Ark. Mat., **39** (2001), 223–243

On the uncertainty principle for M. Riesz potentials

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1. Introduction

Let μ be a Borel charge (real measure) in $\mathbf{R}^d$ satisfying

$$\int_{\mathbf{R}^d} \frac{d|\mu|(t)}{1+|t|^{d-\alpha}} < +\infty \tag{1}$$

for an $\alpha \in (0,d)$. We denote by U^μ_α (or $U_\alpha \mu$) the M. Riesz potential of μ of order α,

$$U^\mu_\alpha(s) = \int_{\mathbf{R}^d} \frac{d\mu(t)}{|t-s|^{d-\alpha}}, \quad s \in \mathbf{R}^d. \tag{2}$$

(Note that $U^\mu_\alpha(s) < +\infty$ m-a.e. in $\mathbf{R}^d$, where $m=m_d$ is Lebesgue measure in $\mathbf{R}^d$.) If μ is m-absolutely continuous, i.e. $\mu = fm$, then we write $U^\mu_\alpha = U^f_\alpha$ (or $U_\alpha f$).

The following result is due to M. Riesz [10]: *if α is not an even integer, then $|\mu|$ and U^μ cannot vanish on the same nonvoid open set unless $\mu=0$*:

$$(E \text{ is open, } E \neq \emptyset,\ |\mu|(E)=0,\ U^\mu_\alpha|_E = 0) \implies \mu = 0.$$

For $d=1$ this "uncertainty principle" applies to sets $E \subset \mathbf{R}$ of positive length (not necessarily open) provided $\mu = fm$ and f is *sufficiently smooth*. To give a precise statement we need the following definition: a charge μ in $\mathbf{R}$ is called *small of order $\gamma>0$ at the point $s_0 \in \mathbf{R}$* if

$$\int_{(s_0-1,s_0+1)} \frac{d|\mu|(t)}{|t-s_0|^\gamma} < +\infty.$$

(1) Research supported in part by RFFI, grant N 99-01-00720 and NSERC.

If $\gamma \geq 1$ and μ is small of order γ at any point of a Borel set E, then $|\mu|(E)=0$ (see [6], p. 513, for a proof), but not vice versa: $|\mu|(E)=0$ does not imply the γ-smallness of μ on E.

Theorem 1. (The uncertainty principle for the Riesz potentials on the line) *Suppose $E \subset \mathbf{R}$ is a Borel set, $m(E)>0$, $\alpha \in (0,1)$, and μ satisfies (1) with $d=1$. If μ is small of order $2-\alpha$ at any point of E and $U_\alpha^\mu|_E=0$, then $\mu=0$.*

(We put $m=m_1$.) This theorem was proved in [5], see also [4], [6, pp. 516–518].

Corollary. *Let μ, α and E be as in Theorem 1. Suppose $\mu=fm$ in a neighborhood of E, f being a Hölder function of order $1-\alpha+\varepsilon$, $\varepsilon>0$. If $f|_E=U_\alpha^\mu|_E=0$ and $m(E)>0$, then $\mu=0$.*

The smallness of μ of order $2-\alpha$ (instead of just $|\mu|(E)=0$) in Theorem 1 and the Hölder condition in Corollary look strange and make the impression of redundant technicalities due to the (possibly inadequate) method of proof.

However, in the present paper we give a *negative* answer to the following question: *is it possible to drop the Hölder condition in the corollary, replacing it by the mere continuity of f?* Our result is the following theorem.

Theorem 2. *For any $\alpha \in (0,1)$ there exists a nonzero function $f \in C(\mathbf{R})$ such that*

$$m(\{x : f(x)=0\} \cap \{x : U_\alpha^f(x)=0\}) > 0. \tag{3}$$

Thus the smallness condition imposed on μ in Theorem 2 and the Hölder condition imposed on f in its corollary are essential. Note that the M. Riesz potentials differ in this respect from their "limit case" (as $\alpha \nearrow 1$), namely, from the logarithmic potentials U_1^μ,

$$U_1^\mu(s) = \int_{\mathbf{R}} \log|s-t| \, d\mu(t)$$

(we assume $\int_{|t|>2} \log|t| \, d|\mu|(t) < +\infty$): if $|\mu|(E)=0$, $U_1^\mu|_E=0$, and $m(E)>0$, then $\mu=0$. No extra smallness of μ is needed here unlike in Theorem 1. (If $\mu=fm$ and f is, say, continuous, then the derivative of U_1^μ coincides with the Hilbert transform of f, and our assertion reduces to the classical boundary uniqueness theorems for functions analytic in the upper half-plane; for the general case see [8]).

Theorem 2 generalizes easily to any value of d (see Section 12 below). However we mainly concentrate on the case $d=1$. It is of special interest, since it is only in this case that our result exhibits the sharpness of a uniqueness theorem (our Theorem 1). The validity of multidimensional analogs of Theorem 1 is an open question. Let us briefly discuss the most important particular case $d \geq 2$ and $\alpha=1$

when U_α^μ can be extended to the ambient space $\mathbf{R}^{d+1}$ ($\mathbf{R}^d=\{(s_1,\dots,s_d,0)\in\mathbf{R}^{d+1}\}$) as the Newtonian potential (with respect to $\mathbf{R}^{d+1}$) of the charge μ carried by the hyperplane $\mathbf{R}^d$. From this point of view $U_1^\mu|_{\mathbf{R}^d}$ and f become the Cauchy data of the function U_1^μ harmonic in the upper half-space $\mathbf{R}_+^{d+1}=\{((s_1,\dots,s_{d+1}):s_{d+1}>0\}$: f is the normal derivative $\partial U/\partial s_{d+1}$ of $U=U_1^f$ on $\mathbf{R}^d$ (up to a constant factor). Our question is now the uniqueness of the solution of the Cauchy problem for the Laplace equation: for which sets $E\subset\mathbf{R}^d$ do the Cauchy data $U|_E$ and $\partial U/\partial s_{d+1}|_E$ uniquely determine the harmonic function U? Bourgain and Wolff proved in [3] that there exists a nonzero function $U\in C^1(\mathbf{R}_+^{d+1}\cup\mathbf{R}^d)$, harmonic in $\mathbf{R}_+^{d+1}$, and such that U and $\operatorname{grad} U$ vanish on the same subset of $\mathbf{R}^d$, whose d-dimensional Lebesgue measure is positive. It is not known for which values of r such an example is possible with $U\in C^r(\mathbf{R}_+^{d+1}\cup\mathbf{R}^d)$; it is an open question whether r can be 2 or even $+\infty$. The one-dimensional result of this paper combined with Theorem 1 suggests that this question probably has a negative answer. (Note that in fact the function U constructed in [3] can be written as U_1^f with an $f\in C(\mathbf{R}^d)$.) The uniqueness properties of the Cauchy problem for the Laplace equation are the theme of [7], [6, part II, Chapter V], and of papers by Mergelyan, Landis, M. M. Lavrent′ev, and N. Rao quoted therein. The theme is closely related to the uniqueness problems for the gradients of harmonic functions, see [12], [1] and [2].

Our proof of Theorem 2 is an adaptation of the method of [3], a nice version of a "correction scheme" used in [9], [1], [2] and going back to Men′shov (see historical remarks in [12]). The method applies smoothly to the formal inverse of U_α (see, however, Remark 1 in Section 10). The case $d=2$ and $\alpha=1$ coincides in fact with the subject of [3]. But even in this classical case it is useful to deal with convolutions ("potentials") in $\mathbf{R}^d$ rather than harmonic functions in $\mathbf{R}^{d+1}$. This point of view simplifies and clarifies the choice of the special functions F_ε of [3], making it almost compulsory (this choice presented a serious difficulty in the initial version of the path breaking paper [12]; it was simplified in [1], but our approach makes it *quite* easy).

Our proof of Theorem 2 yields a function f whose modulus of continuity ω_f satisfies

$$\omega_f(\delta)=O(\delta^{c/\log|\log\delta|}),\quad \delta\to 0,$$

with a positive c. We believe f can be made just Hölder, but the cost is a more complicated construction in the spirit of [12] and [2]. For the time being we prefer a simpler scheme of [3].

Acknowledgment. We are grateful to Paul Koosis who read the manuscript and pointed out some misprints and minor mistakes.

2. The operator W_α

We start with some notation. We need operators D, V_α and W_α defined as follows:

$$(Dh)(s)=h'(s),$$
$$(V_\alpha h)(s)=\frac{1}{\alpha}\int_{-\infty}^{+\infty} h(t)\frac{\operatorname{sgn}(s-t)}{|s-t|^\alpha}\,dt,$$
(4) $$W_\alpha=DV_\alpha$$

(the factor α^{-1} will be convenient in what follows). The operator V_α will be defined on compactly supported C^∞-functions:

$$\operatorname{dom} V_\alpha=\operatorname{dom} W_\alpha=C_0^\infty(\mathbf{R}).$$

In our proof of Theorem 2 we deal with W_α rather than with U_α as defined in Section 1 (see (2) with $d=1$; we assume that the domain of U_α is $\operatorname{dom} U_\alpha=\{f\in C(\mathbf{R}):\mu=fm$ satisfies (1) for $d=1\}$.

Lemma 1. *The operator W_α is the right inverse of U_α in the following sense*:

(5a) $$W_\alpha(C_0^\infty(\mathbf{R}))\subset \operatorname{dom} U_\alpha;$$

(5b) $$U_\alpha W_\alpha\psi=c\psi \quad \textit{for any } \psi\in C_0^\infty(\mathbf{R}),\ c=c(\alpha)\neq 0.$$

This fact is well known. It is a particular case of much more general inversion formulae for the M. Riesz potentials [11] and can be proved by the Fourier transform. We sketch here a direct proof. First,

(6) $$\alpha W_\alpha\psi=-D\psi*\frac{\operatorname{sgn} x}{|x|^\alpha},\quad \psi\in\operatorname{dom} W_\alpha,$$

and

(7) $$(W_\alpha\psi)(s)=(\psi*|x|^{-\beta})(s),\quad \psi\in\operatorname{dom} W_\alpha,\ s\notin\operatorname{supp}\psi,\ \beta=\alpha+1,$$

whence $W_\alpha(\psi)(s)=O(|s|^{-\beta})$, $|s|\to+\infty$, and (5a) follows. It is sufficient to prove (5b) for $s=0$. We have

(8) $$(U_\alpha W_\alpha\psi)(0)=\lim_{N\to+\infty}\int_{-N}^{N} W_\alpha(\psi)(x)\,\frac{dx}{|x|^{1-\alpha}}=\lim_{N\to+\infty}\int_{\mathbf{R}}\psi'(y)K_N(y)\,dy,$$
$$K_N(y):=\frac{1}{\alpha}\int_{-N}^{N}\frac{1}{|x|^{1-\alpha}}\,\frac{\operatorname{sgn}(x-y)}{|x-y|^\alpha}\,dx.$$

Suppose $y>0$ and write $K_N(y)$ as $(\int_{-N}^{-y}+\int_y^N)+\int_{-y}^{y}$. Setting $x=ty$ we see that the sum in brackets is bounded uniformly in N and y; its limit (as $N\to+\infty$) and $\int_{-y}^{y}$ do not depend on y; K_N is odd whence $\lim_{N\to+\infty} K_N(y)=\text{const}\cdot\operatorname{sgn} y$, $y\in\mathbf{R}$. In fact

$$\begin{aligned}\text{const} &= \lim_{N\to+\infty} K_N(1) = \text{p.v.}\frac{1}{\alpha}\int_{-\infty}^{\infty}\frac{1}{|x-1|^{1-\alpha}}\frac{\operatorname{sgn} x}{|x|^{\alpha}}\,dx \\ &= \frac{1}{\alpha}\int_0^{\infty}\frac{1}{|x|^{\alpha}}\left(\frac{1}{|x-1|^{1-\alpha}}-\frac{1}{|x+1|^{1-\alpha}}\right)dx>0.\end{aligned}$$

We conclude from (8) that $U_\alpha W_\alpha=\text{const}\int_{\mathbf{R}}\psi'(y)\operatorname{sgn} y\,dy$, and we are done.

An advantage of W_α (compared to U_α) is its homogeneity of *positive* order α. Denote by C_λ the λ-contraction of the argument:

$$(C_\lambda\psi)(t)=\psi(\lambda t),\quad t\in\mathbf{R},\ \lambda>0. \tag{9}$$

The identity

$$W_\alpha C_\lambda=\lambda^\alpha C_\lambda W_\alpha \tag{10}$$

and the shift invariance of W_α are almost all we need for the proof of Theorem 2.

3. General plan of the proof

From now on $\alpha\in(0,1)$ will be fixed, and we write U and W instead of U_α and W_α. We put $I=(-\frac{1}{2},\frac{1}{2})$.

Given a small number $\sigma>0$ we are going to construct a sequence $(g_n)_{n=1}^{\infty}$ in $C_0^\infty(\mathbf{R})$ and a decreasing sequence $(V_n)_{n=1}^{\infty}$ of subsets of I such that

(A) $\operatorname{supp} g_n\subset 3I$, $n=1,2,\dots$, and $g_n\equiv g_1$ in $3I\setminus I$, $g_1\not\equiv 0$, $g_1\equiv 0$ in I;

(B) $\sum_{n=1}^{\infty}|\operatorname{supp}(g_{n+1}-g_n)|<\sigma$ (we write $|A|$ instead of $m(A)$);

(C) the sequence $(f_n)_{n=1}^{\infty}$, where

$$f_n=Wg_n, \tag{11}$$

converges uniformly on $\mathbf{R}$ to a (continuous!) function $f\in\operatorname{dom} U$, and

$$\lim_{n\to\infty}(Uf_n)(t)=U(f)(t),\quad t\in\mathbf{R}; \tag{12}$$

(D) $|V_n|\ge 1-\sigma$, $n=1,2,\dots$, and $\lim_{n\to\infty}\int_{V_n}|f_n|^p\,dm=0$, where p is a positive number depending only on α.

Suppose this program has been fulfilled. Then we see from (C) and (D) that

$$f|_V = 0, \quad \text{where } V = \bigcap_{n=1}^{\infty} V_n, \ |V| \ge 1-\sigma.$$

Now, by Lemma 1, $cg_n = Uf_n$ (where c is the constant from (5b)), and by (C), $\lim_{n\to\infty} cg_n = Uf$ pointwise on $\mathbf{R}$, whence by (A)

$$Uf \equiv 0 \quad \text{on } I \setminus \bigcup_{n=1}^{\infty} \operatorname{supp}(g_{n+1}-g_n),$$

i.e. on a part of I of length at least $1-\sigma$ (see (B)). Hence

$$|\{x : f(x)=0\} \cap \{x : Uf(x)=0\}| \ge 1-2\sigma > 0 \quad \text{if only } \sigma < \tfrac{1}{2}.$$

Note that $f \not\equiv 0$, since $Uf = \lim_{n\to\infty} g_n \equiv g_1$ on $3I \setminus I$ (by (A)).

4. Some integral means

In the process of constructing $(g_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ the size of various functions appearing underway will often be measured by their integral means. We call

$$M_Q(h) = \left(\frac{1}{|Q|} \int_Q |h|^p \, dm \right)^{1/p}$$

the integral mean of order p of $h \in C(\mathbf{R})$ over the interval Q (p does not figure in the notation, since our $p \in (0,1)$ will remain fixed). Note that $M_Q(h) = h(c)$ for a $c \in Q$, whence for any $\nu > 0$,

$$\sup_Q |h| \le M_Q(h) + \operatorname{osc}_Q h, \tag{13}$$

where $\operatorname{osc}_Q h = \sup\{|h(t)-h(u)| : t, u \in Q\}$. And

$$|Q| = \frac{1}{M_Q(h)^p} \int_Q |h|^p \, dm. \tag{14}$$

We will use the inequality

$$M_Q(h+k)^p \le M_Q(h)^p + M_Q(k)^p.$$

5. Parameters defining the construction. The operator $h \mapsto h_Q$

Our inductive procedure should be preceded by a choice of the following parameters: the positive numbers A, B (big), λ and p (small), and three sequences of positive numbers $\vec{\varepsilon}=(\varepsilon_n)_{n=1}^{\infty}$, $\vec{\delta}=(\delta_n)_{n=1}^{\infty}$, $\vec{K}=(K_n)_{n=1}^{\infty}$.

The parameters could be explicitly defined just now, but to make their choice natural and understandable we postpone the definitions to the moments when they will be really needed and thus explained. We only mention here that $\vec{\varepsilon}$, $\vec{\delta}$ and $\overrightarrow{K^{-1}}$ are infinitesimals, $\overrightarrow{\varepsilon^{-1}}$ and $\vec{K_n}$ growing not too fast (as some powers of n); $1/\delta_n$ will be positive integers, $1/\delta_{n+1}$ being a multiple of $1/\delta_n$. These numbers will be defined inductively (with $\delta_1=1$). We denote by H_n a family of $1/\delta_n$ disjoint open subintervals of I of length δ_n each. Every $Q\in H_{n+1}$ is contained in a unique $Q^*\in H_n$. Let h be a function defined on $\mathbf{R}$. For a positive ε and an interval Q we put

$$h_\varepsilon=\frac{1}{\varepsilon}C_{1/\varepsilon}h, \quad h_Q(t)=h\left(\frac{t-c_Q}{|Q|\lambda}\right), \quad t\in\mathbf{R}, \tag{15}$$

c_Q denotes the center of Q. "The gauge parameter" λ (to be chosen later, but fixed throughout the proof) is incorporated into the definition of h_Q, but we do not include it in the notation. Note an obvious (but important) identity

$$M_Q(h_Q)=M_{\lambda^{-1}I}(h). \tag{16}$$

6. The sequences (g_n), (V_n): description of the recursive process

We start with $V_1=I$ and a nonzero $g_1\in C_0^\infty(\mathbf{R})$ with the support in $3I\setminus I$. At the n-th step we will have constructed $g_n\in C_0^\infty(\mathbf{R})$ and a family $G_n\subset H_n$ of "good" intervals of length δ_n. The set $V_n=\bigcup_{Q\in G_n} Q$ is supposed to satisfy

$$\int_{V_n}|f_n|^p\,dm\le A^p q^{np}, \quad q=(1-B\lambda)^{1/2p}<1 \tag{17_n}$$

(we assume $G_1=H_1=\{I\}$). We define

$$G_{n+1}=\{Q\in H_{n+1}: Q\subset Q^*\in G_n \text{ and } M_Q(f_n)\le K_{n+1}q^n\}. \tag{18}$$

Clearly $V_{n+1}\subset V_n$. Suppose $Q\in H_{n+1}\setminus G_{n+1}$, $Q\subset V_n$. Then by (14) and (18), $|Q|\le K_{n+1}^{-p}q^{-np}\int_Q|f_n|^p\,dm$. Summing over all these Q's and using (17_n) we see that

$$|V_n\setminus V_{n+1}|\le\frac{A^p}{K_{n+1}^p} \quad\text{and}\quad |I\setminus V_{n+1}|\le\sum_{k=1}^{n}|V_k\setminus V_{k+1}|\le A^p\sum_{k=1}^{n+1}\frac{1}{K_{n+1}^p}, \tag{19}$$

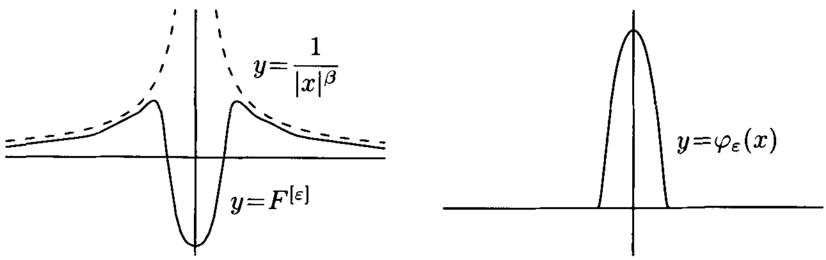

Figure 1.

so that $|I\setminus V_{n+1}|$ is small whenever $\sum_{k=1}^{\infty} K_k^{-p}$ is small, and $G_{n+1}\neq\emptyset$. To define $\vec{\delta}$ put $\delta_1=1$, and assuming f_n has already been defined, find a positive δ_{n+1} satisfying

$$\omega_{f_n}(\delta_{n+1})\leq\frac{\theta q^n}{K_{n+1}},\quad n=1,2,\dots, \tag{20}$$

where ω_{f_n} is the modulus of continuity of f_n. The small constant θ depending only on λ and B will be gradually specified in the process of proof. Note that by (20)

$$\operatorname{osc}_Q f_n\leq\frac{\theta q^n}{K_{n+1}},\quad Q\in H_{n+1},\quad n=1,2,\dots\ . \tag{20$'$}$$

Now we are ready to define g_{n+1} and get (17_{n+1}). We subtract a correcting term r_n from g_n,

$$g_{n+1}=g_n-r_n,$$

so as to make the means $M_Q(f_{n+1})$, $Q\in G_{n+1}$, small, whereas $|\operatorname{supp} g_{n+1}\cap I|$ increases only slightly compared with $|\operatorname{supp} g_n\cap I|$. To define r_n first choose a mollifier $\phi\in C_0^\infty(\mathbf{R})$ with $\operatorname{supp}\phi\subset I$ and $\int_{\mathbf{R}}\phi\,dm=1$ and put

$$r_n=\sum_{Q\in G_{n+1}}(\lambda\delta_{n+1})^\alpha f_n(c_Q)\phi_{\varepsilon_n Q} \tag{21}$$

($\phi_{\varepsilon_n Q}=(\phi_{\varepsilon_n})_Q$, see (15)), so that

$$f_{n+1}=f_n-W(r_n)=f_n-\sum_{Q\in G_{n+1}} f_n(c_Q)F_Q^{[\varepsilon_n]},\quad F^{[\varepsilon]}=W\phi_\varepsilon \tag{22}$$

(we have used the homogeneity of W, see (10)). Thus r_n and W_n are linear combinations of the building blocks $\phi_{\varepsilon_n Q}$ and $F_Q^{[\varepsilon_n]}$, respectively. We have tried to visualize them (very approximately) in Figure 1. It is interesting to note that $F^{[\varepsilon]}$ tends to $|x|^{-\beta}$, as $\varepsilon\searrow 0$, (i.e. to the dotted graph in the left picture), but very reluctantly, diving deep under the x-axis at the origin. This fact does not prevent $F^{[\varepsilon]}$ from being close to $|x|^{-\beta}$ in every $L^p(-N,N)$ with $p<1/\beta$.

Now, $\operatorname{supp} r_n$ is contained in the union of at most δ_{n+1}^{-1} intervals of length $\lambda\varepsilon_n\delta_{n+1}$ each, and

$$|\operatorname{supp} g_{n+1}\cap I|\le|\operatorname{supp} g_n\cap I|+\varepsilon_n \tag{23}$$

(we assume $\lambda<1$). Applying (13) and (20) to $h=f_n$ we get

$$|f_n(c_Q)|\le 2K_{n+1}q^n,\quad Q\in G_{n+1}, \tag{24}$$

(we assume $\theta<\frac{1}{2}$ and $K_{n+1}>1$).

7. Deduction of (17_{n+1}) from (17_n): some philosophy

Our main concern is to pass from (17_n) to (17_{n+1}) (by a proper choice of δ_{n+1}). We have to compare f_{n+1} with f_n, which satisfies (17_n). First note that $F^{[\varepsilon]}(t)\equiv\varepsilon^{-\beta}F^{[1]}(t/\varepsilon)$ (see (22) and (10)), and by (7), $F^{[1]}$ is a bounded function satisfying $|F^{[1]}(t)|\le c|t|^{-\beta}$, $t\in\mathbf{R}$, $c=c(\phi,\alpha)$. Hence

$$\begin{aligned}&|F^{[\varepsilon]}(t)|\le C\min\left\{\frac{1}{\varepsilon^\beta},\frac{1}{|t|^\beta}\right\},\quad t\ne0,\\ &\sup_{\mathbf{R}}|F_Q^{[\varepsilon]}|\le\frac{c}{\varepsilon^\beta},\\ &|F_Q^{[\varepsilon]}(t)|\le c\frac{(\lambda\delta_{n+1})^\beta}{|t-c_Q|^\beta},\qquad t\notin Q,\ Q\in H_{n+1},\ \beta=\alpha+1.\end{aligned} \tag{25}$$

The last estimate means that $F_Q^{[\varepsilon]}$ is "almost concentrated" on Q decaying fast enough as its argument moves away from Q. This observation suggests that on a $Q\in G_{n+1}$,

$$f_{n+1}\approx f_n-f_n(c_Q)F_Q^{[\varepsilon]}. \tag{26}$$

The error of this approximation (to be measured in the "weak" L^p-metric with a $p\in(0,1)$) is likely to be small, since the contributions of intervals $Q'\in G_{n+1}\setminus\{Q\}$ to r_n (i.e. "the tails" of $f_n(c_Q)F_{Q'}^{[\varepsilon_n]}$) are negligible on Q (see Lemma 3 below). The right-hand side in (26) is

$$(f_n-f_n(c_Q))+f_n(c_Q)(1-F_Q^{[\varepsilon_n]}).$$

The first bracket can be made arbitrarily (and uniformly) small on Q if $\delta_{n+1}=|Q|$ is small enough. The L^p-estimate of the second bracket is the heart of the construction (as in [3], [12], [1] and [2]). It turns out that *for any interval* Q,

$$M_Q(1-F_Q^{[\varepsilon]})<q^2 \tag{27}$$

for a constant $B=B(\alpha)$ *and all* $\lambda<\lambda(\alpha)$, $\varepsilon<\varepsilon(\alpha)$ (see Lemma 2 below). So subtracting $F_Q^{[\varepsilon]}$ from 1 reduces the mean $M_Q(1)=1$ by a factor q^2 strictly less than 1. Thus $F_Q^{[\varepsilon]}$ acts as if it were the characteristic function $\chi_{\widetilde{Q}}$ of a large portion $\widetilde{Q}$ of Q, although it is very different from any $\chi_{\widetilde{Q}}$.

8. Main lemma: a generalization of (27)

We may replace 1 in (27) by any constant h just multiplying $F_Q^{[\varepsilon]}$ and q^2 by h. This version of (27) can be generalized even to *a function* h provided $\operatorname{osc}_Q h/|h(c_Q)|$ is majorized by a small constant.

Lemma 2. *There exist positive numbers* $\lambda(\alpha)$, $p=p(\alpha)\in(0,1/\beta)$, *and* $B=B(\alpha)$ *such that* $\lambda(\alpha)B<1$ *and for any* $\lambda\in(0,\lambda(\alpha))$ *and* $\varepsilon\in(0,\varepsilon(\lambda))$,

$$M_Q(h-h(c_Q)F_Q^{[\varepsilon]})\le q^2|h(c_Q)|,\quad q=(1-B\lambda)^{1/2p}, \tag{28}$$

for any bounded interval Q *and* $h\in C(Q)$ *satisfying*

$$\operatorname{osc}_Q h\le(\lambda B)^{1/p}|h(c_Q)|. \tag{29}$$

This is a quantitative property of the functions $F^{[\varepsilon]}$ (or rather of the function $F^{[1]}$ depending only on ϕ, see Section 7). The proof is based on some quite concrete preliminary computations.

Proof. Put $F^{[0]}(t)=|t|^{-\beta}$; $F^{[\varepsilon]}$ is now defined for any nonnegative ε, and

$$\lim_{\varepsilon\to0}F_Q^{[\varepsilon]}(t)=F^{[0]}(t),\quad |F^{[\varepsilon]}(t)|\le\frac{c}{|t|^\beta},\quad t\ne0,\ \varepsilon\ge0, \tag{30}$$

(the equality follows from (7) with $\phi=\phi_\varepsilon$, and the inequality from (25)). Put

$$\begin{aligned} J(p)&=\int_{\mathbf{R}}(|1-F^{[0]}|^p-1)\,dm,\\ J(p,\lambda,\varepsilon)&=\int_{\lambda^{-1}I}(|1-F^{[\varepsilon]}|^p-1)\,dm,\quad \lambda>0,\ \varepsilon\ge0,\\ L&=\int_{\mathbf{R}}\log|1-F^{[0]}|\,dm. \end{aligned} \tag{31}$$

All integrals are finite, since $p\beta<1$ and the integrals are majorized by $c|t|^{-p\beta}$ if $|t|$ is small and by $c|t|^{-\beta}$ if $|t|$ is large, see (25). Now,

$$\lim_{p\searrow0}\frac{J(p)}{p}=L \tag{32}$$

($(a^p-1)/p$ is monotone in p for any $a>0$ and tends to $\log a$ as $p\searrow 0$). But L *is negative*:

$$\begin{aligned}\tfrac{1}{2}L&=\int_0^1\log\frac{1-x^\beta}{x^\beta}\,dx+\int_1^\infty\log\Big(1-\frac{1}{x^\beta}\Big)\,dx\\&=-\beta\int_0^1\log x\,dx+\int_0^1\log(1-x^\beta)\,dx+\int_1^\infty\log\Big(1-\frac{1}{x^\beta}\Big)\,dx\\&=\beta-\sum_{k=1}^\infty\frac{1}{k}\int_0^1x^{\beta k}\,dx-\sum_{k=1}^\infty\frac{1}{k}\int_1^\infty\frac{dx}{x^{\beta k}}\\&=\beta-\sum_{k=1}^\infty\frac{2\beta}{k^2\beta^2-1}=\pi\cot\frac{\pi}{\beta}<0,\end{aligned}$$

since $\beta=1+\alpha\in(1,2)$. By (32), $J(p)<0$ if $p<1/\beta$ is small enough (depending only on α). From now on p is supposed to be fixed and satisfy this condition. For any $\lambda>0$,

$$\lim_{\varepsilon\searrow 0}J(p,\lambda,\varepsilon)=J(p,\lambda,0)\tag{33}$$

by (30) and the dominated convergence theorem. Clearly $\lim_{\lambda\to 0}J(p,\lambda,0)=J(p)$ whence $J(p,\lambda,0)<\frac{1}{2}J(p)<0$ for $\lambda<\lambda(\alpha)$, and by (33),

$$J(p,\lambda,\varepsilon)<\tfrac{1}{2}J(p),\quad \lambda\in(0,\lambda(\alpha)),\ \varepsilon\in(0,\varepsilon(\alpha)).\tag{34}$$

Now we are ready to prove (28) (if (29) is fulfilled). Raising the left part of (28) to the power p we get

$$\begin{aligned}X:&=M_Q(h-h(c_Q)F_Q^{[\varepsilon]})^p\le M_Q(h-h(c_Q))^p+|h(c_Q)|^pM_Q(1-F_Q^{[\varepsilon]})^p\\&\le(\operatorname{osc}_Q h)^p+|h(c_Q)|^pM_{\lambda^{-1}I}(1-F^{[\varepsilon]})^p\end{aligned}$$

(we have used (16)). It is easy to see that $M_{\lambda^{-1}I}(1-F^{[\varepsilon]})^p=1+\lambda J(p,\lambda,\varepsilon)$. We may assume $h(c_Q)\neq 0$ (otherwise $\operatorname{osc}_Q h=0$ by (29)). Put $B=\frac{1}{4}|J(p)|$ and continue: if $\lambda\in(0,\lambda(\alpha))$ and $\varepsilon\in(0,\varepsilon(\alpha))$, then, by (34),

$$\begin{aligned}X&\le|h(c_Q)|^p\bigg(1+\Big(\frac{\operatorname{osc}_Q h}{|h(c_Q)|}\Big)^p+\lambda J(p,\lambda,\varepsilon)\bigg)\\&<|h(c_Q)|^p\Big(1+\lambda\frac{|J(p)|}{4}+\lambda\frac{J(p)}{2}\Big)=|h(c_Q)|^p(1-B\lambda),\end{aligned}$$

and we get (28).

Remark. The integral $J(p)$ is computed explicitly in [1] for any p in $(0,1/\beta)$. We prefer our easy reduction to L, since we need only one p making $J(p)$ negative.

9. Justification of (26)

Here we estimate (following [3]) the contribution of "the tails" of $f_n(c_{Q'})F_{Q'}^{[\varepsilon_n]}$ to $W(r_n)$ on a $Q\in G_{n+1}$ where $Q'\in G_{n+1}(Q):=G_{n+1}\setminus\{Q\}$. Put

$$T_{n+1}^Q=W(r_n)-f_n(c_Q)F_Q^{[\varepsilon_n]}=\sum_{Q'\in G_{n+1}(Q)} f_n(c_{Q'})F_{Q'}^{[\varepsilon_n]}.$$

Lemma 3. *For $t\in Q$ and $Q\in G_{n+1}$,*

$$|T_{n+1}^Q(t)|\le c(\alpha)\lambda^\beta(|f_n(c_Q)|+3q^n) \tag{35}$$

provided (20) *is fulfilled.*

Proof. Recall the elementary fact that if $\Phi\in C([0,+\infty))$ is nonnegative and decreasing, then

$$\sum_{j=0}^\infty \Phi(j)\le\Phi(0)+\int_0^{+\infty}\Phi(t)\,dt. \tag{36}$$

For a $\varrho>0$ we denote by $G^+(\varrho)$ ($G^-(\varrho)$) the set of all $Q'\in G_{n+1}(Q)$ lying to the right (to the left) of Q and satisfying $\operatorname{dist}(c_{Q'},Q)\ge\varrho$. Put

$$\sigma_\varrho^\pm=\sum_{Q'\in G^\pm(\varrho)}|F_{Q'}^{[\varepsilon]}|,\quad \sigma_\varrho^\pm=\sigma_\varrho^++\sigma_\varrho^-.$$

Using (25) we get

$$\sigma_\varrho^+(t)\le c\lambda^\beta\delta_{n+1}^\alpha\sum_{Q'\in G^+(\varrho)}\frac{\delta_{n+1}}{(c_{Q'}-t)^\beta}.$$

Let Q^* be the first (i.e. the closest to Q) element of G_ϱ^+. The last sum is $\sum_{j=0}^\infty\Phi(j)$, where $\Phi(x)=\delta_{n+1}/(c_{Q^*}+\delta_{n+1}x-t)^\beta$, whence by (36),

$$\begin{aligned}\sigma_\varrho^+&\le c\lambda^\beta\delta_{n+1}^\alpha\left(\frac{\delta_{n+1}}{(c_{Q^*}-t)^\beta}+\int_0^{+\infty}\frac{du}{((c_{Q^*}-t)+u)^\beta}\right)\\&\le c\lambda^\beta\delta_{n+1}^\alpha\left(\frac{2}{(c_{Q^*}-t)^\alpha}+\frac{\alpha}{(c_{Q^*}-t)^\alpha}\right)\le c(\alpha)\lambda^\beta\left(\frac{\delta_{n+1}}{\varrho}\right)^\alpha\end{aligned}$$

(we have used the estimates $c_{Q^*}-t\ge\frac12\delta_{n+1}$ and $c_{Q^*}-t\ge\varrho$). A similar estimate holds for $\sigma_\varrho^-(t)$, and

$$\sigma_\varrho(t)\le c(\alpha)\lambda^\beta\left(\frac{\delta_{n+1}}{\varrho}\right)^\alpha. \tag{37}$$

Now, for any $M>0$ and $t\in Q$,

$$|T_{n+1}^Q(t)|\leq \sum_{Q'\in G(M\delta_{n+1})} |f_n(c_{Q'})|\,|F_{Q'}^{[\varepsilon_{n+1}]}(t)| + \sum_{Q'\in G_{n+1}(Q)\setminus G(M\delta_{n+1})} |f_n(c_{Q'})|\,|F_{Q'}^{[\varepsilon_{n+1}]}(t)|.$$

From (37) and (24) we conclude that

$$\sum_{Q'\in G(M\delta_{n+1})} |f_n(c_{Q'})|\,|F_{Q'}^{[\varepsilon_{n+1}]}(t)|\leq 2K_{n+1}q^n\sigma_{M\delta_{n+1}}\leq \frac{2c(\alpha)K_{n+1}q^n\lambda^\beta}{M}$$

(we have used the obvious estimate $\operatorname{dist}(c_{Q'},Q)\geq\frac{1}{2}\delta_{n+1}$, $Q'\in G_{n+1}(Q)$). Now, if $Q'\in G_{n+1}(Q)\setminus G(M\delta_{n+1})$, then $|c_Q-c_{Q'}|\leq(M+1)\delta_{n+1}$ and if $M>2$,

$$|f_n(c_{Q'})-f_n(c_Q)|\leq\omega_{f_n}((M+1)\delta_{n+1})\leq(M+2)\omega_{f_n}(\delta_{n+1})\leq 2M\omega_{f_n}(\delta_{n+1}),$$

so that $|f_n(c_{Q'})|\leq|f_n(c_Q)|+2M\omega_{f_n}(\delta_{n+1})$. We get (again by (37))

$$\sum_{Q'\in G_{n+1}(Q)\setminus G(M\delta_{n+1})} |f_n(c_{Q'})|\,|F_{Q'}^{[\varepsilon_{n+1}]}(t)|\leq c(\alpha)2^\alpha\lambda^\beta[|f_n(c_Q)|+2M\omega_{f_n}(\delta_{n+1})]$$

and thus for any $M>2$,

$$|T_{n+1}^Q(t)|\leq 2c(\alpha)\lambda^\beta\left(|f_n(c_Q)|+2M\omega_{f_n}(\delta_{n+1})+\frac{K_{n+1}q^n}{M}\right).$$

Now recall that $\omega_{f_n}(\delta_{n+1})\leq K_{n+1}^{-1}q^n$, see (20). Choosing $M=K_{n+1}$ we get (35).

Remark. Suppose $t\in\mathbf{R}\setminus V_{n+1}$. Then

$$|f_{n+1}(t)-f_n(t)|\leq 2K_{n+1}q^n\sum_{Q\in G_{n+1}}|F_Q^{[\varepsilon_n]}(t)|\leq c(\alpha)\lambda^\beta K_{n+1}q^n,$$

since the sum can be estimated as in (37) and $\operatorname{dist}(t,c_Q)\geq\frac{1}{2}\delta_{n+1}$ for any $Q\in G_{n+1}$. If $t\in Q$ and $Q\in G_{n+1}$, then by (25) and (35),

$$|f_{n+1}(t)-f_n(t)|\leq|f_n(c_Q)|\,|F_Q^{[\varepsilon_n]}(t)|+|T_{n+1}^Q|\leq\frac{c(\alpha)K_{n+1}q^n\lambda^\beta}{\varepsilon_n^\beta}.$$

10. Deduction of (17_{n+1}) from (17_n): the proof

Now we realize the vague program sketched in Section 7. Assume (17_n) and show that (20) entails (17_{n+1}) if the parameters λ, $\theta=\theta(\lambda,B)$ and A have been defined properly.

Divide G_{n+1} into two parts G^l_{n+1} and G^s_{n+1} on which f_n is "large" or "small" compared with q^n,

$$G^l_{n+1}=\{Q\in G_{n+1}:|f_n(c_Q)|\geq q^n\},\quad G^s_{n+1}=G_{n+1}\setminus G^l_{n+1}.$$

(a) Estimate of $M_Q(f_{n+1})$ for $Q\in G^l_{n+1}$. We prove

$$M_Q(f_{n+1})\leq q^{1.25}M_Q(f_n). \tag{38}$$

It is at this stage that we make the final choice of λ. For $Q\in G^l_{n+1}$ put $P^Q_{n+1}=f_n-f_n(c_Q)F^{[\varepsilon_n]}_Q$ so that

$$f_{n+1}=P^Q_{n+1}+T^Q_{n+1}=P+T$$

for short. The function P is the principal term of this decomposition *off an exceptional part* of Q. In fact P is quite small on a "bad" part Q_b of Q where $F^{[\varepsilon_n]}_Q\approx 1$ (recall that $F^{[\varepsilon_n]}_Q=1$ at some points (see Figure 1)). We define

$$Q_b=\{t\in Q:|F^{[\varepsilon_n]}_Q(t)-1|<\lambda^{\alpha/2}\},\quad Q_g=Q\setminus Q_b.$$

Luckily $|Q_b|/|Q|$ can be made small if λ is small, so that the bad part Q_b contributes very little to $M_Q(f_n)$. To see this, note that for $\mu\in(0,1)$,

$$R^\mu_\varepsilon:=\{t:|F^{[\varepsilon]}(t)-1|<\mu\}\subset\{t:|F^{[0]}(t)-1|<2\mu\}\cup\{t:|F^{[\varepsilon]}(t)-F^{[0]}(t)|>\mu\}.$$

But

$$\big|\{t:|F^{[0]}(t)-1|<2\mu\}\big|\leq C_1(\alpha),\quad\text{and}\quad\big|\{t:|F^{[\varepsilon]}(t)-F^{[0]}(t)|>\mu\}\big|<1$$

if $\varepsilon<\varepsilon(\alpha,\mu)$ (since $\int_I|F^{[0]}-F^{[\varepsilon]}|^p\,dm+\int_{\mathbf{R}\setminus I}|F^{[0]}-F^{[\varepsilon]}|\,dm\to 0$, as $\varepsilon\to 0$, see (30)). Hence $|R^\mu_\varepsilon|<C(\alpha)$ for $\varepsilon<\varepsilon(\alpha,\mu)$. For $\mu=\lambda^{\alpha/2}$,

$$Q_b\subset\{t:|F^{[\varepsilon_n]}_Q(t)-1|<\mu\}=\lambda|Q|R^\mu_\varepsilon+c_Q,$$

and

$$|Q_b|\leq C(\alpha)\lambda|Q| \tag{39}$$

(we assume $\varepsilon_n<\varepsilon(\alpha,\mu)$, $n=1,2,\ldots$). We need the estimate

$$\operatorname{osc}_Q f_n \le \theta|f_n(c_Q)|, \quad Q\in G^l_{n+1}, \tag{40}$$

implied by (20), (20′) and the definition of G^l_{n+1}. For $Q\in G^l_{n+1}$ and $t\in Q_g$ we have

$$|P(t)| \ge |f_n(c_Q)|\,|1-F^{[\varepsilon]}_Q(t)|-\operatorname{osc}_Q f_n \ge |f_n(c_Q)|(\lambda^{\alpha/2}-\tfrac{1}{2}\lambda^{\alpha/2})=\tfrac{1}{2}|f_n(c_Q)|\lambda^{\alpha/2} \tag{41}$$

if $\theta<\frac{1}{2}\lambda^{\alpha/2}$, and, by (35),

$$\begin{aligned}|f_{n+1}(t)| &= |P(t)|\left|1+\frac{T(t)}{P(t)}\right| \le |P(t)|\left(1+\frac{c(\alpha)\lambda^{\beta}[|f_n(c_Q)|+3q^n]}{\frac{1}{2}|f_n(c_Q)|\lambda^{\alpha/2}}\right)\\ &\le |P(t)|(1+c'(\alpha)\lambda^{1+\alpha/2}),\end{aligned} \tag{42}$$

since the square bracket is $\le 4|f_n(c_Q)|$. So far we stayed in Q_g. If $t\in Q_b$ and $\theta<\lambda$, then

$$\begin{aligned}|f_{n+1}(t)|^p &\le |P(t)|^p+|T(t)|^p\\ &\le |f_n(c_Q)|^p\lambda^{\alpha p/2}+(\operatorname{osc}_Q f_n)^p+4^p|f_n(c_Q)|^p c(\alpha)^p\lambda^{\beta p}\\ &\le c'(\alpha)|f_n(c_Q)|^p\lambda^{\gamma p},\end{aligned} \tag{43}$$

where γ is a positive number. Combining (41) and (43) we obtain (for any $Q\in G^l_{n+1}$)

$$\begin{aligned}Y:=\int_Q |f_{n+1}|^p\,dm &= \int_{Q_g}|f_{n+1}|^p\,dm+\int_{Q_b}|f_{n+1}|^p\,dm\\ &\le (1+c''(\alpha)\lambda^{1+\alpha/2})\int_Q |P|^p\,dm+c'(\alpha)^p|f_n(c_Q)|^p\lambda^{\gamma p}|Q_b|.\end{aligned}$$

Now, $\int_Q |P|^p\,dm$ can be estimated using (28) (with f_n as h) provided $\theta^p<\lambda B$ (see (29) with $h=f_n$, and (20′)).

Taking (39) into account we get

$$\begin{aligned}Y &\le (1+c''(\alpha)\lambda^{1+\alpha/2})^p|f_n(c_Q)|^p q^{2p}|Q|+c'(\alpha)^p|f_n(c_Q)|^p\lambda^{\gamma p}\lambda|Q|\\ &\le q^{2p}|f_n(c_Q)|^p|Q|(1+\tilde{c}(\alpha)\lambda^{1+\xi})\end{aligned}$$

for a $\xi>0$ (we assume $\lambda<\frac{1}{2}B$, so that $q^{-2p}=(1-B\lambda)^{-1}<2$; we have also used $(1+c''(\alpha)\lambda^{1+\alpha/2})^p<1+pc''(\alpha)\lambda^{1+\alpha/2})$. By (13) and (40),

$$(1-\theta)|f_n(c_Q)| \le M_Q(f_n),$$

whence $|f_n(c_Q)| \le q^{-0.5} M_Q(f_n)$ if θ is small, and

$$M_Q(f_{n+1}) = \left(\frac{Y}{|Q|}\right)^{1/p} \le q^{1.5} M_Q(f_n)(1+\tilde{c}(\alpha)\lambda^{1+\xi})^{1/p}. \tag{44}$$

The last bracket in (44) is $1+O(\lambda^{1+\xi})$, whereas $q^{-0.25} = (1-B\lambda)^{-0.125/p} = 1+c\lambda+o(\lambda)$, as $\lambda \searrow 0$ ($c=0.125B/p$). Hence the last bracket in (44) is less than $q^{-0.25}$ if $\lambda \le \lambda(\alpha)$, and we get (38).

Remark. In the proof of (38) we have deviated from [3] where no distinction is made between the "bad" and "good" zones Q_b and Q_g. We were unable to follow the argument on p. 258 of that beautiful paper.

(b) Suppose $Q \in G^s_{n+1}$. Then on Q,

$$\begin{aligned} |f_{n+1}|^p &= |f_n(c_Q)(1-F_Q^{[\varepsilon_n]}) + (f_n - f_n(c_Q)) + T^Q_{n+1}|^p \\ &\le q^{np}|1-F_Q^{[\varepsilon_n]}|^p + (\operatorname{osc}_Q f_n)^p + c(\alpha)^p \lambda^{\beta p} 4^p q^{np} \\ &\le q^{np}(|1-F_Q^{[\varepsilon_n]}|^p + 5) \end{aligned}$$

(we have used the estimates $c(\alpha)\lambda^\beta < 1$, $|f_n(c_Q)| \le q^n$ and $\operatorname{osc}_Q f_n \le q^n$). Hence by (27),

$$\int_Q |f_{n+1}|^p \, dm \le q^{np}(q^{2p}+5)|Q| \le 6q^{np}|Q|, \quad Q \in G^s_{n+1}. \tag{45}$$

At last, using (38), (45), and (17_n) we get

$$\begin{aligned} \int_{V_{n+1}} |f_{n+1}|^p \, dm &= \sum_{Q \in G^l_{n+1}} \int_Q |f_{n+1}|^p \, dm + \sum_{Q \in G^s_{n+1}} \int_Q |f_{n+1}|^p \, dm \\ &\le q^{1.25p} \sum_{Q \in G^l_{n+1}} \int_Q |f_n|^p \, dm + 6q^{np} \\ &\le q^{1.25p} A^p q^{np} + 6q^{np} = A^p q^{(n+1)p}\left(q^{0.25p} + \frac{6}{A^p}\right) < A^p q^{(n+1)p}, \end{aligned}$$

if A is sufficiently big (depending on q, i.e. on λ, B and p). Thus we have proved (17_{n+1}).

Remark. Let us review the order of choice of our parameters. We first choose $p=p(\alpha)$ (as in Section 8, see the estimate following (32)) thus fixing $B=\frac{1}{4}|J(p)|$. Our next step is to choose λ (see Section 8 and the end of the proof of (38)) and thus determine q (see (17_n)) and θ. Then we choose A so as to make $q^{0.25}+6A^{-p}<1$, not forgetting $\int_I |f_1|^p \, dm < Aq$ to start the process. We have yet to specify $\vec{\varepsilon}$ and $\vec{K}$.

11. Final steps

Let us go back to our general plan (Section 3). Part (A) is fulfilled. Now, $\operatorname{supp}(g_{n+1}-g_n)\subset\bigcup_{Q\in H_n}(c_Q+\varepsilon_n\delta_n I)$ and $\operatorname{card} H_n=1/\delta_n$ so that $|\operatorname{supp}(g_{n+1}-g_n)|\le \delta_n\varepsilon_n/\delta_n=\varepsilon_n$, and we need only

$$\sum_{n+1}^{\infty}\varepsilon_n<\sigma$$

to get (B). We prove (C). By the remark at the end of Section 9,

$$|f_{n+1}-f_n|\le\frac{cK_{n+1}q^n}{\varepsilon_n^\beta}$$

whereas $K_{n+1}\varepsilon_n^{-\beta}=O(n^A)$ for a positive A. The series

$$f_1+\sum_{n=1}^{\infty}(f_{n+1}-f_n)$$

converges uniformly on $\mathbf{R}$ to an $f\in C(\mathbf{R})$, $f=\lim_{n\to\infty}f_n$. Now,

$$|g_{n+1}-g_n|\le\sum_{Q\in G_{n+1}}|f_n(c_Q)|\,|\phi_{\varepsilon_n Q}|\le\frac{2K_{n+1}q^n\max|\phi|}{\varepsilon_n},$$

the supports of $\phi_{\varepsilon Q}$'s being disjoint. Hence $(g_n)_{n=1}^{\infty}$ converges uniformly on $\mathbf{R}$, and $\operatorname{supp} g_n\subset 3I$, $n=1,2,\dots$. From the identity $f_n=Ug_n$ we conclude that $|f_n(t)|+|f(t)|\le\text{const}|t|^{-\beta}$ for $|t|\ge4$ so that $f\in\operatorname{dom}U$ and for any t

$$\frac{|f_n(s)|}{|s-t|^{1-\alpha}}\le\frac{c}{|s|^{1+\alpha}|s|^{1-\alpha}}=\frac{c}{s^2}$$

if $|s|>2|t|$, $|s|>4$, $n=1,2,\dots$. This estimate justifies the passage to the limit in the integrals $\int_{\mathbf{R}}f_n(s)|t-s|^{1-\alpha}\,ds$ as $n\to\infty$, and we get (12). And, at last, (D) follows from (17_n) and (19) if

$$A^p\sum_{n=1}^{\infty}\frac{1}{K_{n+1}^p}<\sigma.$$

To finish the proof let us specify $\vec{\varepsilon}=(cn^{-\gamma})_{n=1}^{\infty}$ and $\vec{K}=(n^\gamma/c)_{n=1}^{\infty}$, where $\gamma>1/p$ and c is sufficiently small.

12. Concluding remarks

(a) Here we sketch an estimate of ω_f. First $DF_Q^{[\varepsilon]}=\delta_{n+1}^{-1}\varepsilon^{-\beta-1}(DF^{[1]})_Q$, $Q\in H_{n+1}$, and $DF^{[1]}(t)=O(|t|^{-\beta-1})$, $|t|\to\infty$, whence

$$|DF_Q^{[\varepsilon]}(t)|\le\min\left\{\frac{c}{\delta_n\varepsilon_n^{\beta+1}},\frac{c\delta_{n+1}^{\beta}}{|t-c_Q|^{\beta+1}}\right\}.$$

As in Section 9 we deduce $|(T_{n+1}^Q)'|\le CK_{n+1}q^n/\delta_{n+1}$ on $Q\in G_{n+1}$, and

$$|(f_{n+1}-f_n)'|\le|f_n(c_Q)|\,|F_Q^{[\varepsilon_n]\prime}|+|(T_{n+1}^Q)'|\le\frac{CK_{n+1}q^n}{\delta_{n+1}\varepsilon_n^{\beta+1}}+\frac{CK_{n+1}q^n}{\delta_{n+1}}\le\frac{C'K_{n+1}q^n}{\delta_{n+1}\varepsilon_n^{\beta+1}}$$

on Q and actually everywhere (see the remark at the end of Section 9). Hence

$$\max|f'_{n+1}|\le\max|f'_n|+\frac{C'K_{n+1}q^n}{\varepsilon_n^{\beta+1}\delta_{n+1}}. \tag{46}$$

Put $\delta_{n+1}=\theta K_{n+1}^{-1}q^n/\max|f'_n|$ to satisfy (20). By (46)

$$\max|f'_{n+1}|\le\left(1+\frac{K_{n+1}^2}{\theta\varepsilon_n^{\beta+1}}\right)\max|f'_n|\le n^c\max|f'_n|,\quad n=1,2,\dots,$$

so that $\max|f'_n|\le(n!)^c\max|f'_1|\le n^{c_1n}$, $n\ge2$. Now take a small $\delta>0$ and suppose $|t'-t''|\le\delta$. Then

$$|f(t')-f(t'')|\le\delta\max|f'_n|+\sum_{k=n}^{\infty}|f_{k+1}(t)-f_k(t)|\le n^{c_1n}\delta+q_1^n,$$

where $0<q_1<1$. Choose n so as to make $n^{c_1n}\delta=q_1^n$, i.e.

$$\frac{n}{r}\log\frac{n}{r}=\frac{1}{c_1r}\log\frac{1}{\delta},\quad r=q_1^{1/c_1}.$$

If $x\log x=y$ and y is large, then $x\asymp y/\log y$. Thus $n\asymp c_2\log(1/\delta)/\log\log(1/\delta)$ as $\delta\searrow 0$, and $|f(t')-f(t'')|\le c_3\exp(c_4\log\delta/\log|\log\delta|)$, $c_3,c_4>0$. To get rid of the double logarithm in this exponent (that is to get a Hölder f) would require an essential change in the construction.

On the other hand it is not hard to see that $Uf\in C^\alpha$. This can be shown by simple estimates of the Hölder norms $\|Uf_n\|_\alpha$ using the uniform boundedness and decay of f_n's at infinity. But we also can deduce this result from the inequalities

$$\|g_{n+1}-g_n\|_\alpha\le2K_{n+1}q^n\lambda^\alpha\delta_{n+1}^\alpha\sum_{Q\in G_{n+1}}\|\phi_{\varepsilon_nQ}\|_\alpha,$$

$$\|\phi_{\varepsilon_nQ}\|_\alpha=\frac{\|\phi\|_\alpha}{\varepsilon_n^\beta\delta_{n+1}^\alpha\lambda^\alpha}.$$

and

$$\operatorname{dist}(\operatorname{supp}\phi_{\varepsilon_n Q'}, \operatorname{supp}\phi_{\varepsilon_n Q''}) \geq \varepsilon_n \delta_{n+1}, \quad Q' \neq Q'',$$

whence $\|g_{n+1}-g_n\| = O(q_1^n)$ for a $q_1 \in (0,1)$.

(b) Theorem 2 generalizes easily to the M. Riesz potentials in $\mathbf{R}^d$ for $\alpha \in (0,d)$: *there is a nonzero function* $f \in \operatorname{dom} U_\alpha$ *such that* $m_d(\{x: f(x)=U_\alpha f(x)=0\})>0$. The construction is quite similar to the case $d=1$. The role of I is played by the cube of side one centered at the origin, and H_n is now the set of equal subcubes of I of volumes δ_n^d. We only have to write down W_α, "the inverse" of U_α, a well-known operator (see [11]) which we heuristically describe here for the convenience of the reader. Denote by $\mathcal{F}$ the Fourier transform in $\mathbf{R}^d$. Clearly $\mathcal{F}(U_\alpha h)(\xi) = c\mathcal{F}(h)(\xi)/|\xi|^\alpha$, whence

$$\mathcal{F}(W_\alpha h)(\xi) = c|\xi|^\alpha \mathcal{F}(h)(\xi), \quad h \in C_0^\infty(\mathbf{R}^d), \ \xi \in \mathbf{R}^d. \tag{47}$$

If d is even, $d=2m$, then writing $|\xi|^\alpha = |\xi|^d |\xi|^{\alpha-d}$ gives

$$W_\alpha(h) = \Delta^m U_{d-\alpha}. \tag{48}$$

If $d=2m+1$, then W_α can be written using the Riesz transform (multiplication by the vector $\xi/|\xi|$ in the Fourier coordinates), Δ and div. But we prefer a simpler formula. Note that even α's are of no interest for us, since $\Delta^k f$ and $U_{2k}\Delta^k f = cf$ both vanish on the interior of $\{x: f(x)=0\}$, $f \in C_0^\infty(\mathbf{R}^d)$, $k=1,2,\dots,m$. Suppose k is an integer, $0 \leq k \leq m$, and $2k<\alpha<2k+2$. Then $|\xi|^\alpha = |\xi|^{2k+2}/|\xi|^{2k+2-\alpha}$, and (47) gives

$$W_\alpha = c\Delta^{k+1} U_{d-(2k+2-\alpha)}. \tag{48'}$$

We need W_α only on $C_0^\infty(\mathbf{R}^d)$, but let us heuristically try the sign of

$$L = \int_{\mathbf{R}^d} \log|1-cW_\alpha \delta|\, dm_d.$$

From (48) and (48′) we easily conclude that (with a due c and positive C)

$$\begin{aligned} L &= \int_{\mathbf{R}^d} \log\left|1-|x|^{-(d+\alpha)}\right| dx = C\int_0^\infty \log\left|1-r^{-(d+\alpha)}\right| dr^d \\ &= C\int_0^\infty \log\left|1-u^{-(d+\alpha)/d}\right| du, \end{aligned}$$

and $1<(d+\alpha)/d<2$. Applying the computation from the proof of Lemma 2 we see that $L<0$. Therefore

$$J(p)=\int_{\mathbf{R}^d}(|1-cW_\alpha\delta|^p-1)\,dm_d<0$$

if $p>0$ is small. Now we spread the point mass δ slightly, replacing it by a ϕ_ε, and get

$$\int_{\mathbf{R}^d}(|1-F^{[\varepsilon]}|^p-1)\,dm_d<0, \quad F^{[\varepsilon]}=W\phi_\varepsilon,\ 0<\varepsilon<\varepsilon(\alpha).$$

Recall that $J(p)$ is computed in [1].

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Received April 17, 2000

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