C_q;*

(iii) *if $z\in P_{c\varepsilon}(q)\cap\overline{C}_q$, then*

$$\sum_{i,j=1}^n \frac{\partial^2\phi_{q,\varepsilon}}{\partial z_i\partial\bar{z}_j}(z)\xi_i\bar{\xi}_j\gtrsim\sum_{i=1}^n\frac{|\xi_i|^2}{\tau_i(q,\varepsilon)^2}.$$

The next step is to show that the Bergman kernel associated with Ω satisfies estimates related to the polydisc structure in Proposition 2.1. For this, the functions in Proposition 2.2 are used as weight functions in Hörmander's inequality. The proof of the following result is given in [M4], following closely a method developed in [C].

Proposition 2.3. *Let Ω be a smoothly bounded, convex domain of finite type and the notation be as above. There exists a neighborhood $V\Subset U$ so that if $q\in V\cap\Omega$,*

$$B_\Omega(q,q)\approx\prod_{i=1}^n\frac{1}{\tau_i(q,\varepsilon)^2},$$

where $\varepsilon=|r(q)|$ and the constants are independent of q.

At this point, we are in a position to derive the main estimates for the Bergman, Carathéodory, and Kobayashi metrics on convex domains of finite type. In order to state these estimates, we first define a convenient pseudometric on Ω. Let $q\in\Omega$, set $\varepsilon=|r(q)|$, and let $(z_1,\dots,z_n)$ be the coordinates given by Proposition 2.1 relative to q and ε. If $X=\sum_{l=1}^n a_l\partial/\partial z_l$, define

$$M_A(p;X)=\sum_{l=1}^n\frac{|a_l|}{\tau_l(q,\varepsilon)}. \tag{2.9}$$

As mentioned in the introduction, the following result was first obtained by Chen, [Ch], in his Purdue PhD dissertation; we give a proof here for completeness.

Proposition 2.4. *Let Ω be a smoothly bounded, convex domain of finite type and the notation be as above. There exists a neighborhood V of p so that, for all $q \in V \cap \Omega$,*

$$M_I(q;X) \approx M_A(q;X), \quad X \in \mathbf{C}^n,$$

with constants independent of q. Here $I = B, C, K$.

Proof. By a result of Lempert [L]

$$M_K(q;X) = M_C(q;X)$$

in our situation. And, in general, one has the inequality

$$M_C(q;X) \le M_B(q;X),$$

see [Ha]. So we are reduced to showing that $M_B(q;X) \lesssim M_A(q;X)$ and $M_A(q;X) \lesssim M_C(q;X)$. Now fix q, let $\varepsilon = |r(q)|$, and let $(z_1, \dots, z_n)$ be the coordinates given by Proposition 2.1.

To obtain the upper bound on the Bergman metric, observe that Cauchy's estimates imply that for any $f \in L^2(\Omega) \cap H(\Omega, \mathbf{C})$

$$\left| \frac{\partial f}{\partial z_i}(q) \right| \lesssim \frac{[\operatorname{Vol} P_\varepsilon(q)]^{1/2} \|f\|_2}{\tau_i(q,\varepsilon)}$$

with uniform constants. It follows that, if $X = \sum_{l=1}^n a_l \partial/\partial z_l$,

$$N(q;X) \lesssim \sum_{l=1}^n \frac{|a_l|}{\tau_l(q,\varepsilon)} \left(\prod_{l=1}^n \tau_l(q,\varepsilon) \right)^{-1}.$$

Combining this with Proposition 2.3, we obtain

$$M_B(q;X) \lesssim M_A(q;X),$$

with constant independent of q (and X).

For the lower bound on the Carathéodory metric, consider the linear functions

$$L_k(z) = (z - p_k) \cdot \partial r(p_k), \quad k = 1, \dots, n,$$

where $p_1, \dots, p_n$ are given by Proposition 2.1. Define

$$g_k(z) = e^{L_k(z)/\varepsilon}.$$

The estimates in Proposition 2.1 imply that the value of each g_k at q is independent of q and ε. Also, the convexity of Ω implies that $\operatorname{Re} L_k(z) \leq 0$ for $z \in \bar{\Omega}$. Setting $f_k(z) = g_k(z) - g_k(q)$ and considering the mapping $f = (f_1, \dots, f_n)$ we have that

$$(2.10) \qquad f(q) = (0, \dots, 0) \quad \text{and} \quad \|f\|_{L^\infty(\Omega)} \lesssim 1$$

for a constant independent of q.

If $X = \sum_{l=1}^n a_l \partial/\partial z_l$, it follows from the definition of f and the estimates in Proposition 2.1 that

$$(2.11) \qquad |Xf(q)| \gtrsim \max\left\{ \frac{|a_1|}{\tau_1(q,\varepsilon)}, \dots, \frac{|a_n|}{\tau_n(q,\varepsilon)} \right\}.$$

Since (2.10) shows that f is a candidate for the supremum defining $M_C(q;X)$, (2.11) gives the desired lower bound. This completes the proof.

Remarks. 1. For the other types of simple domains the analogs of Propositions 2.3 and 2.4 have been established by various authors. Although the original techniques were somewhat varied, these estimates can all be obtained in the manner outlined above.

If Ω is strongly pseudoconvex, the analog to Proposition 2.3 and the size estimate on $M_B(z;X)$ analogous to Proposition 2.4 were obtained by Diederich [D1], [D2]. We mention that much more precise estimates were later given by Fefferman [F]. The estimates on $M_C(z;X)$ and $M_K(z;X)$, in the strongly pseudoconvex case, were obtained by Graham [Gr]. For Ω a finite type domain in $\mathbf{C}^2$, the estimates on $B(z,z)$ and all three metrics were obtained by Catlin [C]. In the case that Ω is decoupled and of finite type, the estimates on the Bergman kernel and the three metrics were obtained by McNeal [M2].

2. We have stated the estimates in Proposition 2.3 and 2.4 locally, near a fixed point in $b\Omega$. However, the smoothness of $B(z,z)$ and $M_I(z;X)$ for $z \in K \Subset \Omega$ and the boundedness of Ω imply that the estimates hold on all of Ω (after extending the definitions of τ_j and M_A in any reasonable fashion).

We thus have the following result.

Theorem 2.5. *Let $\Omega \Subset \mathbf{C}^n$ be a simple domain defined by a smooth defining function r. Then for $q \in \Omega$,*

$$(2.12) \qquad B_\Omega(q,q) \approx \prod_{i=1}^n \frac{1}{\tau_i(q, |r(q)|)^2}$$

for some positive numbers $\tau_1, \dots, \tau_n$.

Also, for each $q\in\Omega$ there exist coordinates $(z_1^q,\dots,z_n^q)=(z_1,\dots,z_n)$ centered at q, so that if $X=\sum_{l=1}^n a_l\partial/\partial z_l$ and $M_A(q;X)$ is defined by (2.9) *(using the positive numbers $\tau_i(q,|r(q)|)$ above), then*

$$M_I(q;X)\approx M_A(q;X) \tag{2.13}$$

for $I=B$, C, K. In both (2.12) *and* (2.13), *the constants are independent of q (and X).*

In the next section, we shall also need estimates on the derivatives of the Bergman kernel.

Theorem 2.6. *Let $\Omega\Subset\mathbf{C}^n$ be a simple domain defined by a smooth defining function r. Then for β any multi-index, there exists a constant C_β so that for all $q\in\Omega$, there are coordinates such that if $\partial/\partial z_i$ are the associated coordinate vector fields and $D^\beta=(\partial/\partial z_1)^{\beta_1}\dots(\partial/\partial z_n)^{\beta_n}$, it holds that*

$$|D^\beta B_\Omega(q,q)|\le C_\beta\prod_{j=1}^n\frac{1}{\tau_j(q,|r(q)|)^{2+\beta_j}}.$$

The techniques that establish Theorem 2.6 are somewhat different than those mentioned above and so we simply give references to the proofs: for strongly pseudoconvex domains, see Diederich [D1] (or Fefferman [F]); for finite type domains in $\mathbf{C}^2$, see McNeal [M1] or Nagel–Rosay–Stein–Wainger [NRSW]; for convex, finite type domains, see McNeal [M4]; and for decoupled domains, see [M2].

The Carathéodory and Kobayashi metrics also act naturally on 1-forms, by duality. For instance, if ω is a $(1,0)$-form and $I=C$, K, define

$$M_I(z;\omega)=\sup\{|\omega(X)|:M_I(z;X)=1\}, \tag{2.14}$$

where $\omega(X)$ represents the action of ω on the $(1,0)$-vector X. In a similar way, we obtain a natural definition for the length of a $(0,1)$-form, using complex conjugation in (2.1), (2.2), (2.5), and (2.14) in the obvious way. However, the Carathéodory and Kobayashi metrics are simply Finsler metrics (not necessarily Hermitian) and their natural extension to forms of higher order is not obvious.

In view of the comparability of the three metrics given by Theorem 2.5, we will base the length of a (p,q)-form in the Carathéodory and Kobayashi norms on the Bergman length of a (p,q)-form. The Bergman metric does have a natural extension to higher order forms and we recall this standard lifting trick from Hermitian geometry. If $(b_{lk}(z))_{l,k=1}^n$ is the matrix with entries given by (2.6), define

$$M_B(z;\omega)=\left(\sum_{l,k=1}^n b^{lk}(z)\omega_l\bar{\omega}_k\right)^{1/2},$$

where $\omega=\sum_{j=1}^n \omega_j\,dz_j$ and $(b^{lk}(z))_{l,k=1}^n$ is the inverse of $(b_{lk}(z))_{l,k=1}^n$. For notational ease, if $\alpha=\sum_{j=1}^n \alpha_j\,dz_j$ and $\beta=\sum_{j=1}^n \beta_j\,dz_j$, let $\langle\alpha,\beta\rangle=\sum_{l,k=1}^n b^{lk}\alpha_l\bar\beta_k$. The spaces of $(1,0)$-forms and $(0,1)$-forms are declared to be orthogonal. Then, if $I=(i_1,\dots,i_p)$ and $J=(j_1,\dots,j_q)$ are multi-indices, $dz^I=dz_{i_1}\wedge\dots\wedge dz_{i_p}$, and $d\bar z^J=d\bar z_{j_1}\wedge\dots\wedge d\bar z_{j_q}$, define

$$\langle dz^I\wedge d\bar z^J, dz^I\wedge d\bar z^J\rangle=\det\begin{pmatrix}(\langle dz_{i_l},dz_{i_k}\rangle)_{l,k=1}^p & 0\\ 0 & (\langle d\bar z_l, d\bar z_{j_k}\rangle)_{l,k=1}^q\end{pmatrix}. \tag{2.15}$$

(The right-hand side of (2.15) is a $(p+q)\times(p+q)$ matrix with large blocks of 0 entries.) Linearity and (2.15) give the definition of the Bergman inner product on (p,q)-forms. Forms of different bi-degree are declared to be orthogonal (thus extending $\langle\cdot,\cdot\rangle$ to the full Grassman ring), though this will play no role here.

For the reason mentioned above, the Carathéodory and Kobayashi metrics do not have naturally associated volume forms; indeed, there are several reasonable choices, see [K]. The choice made here is motivated by simple affine volume considerations. For $z\in\Omega$ and $I=C,\ K$, define

$$\lambda_I^1(z)=\sup\{M_I(z;X)^2 : |X|=1\}.$$

Here $|X|$ denotes the euclidean length of X. It follows from (2.13) and (2.8) that

$$\lambda_I^1(z)=\frac{1}{\tau_1(z,|r(z)|)^2}.$$

Let l_1 be a complex line such that $M_I(z;l_1)^2=\lambda_I^1(z)$, and denote by O_1 the (euclidean) orthogonal complement of l_1 in $\mathbf{C}^n$. Define

$$\lambda_I^2(z)=\sup\{M_I(z;X)^2 : X\in O_1 \text{ and } |X|=1\},$$

and continue inductively to define $\lambda_I^3,\dots,\lambda_I^n$. As a pointwise definition, set

$$dV_I(z)=\left(\prod_{j=1}^n \lambda_I^j(z)\right)dV_E,\quad I=C,\ K,$$

where dV_E is the euclidean volume form.

Since it is Hermitian, the Bergman metric does have a naturally defined volume element

$$dV_B(z)=\det(b_{lk}(z))\,dV_E,$$

where, as before, $(b_{lk}(z))_{l,k=1}^n$ is the matrix given by the entries in (2.6). Let $e_1(z), \dots, e_n(z)$ be the eigenvalues of the matrix $(b_{lk}(z))_{l,k=1}^n$, ordered (at the temporarily fixed z) $e_1(z) \geq \dots \geq e_n(z)$. If V is a linear subspace of $\mathbf{C}^n$, set

$$\varkappa(V) = \sup\left\{ \sum_{l,k=1}^n b_{lk}(z) v_l \bar{v}_k : v = \sum_{l=1}^n v_l \frac{\partial}{\partial z_l} \in V \text{ and } \sum_{l=1}^n |v_l|^2 = 1 \right\}.$$

The minimax theorem from linear algebra implies that $e_k(z) = \inf\{\varkappa(V) : \dim V = n-k+1\}$. It thus follows from the estimates on $M_B(z;X)$ in Theorem 2.5 that for simple domains $e_j(z) \approx \tau_{n+2-j}(z, |r(z)|)^{-2}$ for $j=2,\dots,n$ and $e_1(z) \approx \tau_1(z, |r(z)|)^{-2}$, with constants independent of z.

Thus, we have the following result.

Theorem 2.7. *Let $\Omega \Subset \mathbf{C}^n$ be a simple domain defined by a smooth defining function r. For $I = B$, C, K, the pointwise inequalities*

$$dV_I(z) \approx \prod_{l=1}^n \frac{1}{\tau_l(z, |r(z)|)^2}, \quad z \in \Omega,$$

hold, with constants independent of z.

Finally, we define the L^2 norms with respect to the three metrics. For the Bergman metric this is standard: if α is a (p,q)-form, define

$$\|\alpha\|_B^2 = \int_\Omega \langle \alpha, \alpha \rangle \, dV_B,$$

where $\langle \cdot, \cdot \rangle$ is the extended Bergman inner product defined above. Notice that if α is an $(n,0)$-form, (2.15) implies that

$$\|\alpha\|_B^2 = \int_\Omega |\alpha(z)|^2 \, dV_E(z),$$

and, if β is an $(n,1)$-form, that

$$\|\beta\|_B^2 = \int_\Omega M_B(z;\beta)^2 \, dV_E(z).$$

As before, dV_E is the euclidean volume form. The cancellation of metric volume form that occurs in this case, for these form levels, suggests the following as natural definitions: if $I = B$, C, K,

$$\begin{aligned} \|\alpha\|_I^2 &= \int_\Omega |\alpha(z)|^2 \, dV_E(z), && \alpha \text{ an } (n,0)\text{-form}, \\ \|\beta\|_I^2 &= \int_\Omega M_I(z;\beta)^2 \, dV_E(z), && \beta \text{ an } (n,1)\text{-form}. \end{aligned} \tag{2.16}$$

3. Estimates in the Bergman metric

From the point of view of L^2 estimates for $\bar\partial$, the Bergman metric is easier to work with than the Carathéodory and Kobayashi metrics as it is defined in terms of a global potential. Using this potential function as a weight in Hörmander's identity, for example, one produces the Bergman metric as a certain curvature term in the basic inequality. However, this technique also distorts the volume element by the factor $e^{-\phi}$, where ϕ is the potential function, and so the usual duality argument does not give (1.1) (for $I=B$).

We show that a slight variation of the standard argument does, however, imply (1.1) for the Bergman metric in the case of simple domains. There are also similar, related variations which imply the estimate, see Donnelly [Do] and, especially, Berndtsson–Charpentier [BC], and could be invoked in our situation. The argument given here, especially its explicit use of the two perturbation factors, is very simple and suggestive of further applications, perhaps justifying its inclusion.

We first introduce some notation. If λ is a real-valued function in $L^1_{\mathrm{loc}}(\Omega)$, for $\Omega=\{r<0\}$ a domain in $\mathbf{C}^n$, define the weighted L^2 norm of a function f by

$$\|f\|^2_\lambda=\int_\Omega |f|^2e^{-\lambda}\,dV_E,$$

and let $L^2(\Omega,\lambda)$ be the functions on Ω for which $\|f\|_\lambda<\infty$. The volume form dV_E will usually be suppressed when we write integrals below. We denote the associated inner product by $(\cdot,\cdot)_\lambda$ and extend these definitions to forms by linearity. Let $\Lambda^{p,q}(\bar\Omega)$ denote the (p,q)-forms with coefficients which are smooth on $\bar\Omega$ and let $\bar\partial^*_\lambda$ denote the L^2 adjoint of $\bar\partial$ relative to the above inner product. The formal adjoint of $\bar\partial$, ϑ_λ, is defined by the equation $(\bar\partial u,v)_\lambda=(u,\vartheta_\lambda v)_\lambda$ when $u\in\Lambda^{p,q-1}(\bar\Omega)$, $v\in\Lambda^{p,q}(\bar\Omega)$, and at least one of them has compact support in Ω. The first $\bar\partial$-Neumann boundary condition on a form $\phi\in\Lambda^{p,q}(\bar\Omega)$ is $\sigma(\vartheta_\lambda,dr)\phi=0$ on $b\Omega$, where $\sigma(A,b)$ is the principal symbol of A in the direction b. Set

$$\mathcal{D}^{p,q}=\{\phi\in\Lambda^{p,q}(\bar\Omega):\sigma(\vartheta_\lambda,dr)\phi=0 \text{ on } b\Omega\}.$$

When $\phi\in\mathcal{D}^{p,q}$, it happens that $\bar\partial^*_\lambda\phi=\vartheta_\lambda\phi$.

If $\phi\in\Lambda^{0,1}(\Omega)$, let

$$\|\sqrt{l}\,\phi\|^2_{\lambda,\bar z}=\sum_{j,k=1}^n\left\|\sqrt{l}\,\frac{\partial\phi_j}{\partial\bar z_k}\right\|^2_\lambda.$$

If $k\in C^\infty(\Omega)$ and $z\in\Omega$, let

$$\partial\bar\partial k(z;\phi)=\sum_{j,k=1}^n\frac{\partial^2k}{\partial z_j\partial\bar z_k}(z)\phi_j(z)\bar\phi_k(z)$$

and

$$\partial k(z;\phi)=\sum_{j=1}^{n}\frac{\partial k}{\partial z_j}(z)\phi_j(z).$$

Proposition 3.1. *Let $\Omega\Subset\mathbf{C}^n$ be smoothly bounded and let $l,\lambda\in C^\infty(\overline{\Omega})$, with $l\geq 0$ on Ω. Then, if $\phi\in\mathcal{D}^{0,1}$,*

$$\begin{aligned}(l\bar\partial\phi,\bar\partial\phi)_\lambda+(l\bar\partial^*_\lambda\phi,\bar\partial^*_\lambda\phi)_\lambda=\|\sqrt{l}\phi\|^2_{\lambda,\bar z}+\int_\Omega[l\partial\bar\partial\lambda-\partial\bar\partial l](\phi)\,e^{-\lambda}\\ +\int_{b\Omega}l\partial\bar\partial r(\phi)\,e^{-\lambda}-2\,\mathrm{Re}\int_\Omega\partial l(\phi)\cdot\overline{\vartheta_\lambda\phi}\,e^{-\lambda}.\end{aligned}\tag{3.1}$$

A proof of Proposition 3.1 is given in [M5]—displayed (2.14) in that paper—and also appears in [S] and (as a differential identity) in [B]. This type of energy identity for the $\bar\partial$-complex goes back to [DF] and [OT].

In order to exploit Proposition 3.1, the last term in the right-hand side of (3.1) needs to be controlled.

Definition 3.2. Let $\lambda,l\in\Lambda^{0,0}(\Omega)$, $l>0$. Then l is *quasi-hyperbolic to the pair* (λ,l) if there exists a constant $c>0$ such that

$$\frac{1}{l}|\partial l(z;V)|^2\leq c[l\partial\bar\partial\lambda-\partial\bar\partial l](z;V)\tag{3.2}$$

for all $z\in\Omega$ and all vectors $V\in\mathbf{C}^n$.

The main L^2 estimate in this paper is the following result.

Proposition 3.3. *Let $\Omega\Subset\mathbf{C}^n$ be a simple domain. There exists a constant $C>0$ so that, if α is a $\bar\partial$-closed $(0,1)$-form on Ω, there exists a solution to $\bar\partial u=\alpha$ which satisfies*

$$\int_\Omega|u(z)|^2\,dV_E(z)\leq C\int_\Omega M_B(z;\alpha)^2\,dV_E(z),$$

assuming the right-hand side is finite.

Proof. We will first work on a relatively compact subdomain, Ω_c, of Ω and we will superscript the norms and inner products with c to indicate that the integrations occur over Ω_c.

Let $\nu>0$ and define

$$l(z)=B_\Omega(z,z)^{-\nu}.$$

Note $l\in C^\infty(\bar{\Omega}_k)$. If $\phi\in\Lambda^{0,1}(\bar{\Omega}_c)$,

$$\partial\bar{\partial}l(z;\phi)=-\nu l(z)\sum_{j,k=1}^{n}b_{jk}(z)\phi_j\bar{\phi}_k+\nu^2 l(z)|\partial[\log B](z;\phi)|^2 \tag{3.3}$$

and

$$|\partial l(z;\phi)|^2=\nu^2 l^2(z)|\partial[\log B](z;\phi)|^2. \tag{3.4}$$

Theorems 2.5 and 2.6 imply that

$$|\partial[\log B](z;\phi)|^2\lesssim\sum_{j,k=1}^{n}b_{jk}(z)\phi_j\bar{\phi}_k,\quad z\in\Omega_c, \tag{3.5}$$

for a constant independent of z and Ω_c. Choosing ν small enough, (3.3)–(3.5) show that l is quasi-hyperbolic on Ω_c to the pair (λ,l) for any plurisubharmonic function λ. The exponent ν is now fixed. Using Cauchy–Schwarz on the last term in (3.1) then yields, for $\lambda\in C^\infty(\bar{\Omega}_c)$, the inequality

$$(l\bar{\partial}\phi,\bar{\partial}\phi)^c_\lambda+2(l\bar{\partial}^*_\lambda\phi,\bar{\partial}^*_\lambda\phi)^c_\lambda\geq\int_{\Omega_c}l\partial\bar{\partial}\lambda(\phi)\,e^{-\lambda},\quad\phi\in\mathcal{D}^{0,1}(\Omega_c). \tag{3.6}$$

Let $\lambda(z)=\nu\log B_\Omega(z,z)$ and note that $\lambda\in C^\infty(\bar{\Omega}_c)$. The inequality (3.6) implies

$$(l\bar{\partial}\phi,\bar{\partial}\phi)^c_\lambda+2(l\bar{\partial}^*_\lambda\phi,\bar{\partial}^*_\lambda\phi)^c_\lambda\geq\nu\int_{\Omega_c}l\sum_{j,k=1}^{n}b_{jk}\phi_j\bar{\phi}_k\,e^{-\lambda},\quad\phi\in\mathcal{D}^{0,1}(\Omega_c). \tag{3.7}$$

Now let α be a $(0,1)$-form, $\bar{\partial}\alpha=0$ in Ω. On the subspace $\{\sqrt{l}\,\bar{\partial}^*_\lambda g:g\in\mathcal{D}^{0,1}(\Omega_c)\}$, define the linear functional

$$\sqrt{l}\,\bar{\partial}^*_\lambda g\longmapsto(g,\alpha)^c_\lambda.$$

If $g\in\mathrm{Null}(\bar{\partial})$, we obtain

$$\begin{aligned}|(g,\alpha)^c_\lambda|&\leq\left(\int_{\Omega_c}l\sum_{j,k=1}^{n}b_{jk}g_j\bar{g}_k\,e^{-\lambda}\right)^{1/2}\left(\int_{\Omega_c}\frac{1}{l}\sum_{j,k=1}^{n}b^{jk}\alpha_j\bar{\alpha}_k\,e^{-\lambda}\right)^{1/2}\\&\leq\frac{2}{\nu}\|\sqrt{l}\,\bar{\partial}^*_\lambda g\|^c_\lambda\left(\int_{\Omega_c}\sum_{j,k=1}^{n}b^{jk}\alpha_j\bar{\alpha}_k\right)^{1/2},\end{aligned} \tag{3.8}$$

by (3.7) and the choice of l and λ. On the other hand, (3.8) obviously holds if g is orthogonal (in the λ inner product) to $\mathrm{Null}(\bar{\partial})$, since in that case $(g,\alpha)_\lambda=0$.

Also, a standard approximation argument, see [H], shows that $\mathcal{D}^{0,1}(\Omega_c)$ is dense in $\mathrm{Dom}(\sqrt{l}\,\bar\partial^*_\lambda)$. Thus (3.8) actually holds for all $g\in\mathrm{Dom}(\sqrt{l}\,\bar\partial^*_\lambda)$. The Riesz representation theorem implies that there exists a $v\in L^2(\Omega_c,\lambda)$ such that $\left(\sqrt{l}\,\bar\partial^*_\lambda g, v\right)_\lambda = (g,\alpha)_\lambda$ with a norm estimate given by (3.8), i.e. $\bar\partial\sqrt{l}\,(v)=\alpha$ and

$$\int_{\Omega_c} |v_c|^2 e^{-\lambda} \le \left(\frac{2}{\nu}\right)^2 \int_{\Omega_c} \sum_{j,k=1}^{n} b^{jk}\alpha_j\bar\alpha_k.$$

Setting $u_c=\sqrt{l}\,v_c$ gives $\bar\partial u_c=\alpha$ and

$$\int_{\Omega_c} |u_c|^2 \le \left(\frac{2}{\nu}\right)^2 \int_{\Omega_c} \sum_{j,k=1}^{n} b^{jk}\alpha_j\bar\alpha_k. \tag{3.9}$$

To obtain the desired estimate on Ω, let $\{\Omega_c\}$, $c\in\mathbf{N}$, be an exhausting sequence of domains as above. Inequality (3.9) and a diagonal argument give a sequence of functions $\{u_j\}$ which converge weakly to a function u in each $L^2(\Omega_c)$. Obviously $\bar\partial u=\alpha$. Furthermore, if c is fixed,

$$\int_{\Omega_c} |u|^2 = \lim_{j\to\infty} \int_{\Omega_c} |u_j|^2 \le \left(\frac{2}{\nu}\right)^2 \int_{\Omega_c} \sum_{j,k=1}^{n} b^{jk}\alpha_j\bar\alpha_k, \tag{3.10}$$

by dominated convergence. Letting $c\to\infty$ in (3.10) completes the proof.

Proof of Theorem 1.1. Suppose that α is an $(n,1)$-form which satisfies $\bar\partial\alpha=0$ in Ω. If $\alpha=\sum_{j=1}^n \alpha_j\, dz_1\wedge\ldots\wedge dz_n\wedge d\bar z_j$ in some coordinate system, note that the $(0,1)$-form $\sum_{j=1}^n \alpha_j\, d\bar z_j$ is also $\bar\partial$-closed. The desired estimates now follow directly from (2.16), (2.13), and Proposition 3.3.

Remarks. 1. For the finite type domains which make up our class of simple domains, the inequality (1.1) implies the (sharp) subelliptic, L^2 Sobolev, estimate on a solution to $\bar\partial u=\alpha$. For example, if $\Omega=\{r<0\}$ is strongly pseudoconvex, (1.1) says

$$\int_{\Omega} |u|^2 \lesssim \int_{\Omega} |r|\,|\alpha_T|^2 + |r|^2|\alpha_N|^2, \tag{3.10}$$

where $\alpha=\alpha_T+\alpha_N$ is a decomposition of α into tangential and normal components relative to $b\Omega$. The inequality (3.10) implies that half the L^2 Sobolev norm of u is dominated by the L^2 norm of α. For information about relationships between Sobolev norms and L^2 norms weighted by factors of r, see Kohn [Koh].

2. Theorem 3.2 may perhaps hold on a general smooth, pseudoconvex domain, but Theorem 2.5 definitely does not. See Diederich–Fornæss–Herbort [DFH] for an

example where the three invariant metrics are not comparable to each other and Herbort [He] for a (related) example where the Bergman kernel has log factors in the principal term of its asymptotic series.

3. Our definition of simple domains is clearly artificial and what we have really presented is a recipe for obtaining (1.1) in the presence of good metric estimates. For example, the class of finite type domains in $\mathbf{C}^2$ could be enlarged to finite type domains in $\mathbf{C}^n$ whose Levi form has at most one degenerate eigenvalue. It also seems likely that h-extendible domains or semi-regular domains, see [Y] and [DH], and finite type domains where all of the (possibly degenerate) eigenvalues are comparable, see Koenig [Koe], are cases to which the recipe is applicable.

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