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Closures of finitely generated ideals in Hardy spaces

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Abstract. Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbf{D}$. Let $I=I(f_1, \dots, f_N)$ be the ideal generated by $f_1, \dots, f_N \in H^\infty$ and $J=J(f_1, \dots, f_N)$ the ideal of the functions $f \in H^\infty$ for which there exists a constant $C=C(f)$ such that $|f(z)| \le C(|f_1(z)|+\dots+|f_N(z)|)$, $z\in\mathbf{D}$. It is clear that $I\subseteq J$, but an example due to J. Bourgain shows that J is not, in general, in the norm closure of I. Our first result asserts that J is included in the norm closure of I if I contains a Carleson–Newman Blaschke product, or equivalently, if there exists $s>0$ such that

$$\inf_{z\in\mathbf{D}} \sum_{k=0}^{s} (1-|z|)^k \sum_{j=1}^{N} |f_j^{(k)}(z)| > 0.$$

Our second result says that there is no analogue of Bourgain's example in any Hardy space H^p, $1\le p<\infty$. More concretely, if $g\in H^p$ and the nontangential maximal function of $|g(z)|/\sum_{j=1}^N |f_j(z)|$ belongs to $L^p(\mathbf{T})$, then g is in the H^p-closure of the ideal I.

1. Introduction

Let H^∞ be the algebra of bounded analytic functions in the unit disc $\mathbf{D}$. Given functions $f_1, \dots, f_N$ in H^∞, let $I=I(f_1, \dots, f_N)$ denote the ideal generated by $\{f_1, \dots, f_N\}$, that is,

$$I=I(f_1, \dots, f_N)=\left\{\sum_{j=1}^{N} f_j g_j : g_j \in H^\infty\right\}.$$

The celebrated Corona theorem of L. Carleson [3] asserts that the ideal I is the whole algebra H^∞ if

$$\inf\left\{\sum_{j=1}^{N} |f_j(z)| : z\in\mathbf{D}\right\} > 0.$$

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Let $J=J(f_1,\dots,f_N)$ denote the ideal of the functions $f\in H^\infty$ for which there exists a constant $C=C(f)>0$ such that

$$(1)\qquad |f(z)|\le C\sum_{j=1}^{N}|f_j(z)|,\quad z\in\mathbf{D}.$$

It is clear that I is contained in J. However, an example of Rao shows that, in general, the two ideals are different. Actually, one may take $f=B_1B_2$, $f_1=B_1^2$, $f_2=B_2^2$, where B_1 and B_2 are two Blaschke products with disjoint zero sets satisfying

$$\inf\{|B_1(z)|+|B_2(z)|:z\in\mathbf{D}\}=0.$$

Then (1) holds but an easy factorization argument shows that f does not belong to $I(f_1,f_2)$. In fact, it has been proved in [8] that $I(f_1,f_2)=J(f_1,f_2)$ if and only if

$$\inf\{|f_1(z)|+|f_2(z)|+(1-|z|)(|f_1'(z)|+|f_2'(z)|):z\in\mathbf{D}\}>0.$$

It is worth mentioning that condition (1) implies that f^3 belongs to the ideal I, while for f^2 the question is open (see [6]).

J. Bourgain [2] has shown that one can construct the Blaschke products B_1 and B_2 in Rao's example such that B_1B_2 does not belong to the (norm) closure of $I(B_1^2,B_2^2)$. So condition (1) is not even sufficient to assure that the function f is in the (norm) closure of the ideal I. On the other hand, he also showed that if instead of (1) one requires

$$|f(z)|\le\alpha(|f_1(z)|+\dots+|f_N(z)|),\quad z\in\mathbf{D},$$

where α is a positive function satisfying

$$\lim_{t\to0}\frac{\alpha(t)}{t}=0,$$

one can conclude that f belongs to the norm closure of the ideal I.

Our first result states that condition (1) is sufficient if the ideal I contains a Carleson–Newman Blaschke product.

A Blaschke product with zero set $\{z_n\}_{n=1}^\infty$ is called a Carleson–Newman Blaschke product if the measure

$$\sum_{n=1}^{\infty}(1-|z_n|)\delta_n$$

is a Carleson measure. Here δ_n denotes the Dirac mass at the point z_n. Equivalently, the Blaschke product B with zero set $\{z_n\}_{n=1}^\infty$ is a Carleson–Newman Blaschke product if and only if for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$ such that $|B(z)|>\eta$ for any z such that $\inf_{n\ge1}|(z-z_n)/(1-\bar z_nz)|>\varepsilon$.

Theorem 1.1. *Let $f_1, \dots, f_N$ be functions in H^∞. Assume that $I(f_1, \dots, f_N)$ contains a Carleson–Newman Blaschke product. Then, the ideal $J(f_1, \dots, f_N)$ is contained in the norm closure of the ideal $I(f_1, \dots, f_N)$.*

This result has been proved previously by P. Gorkin and R. Mortini (see [7]). However, the methods are completely different. Their approach is based on subtle properties of the maximal ideal space, while we use a variation due to J. Bourgain of the $\bar\partial$-techniques in the proof of the classical Corona theorem.

The assumption on the ideal $I(f_1, \dots, f_N)$ may not look very natural. However it is really a condition on the structure of the ideal. To explain it, let $M(H^\infty)$ denote the maximal ideal space of H^∞, that is, the space of multiplicative linear functionals on H^∞, endowed with the weak-star topology. If $x, m \in M(H^\infty)$, then the pseudohyperbolic distance from x to m is defined as

$$\varrho(x,m) = \sup\{|m(f)| : f \in H^\infty,\ \|f\|_\infty \le 1,\ x(f) = 0\}.$$

By Schwarz's lemma this is the extension of the function

$$\varrho(z,w) = \left| \frac{z-w}{1-\bar w z} \right|$$

for $z, w \in \mathbf{D}$. It is well known that $M(H^\infty)$ can be partitioned into equivalence classes defined through the relation $x \sim m$ if and only if $\varrho(x,m) < 1$. The equivalence classes are called Gleason parts.

Given a function $f \in H^\infty$, its zero set $Z(f)$ is defined as

$$Z(f) = \{m \in M(H^\infty) : m(f) = 0\}.$$

The hull or zero set $Z(I)$ of an ideal I in H^∞ is

$$Z(I) = \bigcap_{f \in I} Z(f).$$

The following result was proved by V. Tolokonnikov [12].

Theorem A. *Let $f_1, \dots, f_N$ be functions in H^∞. Then, the following properties are equivalent:*

(a) *The ideal $I = I(f_1, \dots, f_N)$ contains a Carleson–Newman Blaschke product.*

(b) *The zero set $Z(I)$ is contained in the set G of points in $M(H^\infty)$ whose Gleason part contains more than one point.*

(c) *There exists a natural number $s \ge 0$ such that*

$$\inf_{z \in \mathbf{D}} \sum_{k=0}^{s} (1-|z|)^k \sum_{j=1}^{N} |f_j^{(k)}(z)| > 0.$$

Let $0<p<\infty$ and let H^p be the space of analytic functions f in the unit disk such that

$$\sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = \|f\|_p^p < \infty.$$

It is well known that an analytic function f belongs to H^p if and only if the non-tangential maximal function

$$Mf(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma(\theta)\}$$

belongs to the usual Lebesgue space $L^p(\mathbf{T})$, $\Gamma(\theta)$ being the Stolz angle with vertex at $e^{i\theta}$. Here $\mathbf{T}$ denotes the unit circle. Recently, several H^p versions of the Corona theorem have been considered. Given $f_1, \dots, f_N \in H^\infty$, one wants to study the Bezout equation

$$1 = f_1 g_1 + \dots + f_N g_N, \tag{2}$$

where $g_1, \dots, g_N$ are functions in H^p. Concretely, one is interested in conditions on $f_1, \dots, f_N$ so that solutions $g_1, \dots, g_N$ in H^p exist. If $|f|^2 = |f_1|^2 + \dots + |f_N|^2$ and $|g|^2 = |g_1|^2 + \dots + |g_N|^2$, it follows from (2) that $1 \le |f|\,|g|$ and hence

$$M(|f|^{-1}) \in L^p(\mathbf{T}) \tag{3}$$

is a necessary condition. Observe that when $p=\infty$, this is the usual Corona condition. However, for $p<\infty$, this condition is far from being sufficient. Actually, it is shown in [1] that for any $\varepsilon>0$, the stronger condition

$$M(|f|^{-2+\varepsilon}) \in L^p(\mathbf{T})$$

is not sufficient. Our next result says that condition (3) is sufficient to conclude that 1 is in the H^p-closure of the ideal $I(f_1, \dots, f_N)$. So, in this H^p-context there is no analogue of Bourgain's example.

Theorem 1.2. *Let $f_1, \dots, f_N$ be functions in H^∞. Let $1 \le p < \infty$. Let $g \in H^p$ such that*

$$M(g/|f|) \in L^p(\mathbf{T}).$$

Then, given any $\gamma>0$, there exist functions $g_1, \dots, g_N \in H^p$ such that

$$\|g - (f_1 g_1 + \dots + f_N g_N)\|_p < \gamma.$$

It is worth mentioning that for $\varepsilon>0$, the condition

$$M(|f|^{-2} \big|\log|f|\big|^{2+\varepsilon}) \in L^p(\mathbf{T})$$

is sufficient to solve the Bezout equation (see [1]). This result can be slightly improved (replacing the second 2 by $\frac{3}{2}$) but the question if

$$M(|f|^{-2}) \in L^p(\mathbf{T})$$

is sufficient to solve the Bezout equation in H^p, remains open.

2. Preliminaries

Recall that a positive Borel measure μ on $\mathbf{D}$ is called a *Carleson measure* if there exists a constant C so that

$$\int_{\mathbf{D}} |f|\, d\mu \le C\|f\|_1$$

for every function f in the Hardy space H^1. It is well known that Carleson measures are those positive measures μ for which there exists a constant K such that

$$\mu(Q) \le K l(Q)$$

for every *Carleson square* Q defined by

$$Q = \{re^{i\theta} \in \mathbf{D} : 1-r < l(Q),\ |\theta-\theta_0| < l(Q)\}.$$

Another equivalent condition is that

$$\sup_{z\in\mathbf{D}} \int_{\mathbf{D}} \frac{1-|z|^2}{|1-\bar{z}w|^2}\, d\mu(w) < +\infty.$$

Let

$$N(\mu) = \sup\left\{ \frac{\mu(Q)}{l(Q)} : Q \text{ is a Carleson square} \right\}$$

denote the *Carleson norm* of μ. We will use two results, the first due to Carleson on the existence of bounded solutions of $\bar{\partial}$-equations (see, for example, [6]) and the second is a result on approximation due to Garnett (in a weaker form) and Dahlberg (see [4]).

Proposition 2.1. *Let G be a bounded and continuous function in $\mathbf{D}$. Assume that $|G|\, dx\, dy$ is a Carleson measure on $\mathbf{D}$. Then, the $\bar{\partial}$-equation $\bar{\partial}h=G$ admits a solution $h\in\mathcal{C}(\overline{\mathbf{D}})\cap\mathcal{C}^1(\mathbf{D})$ with $\|h\|_{L^\infty(\partial\mathbf{D})}\le C_1 N(|G|\, dx\, dy)$, where C_1 is an absolute constant.*

Proposition 2.2. *Let u be a bounded harmonic function on $\mathbf{D}$. For each $\varepsilon>0$, there exists a $\mathcal{C}^\infty$-function φ on $\mathbf{D}$ satisfying $|\varphi(z)-u(z)|<\varepsilon$, $z\in\mathbf{D}$, and such that $\nu=|\nabla\varphi|\, dx\, dy$ is a Carleson measure with norm $N(\nu)<C_2/\varepsilon$, where C_2 is an absolute constant.*

The level curve of a bounded analytic function is, in general, not rectifiable. However, given a bounded analytic function f, L. Carleson constructed a system of rectifiable curves which act as level sets, in the sense that they separate the sets where $|f|$ is small from those where it is big. We will use a refinement of the Carleson construction due to J. Bourgain [2].

Proposition 2.3. *Let B be a Blaschke product. Given $\varepsilon>0$ there exists an open set R on* $\mathbf{D}$ *such that ∂R is a union of rectifiable curves and*

(i) $|B(z)|<\varepsilon$, *if* $z\in R$;

(ii) $|B(z)|>\delta(\varepsilon)$, *if* $z\in\mathbf{D}\setminus R$;

(iii) $N(\lambda_{\partial R})<C$, *where $\delta(\varepsilon)$ only depends on ε (not on B), $\lambda_{\partial R}$ is the linear measure on the boundary of R and C denotes a universal constant.*

We use this proposition to prove Theorem 1.2.

3. Proof of Theorem 1.1

We can assume that the ideal $I=I(f_1,\dots,f_N)$ is generated by $N+1$ Carleson–Newman Blaschke products. Actually, if $B\in I$ is a Carleson–Newman Blaschke product and $\varepsilon>0$ is sufficiently small, one has $B-\varepsilon f_j=B_jh_j$, $j=1,\dots,N$, where B_j is also a Carleson–Newman Blaschke product and $h_j^{-1}\in H^\infty$. So, $B,B_1,\dots,B_N$ generate the ideal I.

So, assume that $f,B_1,\dots,B_{N+1}\in H^\infty$ satisfy

$$|f(z)|\le C\sum_{j=1}^{N+1}|B_j(z)|,\quad z\in\mathbf{D}.$$

Fix $\gamma>0$. Let $D_H(z,r)=\{w:\varrho(z,w)<r\}$. Since $B_1,\dots,B_{N+1}$ are Carleson–Newman Blaschke products, for $j=1,\dots,N+1$ one has

$$|B_j(z)|\ge\eta_j,\quad\text{if } z\notin\bigcup_{n=1}^{\infty}D_H(z_{n,j},\gamma)=:R_j, \tag{4}$$

$$|B_j(z)|<\gamma,\quad\text{if } z\in R_j. \tag{5}$$

Here $\{z_{n,j}\}_{n=1}^\infty$ is the zero sequence of B_j. Let $\delta=\min_{1\le j\le N+1}\eta_j$, and define $R=\bigcap_{j=1}^{N+1}R_j$. Then by (4) and (5) one has

$$\sum_{j=1}^{N+1}|B_j(z)|\ge\delta\qquad\text{for } z\in\mathbf{D}\setminus R, \tag{6}$$

$$\sum_{j=1}^{N+1}|B_j(z)|<(N+1)\gamma\quad\text{for } z\in R. \tag{7}$$

Let $\tau>0$, to be defined later, and apply Proposition 2.2 to each of the functions B_j. We obtain $\mathcal{C}^\infty$-functions v_j on $\mathbf{D}$ such that for any $j=1,\dots,N+1$, one has

$$|B_j(z)-v_j(z)|<\tau,\quad z\in\mathbf{D}, \tag{8}$$

$$N(|\nabla v_j|\,dx\,dy) < \frac{C}{\tau}.$$

For $1 \le j \le N+1$ we define

$$g_j = \frac{\bar{v}_j}{\sum_{j=1}^{N+1} B_j \bar{v}_j} \chi_{\mathbf{D}\setminus R},$$

where χ_E denotes the characteristic function of the set E. Then

$$1 - \sum_{j=1}^{N+1} g_j B_j = \chi_R \quad \text{and} \quad \sum_{j=1}^{N+1} B_j \bar{\partial} g_j = -\bar{\partial}\chi_R. \tag{9}$$

Consider solutions $a_{j,k}$, $j,k=1,\dots,N+1$, of the respective $\bar{\partial}$-equations

$$\bar{\partial} a_{j,k} = f g_j \bar{\partial} g_k \tag{10}$$

and solutions b_j, $j=1,\dots,N+1$, of the $\bar{\partial}$-equations

$$\bar{\partial} b_j = \frac{f \bar{v}_j}{\sum_{k=1}^{N+1} B_k \bar{v}_k} \bar{\partial}\chi_R \tag{11}$$

and assume, momentarily, that $\sum_{j,k=1}^{N+1} \|a_{j,k}\|_{L^\infty} \le C$ and $\sum_{j=1}^{N+1} \|b_j\|_{L^\infty}$ is small. Put, for $j=1,\dots,N+1$,

$$h_j = f g_j + \sum_{k=1}^{N+1} (a_{j,k} - a_{k,j}) B_k + b_j. \tag{12}$$

By construction

$$f - \sum_{j=1}^{N+1} h_j B_j = f\chi_R - \sum_{j=1}^{N+1} b_j B_j$$

and then

$$\left\| f - \sum_{j=1}^{N+1} h_j B_j \right\|_{L^\infty} \le C(N+1)\gamma + \sum_{j=1}^{N+1} \|b_j\|_{L^\infty}. \tag{13}$$

Next, we verify that the functions h_j are analytic. By (9), (10), (11) and (12), we have

$$\begin{aligned}\bar{\partial} h_j &= f \bar{\partial} g_j + f \sum_{k=1}^{N+1} (g_j \bar{\partial} g_k - g_k \bar{\partial} g_j) B_k + \bar{\partial} b_j \\ &= f \bar{\partial} g_j \chi_R + f g_j \sum_{k=1}^{N+1} B_k \bar{\partial} g_k + \bar{\partial} b_j = -f g_j \bar{\partial}\chi_R + \bar{\partial} b_j = 0.\end{aligned}$$

Choose $\tau \le \delta^2/2(N+1)^2$. Next, we will find solutions of (10) and (11) with convenient L^∞-norms. Since by (8), one has

$$|fg_j| \le C \frac{(|B_j|+\tau)\sum_{k=1}^{N+1}|B_k|}{\sum_{k=1}^{N+1}|B_k|^2-(N+1)\tau}\chi_{\mathbf{D}\setminus R}$$

and because $\delta \le \sum_{k=1}^{N+1}|B_k|$ outside the region R, it follows that

$$\|fg_j\|_{L^\infty} \le C, \quad 1 \le j \le N+1.$$

Since

$$\bar\partial g_j \doteq \frac{\bar\partial v_j}{\sum_{k=1}^{N+1} B_k v_k}\chi_{\mathbf{D}\setminus R} - \frac{\bar v_j \sum_{k=1}^{N+1} B_k \bar\partial v_k}{\left(\sum_{k=1}^{N+1} B_k \bar v_k\right)^2}\chi_{\mathbf{D}\setminus R} - \frac{\bar v_j}{\sum_{k=1}^{N+1} B_k \bar v_k}\bar\partial \chi_R,$$

it follows that $|fg_j\bar\partial g_k|\,dx\,dy$ is a Carleson measure and

$$N(|fg_j\bar\partial g_k|\,dx\,dy) \le \frac{1}{\delta^2}N\left(\sum_{k=1}^{N+1}|\nabla v_k|\,dx\,dy\right)+\frac{1}{\delta}N(\lambda_{\partial R}).$$

From (4), and the fact that the measures $\mu_j = \sum_{n=1}^{\infty}(1-|z_{n,j}|^2)\delta_n$ are Carleson measures, we obtain that

$$N(\lambda_{\partial R}) \le C(N,\gamma).$$

Now, from Proposition 2.1, we may conclude the existence of solutions $a_{j,k}$ to (10) satisfying

$$\|a_{j,k}\|_{L^\infty(\partial\mathbf{D})} \le \frac{C}{\tau\delta^2}.$$

It remains to analyze the right member of equations (11). Since $\bar\partial\chi_R$ is supported in ∂R and on ∂R it holds that $\left|f\bar v_j\left(\sum_{k=1}^{N+1}B_k\bar v_k\right)^{-1}\right| \le C$, one has

$$N\left(\frac{f\bar v_j}{\sum_{k=1}^{N+1}B_k\bar v_k}\bar\partial\chi_R\right) \le CN(\lambda_{\partial R}) \le C(N,\gamma)$$

and $C(N,\gamma)$ tends to zero when γ tends to zero. Again by Proposition 2.1, this allows us to obtain b_j satisfying (11) with $\|b_j\|_{L^\infty(\partial\mathbf{D})}$ as small as desired. In particular, we have

$$h_j \in H^\infty \quad \text{and} \quad \|h_j\|_\infty \le \frac{C(N)}{\delta^4}.$$

As a consequence of (13), $\|f-h_1f_1-h_2f_2\|_\infty$ can be arbitrarily small. Hence $\operatorname{dist}_{H^\infty}(f,I)=0$, completing the proof.

4. Proof of Theorem 1.2, preliminary results

In this section we consider the H^p analogous of Bourgain's theorem on closures of ideals in H^∞. In contrast with the H^∞ case, the condition

$$M(g/|f|)\in L^p$$

is sufficient. First, we need some previous results.

Lemma 4.1. *Let ψ and φ be continuous functions defined on* $\mathbf{D}$. *Assume that $\sigma=|\psi|\,dx\,dy$ is a Carleson measure. Then*

$$\sigma(\{z\in\mathbf{D}:|\varphi(z)|>\lambda\})\le N(\sigma)|\{e^{i\theta}\in A:M\varphi(e^{i\theta})>\lambda\}|,$$

where $A=\{e^{i\theta}:\psi(z)\neq 0$ for some $z\in\Gamma(\theta)\}$.

Proof. We follow the standard proof in the upper halfplane $\mathbf{H}$. Let

$$R=\{z\in\mathbf{H}:|\varphi(z)|>\lambda\}\cap\{z:\psi(z)\neq 0\}$$

and

$$S=\{t:M\varphi(t)>\lambda\}\cap\{t:\psi(z)\neq 0 \text{ for some } z\in\Gamma(t)\}.$$

Let z be such that $\psi(z)\neq 0$ and $|\varphi(z)|>\lambda$. Then $M\varphi(t)>\lambda$ in the interval $I_z=\{t\in\mathbf{R}:|t-\operatorname{Re} z|<\operatorname{Im} z\}$. From this it follows that

$$\bigcup_{z\in R} I_z\subset S.$$

Since S is an open set, it is a union of a sequence of pairwise disjoint intervals $\{I_j\}$ whose centers are denoted by $c(I_j)$. Consider the sets T_j defined by

$$T_j=\bigl\{z=x+iy\in\mathbf{H}:|x-c(I_j)|+y<\tfrac{1}{2}|I_j|\bigr\}.$$

It is clear that $R\subset\bigcup_j T_j$, and since σ is a Carleson measure, one deduces

$$\sigma(R)\le\sum_j\sigma(T_j)\le N(\sigma)\sum_j|I_j|=N(\sigma)|S|.\quad\square$$

The next lemma is the version for H^p-spaces of the Carleson criterion on existence of solutions of the equation $\bar\partial b=g$.

Lemma 4.2. *Let $1 \le p < \infty$. Let G be a continuous function in $\mathbf{D}$ such that $G = \varphi\psi$, where $M\varphi \in L^p(\mathbf{T})$ and $d\sigma = |\psi|\, dx\, dy$ is a Carleson measure. Then there exists a function $b \in \mathcal{C}^1(\mathbf{D}) \cap \mathcal{C}(\overline{\mathbf{D}})$ such that $\bar\partial b = G$ and*

$$\int_0^{2\pi} |b(e^{i\theta})|^p \, d\theta \le N(\sigma) \int_A (M\varphi(e^{i\theta}))^p \, d\theta,$$

where $A = \{e^{i\theta} : \psi(z) \neq 0 \text{ for some } z \in \Gamma(\theta)\}$.

Proof. Let q be the conjugate exponent of p, that is $1 < q \le \infty$ and $1/p + 1/q = 1$. By duality,

$$\inf\{\|b\|_p : \bar\partial b = G\} = \sup\left\{ \left| \frac{1}{2\pi} \int_0^{2\pi} Fk \, d\theta \right| : k \in H_0^q \text{ and } \|k\|_q \le 1 \right\},$$

where F is an a priori solution of $\bar\partial F = G$ and H_0^q denotes the subspace of functions $f \in H^q$ such that $f(0) = 0$. By Green's formula, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} Fk \, d\theta = -\frac{1}{2\pi i} \int_{|z| \le 1} G(z) \frac{k(z)}{z} \, dz \wedge d\bar z.$$

Then

$$\inf\{\|b\|_p : \bar\partial b = G\} \le \sup\left\{ \frac{1}{\pi} \int_{\mathbf{D}} |G(z)|\, |k(z)| \, dx\, dy : k \in H^q \text{ and } \|k\|_q \le 1 \right\}.$$

Since $G = \varphi\psi$, Hölder's inequality yields

$$\begin{aligned} \int_{\mathbf{D}} |\varphi(z)|\, |k(z)|\, |\psi(z)| \, dx\, dy &\le \left(\int_{\mathbf{D}} |\varphi(z)|^p \, d\sigma \right)^{1/p} \left(\int_{\mathbf{D}} |k(z)|^q \, d\sigma \right)^{1/q} \\ &\le C \|k\|_q N(\sigma)^{1/q} \left(\int_{\mathbf{D}} |\varphi(z)|^p \, d\sigma \right)^{1/p} \end{aligned}$$

because σ is a Carleson measure. Now, the previous lemma finishes the proof. □

The following well-known lemma (see [5]) shows that is sufficient to prove the theorem when f_1 and f_2 are finite Blaschke products with simple zeros.

Lemma 4.3. *Let f be an analytic function in $\mathbf{D}$ which is continuous in $\overline{\mathbf{D}}$. Suppose that $0 < |f(z)| \le 1$ if $|z| = 1$. Let $E = \{z \in \mathbf{T} : |f(z)| < 1\}$. If E is non-empty, then there exists a sequence $\{B_n\}_{n=1}^\infty$ of finite Blaschke products with simple zeros such that $|B_n(z)| \to |f(z)|$ uniformly on compact subsets of $\overline{\mathbf{D}} \setminus \overline{E}$, and $B_n(z) \to f(z)$ uniformly on compact subsets of $\mathbf{D}$.*

5. Completion of the proof of Theorem 1.2

By a standard normal families argument, we can assume that the functions g, f_1 and f_2 are analytic in a neighbourhood of the closed unit disk. For simplicity, we will assume that $N=2$ and that $\|f_j\|_\infty \le 1$, $j=1, 2$. Since we can write $f_j \in H^\infty$ as a product $f_j = B_j F_j$ where B_j is a Blaschke product and F_j is an invertible function with $|F_j| \le 1$, the approximation result of Lemma 4.3 permits us to obtain Proposition 2.3 for functions in the unit ball of H^∞. By Proposition 2.3, given $\varepsilon > 0$ we obtain regions R_1 and R_2 and numbers $\delta_1(\varepsilon), \delta_2(\varepsilon) > 0$, such that $|f_j(z)| < \varepsilon$ if $z \in R_j$, $|f_j(z)| > \delta_j(\varepsilon)$ if $z \in \bar{\mathbf{D}} \setminus R_j$ and $N(\lambda_{\partial R_j}) < C_j$ for $j=1, 2$.

Let $\delta := \min(\delta_1(\varepsilon), \delta_2(\varepsilon))$ and $R = R_1 \cap R_2$, then

$$|f_1(z)| + |f_2(z)| < 2\varepsilon, \quad \text{if } z \in R, \tag{14}$$

$$|f_1(z)| + |f_2(z)| > \delta, \quad \text{if } z \in \bar{\mathbf{D}} \setminus R, \tag{15}$$

$$N(\lambda_{\partial R}) < C. \tag{16}$$

By Proposition 2.2 there exist functions $v_1, v_2 \in \mathcal{C}^\infty(\mathbf{D})$ such that

$$|v_j(z) - f_j(z)| < \tfrac{1}{10}\delta^2, \quad z \in \mathbf{D}, \tag{17}$$

$$N(|\nabla v_j|\, dx\, dy) < \frac{c_0}{\delta^2}. \tag{18}$$

For $j=1, 2$, define

$$h_j = \frac{\bar{v}_j}{f_1 \bar{v}_1 + f_2 \bar{v}_2} \chi_{\bar{\mathbf{D}} \setminus R},$$

then $1 - (h_1 f_1 + h_2 f_2) = \chi_R$ and $f_1 \bar{\partial} h_1 + f_2 \bar{\partial} h_2 = -\bar{\partial} \chi_R$. We will consider functions a_{12}, a_{21}, b_1 and b_2 satisfying

$$\bar{\partial} a_{jk} = g h_j \bar{\partial} h_k \quad \text{and} \quad \bar{\partial} b_k = \frac{g \bar{v}_k}{f_1 \bar{v}_1 + f_2 \bar{v}_2} \bar{\partial} \chi_R, \quad j, k = 1, 2. \tag{19}$$

Then, the functions

$$\begin{cases} g_1 = g h_1 + (a_{12} - a_{21}) f_2 + b_1, \\ g_2 = g h_2 + (a_{21} - a_{12}) f_1 + b_2 \end{cases}$$

are analytic. Thus $F = g - (f_1 g_1 + f_2 g_2)$ is analytic and $F = g\chi_R - b_1 f_1 - b_2 f_2$. Also

$$\|M(g\chi_R)\|_{L^p(\mathbf{T})} \to 0 \quad \text{when } \varepsilon \to 0$$

because $|g| \chi_R \le (2\varepsilon g / |f|) \chi_R$ and $M(g/|f|) \in L^p(\mathbf{T})$. Then

$$\|F\|_{L^p} \le \|M(g\chi_R)\|_{L^p} + \|b_1\|_{L^p} + \|b_2\|_{L^p}.$$

So it only remains to show that there exist solutions a_{jk} and b_k to the equations (19) with $\|a_{jk}\|_{L^p(\mathbf{T})}$ bounded and $\|b_k\|_{L^p(\mathbf{T})}$ as small as desired.

To show that solutions a_{jk} exist, we put $gh_j\bar{\partial}h_k=\Phi\Psi$, where $\Phi=g/|f|$ and $\Psi=|f|h_j\bar{\partial}h_k$. Now, applying Lemma 4.2, we can deduce the existence of these solutions, because by hypothesis $M\Phi\in L^p$, and to show that $|\Psi|\,dx\,dy$ is a Carleson measure, we can repeat the argument given in the proof of Theorem 1.1. Fix $j=1$, 2 and consider

$$G=\frac{g\bar{v}_j}{f_1\bar{v}_1+f_2\bar{v}_2}\bar{\partial}\chi_R.$$

We have $G=\varphi\psi$, where

$$\varphi=\frac{g}{|f|} \quad\text{and}\quad \psi=\frac{|f|\bar{v}_j}{f_1\bar{v}_1+f_2\bar{v}_2}\bar{\partial}\chi_R:=\psi_1\bar{\partial}\chi_R.$$

Since $\bar{\partial}\chi_R$ is supported on ∂R, on ∂R one has

$$|\psi_1|\le|f|(|f_j|+c_1\delta^2)|f|^{-2}\le 1+c_1\delta.$$

Hence $N(|\psi|\,dx\,dy)\le C_1N(\lambda_{\partial R})\le C_2$. By Lemma 4.2 there exists a solution b_j to the equation $\bar{\partial}b_j=G$ such that

$$\int_0^{2\pi}|b_j(e^{i\theta})|^p\,d\theta\le C\int_A(M\varphi)^p\,d\theta,$$

where $A=\{e^{i\theta}:\Gamma(\theta)\cap\partial R\neq\emptyset\}$. Then, choosing $\varepsilon=c_2\gamma$, we have that, when $\gamma\to0$, the set A tends to the set $Z(f_1)\cap Z(f_2)\cap\mathbf{T}$, which has Lebesgue measure zero. Then, by the absolute continuity of the integral, we have that $\|b_j\|_p\to0$ when $\gamma\to0$ and this finishes the proof.

References

1. AMAR, É., BRUNA, J. and NICOLAU, A., On H^p-solutions of the Bezout equation, *Pacific J. Math.* **171** (1995), 297–307.
2. BOURGAIN, J., On finitely generated closed ideals in $H^\infty(\mathbf{D})$, *Ann. Inst. Fourier* (*Grenoble*) **35**:4 (1985), 163–174.
3. CARLESON, L., Interpolation by bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547–559.
4. DAHLBERG, B., Approximation by harmonic functions, *Ann. Inst. Fourier* (*Grenoble*) **30**:2 (1980), 97–101.
5. DUREN, P., *Theory of H^p-spaces*, Academic Press, New York, 1970.
6. GARNETT, J. B., *Bounded Analytic Functions*, Academic Press, Orlando, Fla., 1981.

7. GORKIN, P., IZUCHI, K. and MORTINI, R., Higher order hulls in H^∞, II, *J. Funct. Anal.* **177** (2000), 107–129.
8. GORKIN, P., MORTINI, R. and NICOLAU, A., The generalized corona theorem, *Math. Ann.* **301** (1995), 135–154.
9. MORTINI, R., Ideals generated by interpolating Blaschke products, *Analysis* **14** (1994), 67–73.
10. MORTINI, R., On an example of J. Bourgain on closures of finitely generated ideals, *Math. Z.* **224** (1997), 655–663.
11. TOLOKONNIKOV, V., Interpolating Blaschke products and ideals of the algebra H^∞, *Zap. Nauchn. Sem. Leningrad. Otdel. Math. Inst. Steklov.* (*LOMI*) **126** (1983), 196–201 (Russian). English transl.: *J. Soviet Math.* **27** (1984), 2549–2553.
12. TOLOKONNIKOV, V., Blaschke products satisfying the Carleson–Newman condition and ideals of the algebra H^∞, *Zap. Nauchn. Sem. Leningrad. Otdel. Math. Inst. Steklov.* (*LOMI*) **149** (1986), Issled. Linein. Teor. Funktsii. **15**, 93–102, 188 (Russian). English transl.: *J. Soviet Math.* **42** (1988), 1603–1610.

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