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Deficient rational functions and Ahlfors's theory of covering surfaces

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Abstract. We prove a second fundamental theorem in the sense of Nevanlinna's theory of meromorphic functions replacing the constants a in $\overline{N}(r,f,a)$ by rational functions R with $R(\infty)=a$. The key argument is Ahlfors's second fundamental theorem from his theory of covering surfaces.

1. Introduction

The central result in Nevanlinna's theory of meromorphic functions in the plane is his second fundamental theorem

$$(q-2)T(r,f)\le\sum_{k=1}^{q}\overline{N}(r,f-a_k,0)+S(r,f), \tag{1}$$

where the a_k are distinct points on the sphere. (From now on we assume that the case $a_k=\infty$ is interpreted in an appropriate manner.) For notation and results of this theory we refer to [H], [N] and [Y].

Many efforts have been made to generalize this theorem in the sense that the constants a_k in (1) can be replaced by so called small meromorphic functions, i.e. meromorphic functions with $T(r,a_k)=o(T(r,f))$. The first theorem of this type was proved by R. Nevanlinna for the case $q=3$ using a Möbius transformation which makes it possible to apply (1). The development culminated in Steinmetz's theorem [S]:

$$(q-2-\varepsilon)T(r,f)\le\sum_{k=1}^{q}N(r,f-a_k,0)+S(r,f) \tag{2}$$

for distinct small functions a_k. An independent proof was given by Osgood [O2]. Earlier this was proved for rational functions a_k by Frank and Weissenborn [FW]. For the history of these questions and a proof of (2) we refer to [Y].

It is an open question whether (2) holds if N is replaced by $\overline{N}$. If this was true it would have interesting consequences, e.g. Nevanlinna's theorem on five shared values would carry over to the "small function case".

In this note we will not treat the case of arbitrary small functions. We consider a simpler situation, namely we replace the constants a_k in (1) by rational functions R_k with $R_k(\infty)=a_k$. It turns out that (1) is true in this case. Therefore the usual corollaries of (1) remain valid. We will give no proofs for these corollaries. Keeping in mind that rational functions have only finitely many poles the classical proofs for constants a_k work without any changes.

2. Results

2.1. Theorem. *Let f be a transcendental meromorphic function in the plane and R_k be rational functions with distinct values at ∞. Then*

$$(3)\qquad (q-2)T(r,f)\leq\sum_{k=1}^{q}\overline{N}(r,f-R_k,0)+S(r,f).$$

We admit that Theorem 2.1 is almost a direct consequence of Ahlfors's theory of covering surfaces (see [A], [H] and [N]). Nonetheless, it seems to us that the simple combination of Ahlfors's second fundamental theorem and Rouché's theorem is an interesting argument. It can certainly be applied in various situations. Further the statement seems to be unknown and could be of some value for applications. Probably our method cannot handle the case when many R_k have the same value at ∞. Note that the error term in (3) is a bit worse than the original $S(r,f)$ in (1). See [M] for details. We remark that Osgood proved in [O1] that a transcendental meromorphic function can have at most four completely ramified small functions, pointing in the same direction.

Proof of Theorem 2.1. Let $a_k:=R_k(\infty)$. Without loss of generality we can assume that all a_k are finite. Indeed, considering $1/(f-c)$ and $1/(R_k-c)$ with suitable $c\in\mathbf{C}$ will cause no problems in the following reasoning. Choose disks D_k around the a_k with disjoint closures. According to Ahlfors's second fundamental theorem it holds

$$(4)\qquad (q-2)A(r,f)\leq\sum_{k=1}^{q}\bar{n}(r,f,D_k)+O(L(r,f)).$$

An easy application of Rouché's theorem shows that, except for finitely many, all islands of f over D_k contain a zero of $f-R_k$, i.e. $\bar{n}(r,f,D_k)\leq\bar{n}(r,f-R_k,0)+O(1)$.

Combining this with (4) and logarithmic integration yields the result. The statement about the error term follows from the result of Miles [M] concerning the logarithmic integration of $L(r,f)$. □

It is natural to introduce

$$\Theta_N(f,a):=\liminf_{r\to\infty}\left(1-\frac{\overline{N}(r,f-a,0)}{T(r,f)}\right) \tag{5}$$

for small functions a.

2.2. Corollary. *Let f be a transcendental meromorphic function and R_a be a family of rational functions with $R_a(\infty)=a$, $a\in\widehat{\mathbf{C}}$. Then*

$$\sum_{a\in\widehat{\mathbf{C}}}\Theta_N(f,R_a)\leq 2.$$

2.3. Corollary. *Let f be a transcendental meromorphic function. The set of values $a\in\widehat{\mathbf{C}}$ such that there exists a rational function with $R_a(\infty)=a$ and $\Theta_N(f,R_a)>0$ is at most countable.*

For $a\in\widehat{\mathbf{C}}$ let $\overline{N}(r,f,\mathbf{D}_\varepsilon(a))$ be the logarithmically integrated counting function of islands of f over $\mathbf{D}_\varepsilon(a):=\{z:|z-a|<\varepsilon\}$ (with obvious modification for $a=\infty$). We define for $a\in\widehat{\mathbf{C}}$,

$$\Theta_A(f,a,\varepsilon):=\liminf_{r\to\infty}\left(1-\frac{\overline{N}(r,f,\mathbf{D}_\varepsilon(a))}{T(r,f)}\right)$$

and

$$\Theta_A(f,a):=\lim_{\varepsilon\to0}\Theta_A(f,a,\varepsilon). \tag{6}$$

Inequality (4) gives $\sum_{a\in\widehat{\mathbf{C}}}\Theta_A(f,a)\leq 2$, hence $\Theta_A(f,a)>0$ for at most countably many $a\in\widehat{\mathbf{C}}$. From $\bar{n}(r,f,a)\geq\bar{n}(r,f,\mathbf{D}_\varepsilon(a))$ it follows $\Theta_N(f,a)\leq\Theta_A(f,a)$. Strict inequality is possible as $f(z):=e^z-1/z$ shows. One easily checks $\Theta_N(f,0)=0$ and $\Theta_A(f,0)=1$. Considering $f_\alpha(z):=e^z-\alpha/z$ shows that a generalized defect relation for Θ_A does not hold since $\sum_{\alpha\in\mathbf{C}}\Theta_A(f_\alpha,0)$ diverges. On the other hand Θ_N in Corollary 2.2 can be replaced by Θ_A.

The same argumentation as in the proof of Theorem 2.1 shows that the countable set in Corollary 2.3 is contained in the countable set of all $a\in\widehat{\mathbf{C}}$ with $\Theta_A(f,a)>0$. By the above example strict inclusion is possible.

2.4. Theorem. *Let f be a transcendental meromorphic function and $a \in \widehat{\mathbf{C}}$ with $\Theta_A(f,a)=0$. Then $\Theta_N(f,R_a)=0$ for all rational functions R_a with $R_a(\infty)=a$.*

We conclude: If f has no deficient value in the sense of (6) then f has no deficient rational function in the sense of (5).

It is known that two transcendental meromorphic functions that share seven small functions are identical [T] and that for entire functions seven can be replaced by four (even in $\mathbf{C}^n$) [L]. From Theorem 2.1 we obtain the following result.

2.5. Corollary. *Let f and g be transcendental meromorphic functions that share five rational functions with distinct values at ∞ then $f=g$.*

We denote by $N_{1)}$ the counting function of simple zeros. A well-known variation of (1) also carries over to our situation.

2.6. Theorem. *Let f be a transcendental meromorphic function in the plane and R_k be rational functions with distinct values at ∞. Then*

$$(q-4)T(r,f) \le \sum_{k=1}^{q} N_{1)}(r,f-R_k,0)+S(r,f).$$

Proof. The proof is almost the same as for Theorem 2.1. Instead of (4) we use

$$(q-4)A(r,f) \le \sum_{k=1}^{q} n_s(r,f,D_k)+O(L(r,f)),$$

where n_s is the counting function of schlicht islands. The zeros obtained from Rouché's theorem in schlicht islands are simple. □

We therefore have the following generalization of the theorem on the number of completely ramified values.

2.7. Corollary. *Let f be a transcendental meromorphic function and R_a be a family of rational functions with $R_a(\infty)=a$, $a \in \widehat{\mathbf{C}}$. Then, except for possibly four a-values, the equation $f=R_a$ has infinitely many simple roots.*

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