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On the Siciak extremal function for real compact convex sets

Len Bos, Jean-Paul Calvi and Norman Levenberg

1. Introduction

Let E be a bounded Borel set in $\mathbf{C}^N$. Define

$$V_E(z) := \sup\{u(z) : u \in L,\ u \le 0 \text{ on } E\}, \tag{1.1}$$

where

$$L := \{u \text{ plurisubharmonic in } \mathbf{C}^N : u(z) \le \log^+ |z| + C \text{ for some } C\}$$

is the class of plurisubharmonic functions of logarithmic growth (here we have $|z| = (\sum_{j=1}^N |z_j|^2)^{1/2}$ and $\log^+ |z| = \max\{0, \log |z|\}$). Then the upper semicontinuous regularization $V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta)$ is called the (Siciak) extremal function of E. If K is a compact set in $\mathbf{C}^N$, then the extremal function in (1.1) can be gotten via the formula

$$V_K(z) := \max\left\{0, \sup\left\{\frac{1}{\deg p} \log |p(z)| : p \text{ holomorphic polynomial, } \|p\|_K \le 1\right\}\right\} \tag{1.2}$$

(Theorem 5.1.7 in [K1]). Here, $\|p\|_K := \sup_{z \in K} |p(z)|$ denotes the uniform norm on K. We say that K is regular if and only if $V_K^* = V_K$. Note that if we let

$$\widehat{K} := \{z \in \mathbf{C}^N : |p(z_1, \dots, z_N)| \le \|p\|_K \text{ for all polynomials } p\}$$

denote the polynomial hull of K, then

1. $\widehat{K} = \{z \in \mathbf{C}^N : V_K(z) = 0\}$;
2. $V_{\widehat{K}} = V_K$.

For future use, we say that K is polynomially convex if $K=\widehat{K}$.

In one complex variable ($N=1$), the function V_K (or, in general, V_K^*), is the classical Green function of the planar compact set K with logarithmic pole at infinity. The theory of conformal mapping can be used to find explicit formulas for V_K in many cases. We recall the case of an interval: since $h(\zeta):=\zeta+\sqrt{\zeta^2-1}$ is a conformal map of the complement of the interval $[-1,1]$ onto the complement of the closed unit disk, we have $V_{[-1,1]}(\zeta)=\log|h(\zeta)|$. In $\mathbf{C}^N$ for $N>1$, examples of explicit (or even semi-explicit!) formulas for V_K are severely lacking. The first interesting formulas, due to Siciak, dealt with product sets and circled sets (cf., [S]). In [L], [B1] and [B2], Lundin and Baran have given simplifications of formula (1.2) in the case where K is a convex set in $\mathbf{R}^N$—considered as a subset of $\mathbf{C}^N$—which is symmetric with respect to the origin, i.e., $x\in K$ implies $-x\in K$. For example, if E_N is the closed unit ball in $\mathbf{R}^N$, i.e.,

$$E_N:=\{z\in\mathbf{C}^N:\operatorname{Im} z_1=...=\operatorname{Im} z_N=0,\ (\operatorname{Re} z_1)^2+...+(\operatorname{Re} z_N)^2\le 1\},$$

then $V_{E_N}(z)=\frac{1}{2}\log h(|z|^2+|z^2-1|)$, where $z^2=z_1^2+...+z_N^2$ (note $E_1=[-1,1]$). A few more explicit examples can be obtained using the following result of Klimek.

Proposition 1.1. ([K1]) *Let $f=(f_1,...,f_N)$ be a polynomial mapping of $\mathbf{C}^N$ into $\mathbf{C}^N$ with the properties that* $\deg f_1=...=\deg f_N:=d\ge 1$ *and $\hat{f}^{-1}(0)=\{0\}$ (where $\hat{f}:=(\hat{f}_1,...,\hat{f}_N)$ denotes the top degree (d) homogeneous piece of f). Then for any compact set K,*

$$V_{f^{-1}(K)}(z)=\frac{1}{d}V_K(f(z)).$$

In $\mathbf{C}^2$, if we set $f(z_1,z_2):=(z_1^2,z_2^2)$, and if we take $K=S_2$, where

$$S_2:=\{(x_1,x_2)\in\mathbf{R}^2:x_1,x_2\ge 0,\ x_1+x_2\le 1\}$$

is the standard simplex, then $f^{-1}(S_2)=E_2$ and we obtain

$$V_{S_2}(z_1,z_2)=\log h(|z_1|+|z_2|+|z_1+z_2-1|).$$

We will use this fact later in the paper.

In the examples of E_N and S_2, the extremal functions were gotten from one-variable functions. More generally, the following statement is a consequence of the results of Baran and Lundin.

Proposition 1.2. ([B1], [B2], [B3], [L]) *Let $K\subset\mathbf{R}^N$ be a convex body (i.e., a compact, convex set with non-empty interior in $\mathbf{R}^N$) which is symmetric with respect to the origin. Then for all $z\in\mathbf{C}^N$,*

$$V_K(z)=\widetilde{V}(z):=\sup\{V_{l(K)}(l(z)):l\in\mathbf{R}^{N^*}\}. \tag{1.3}$$

Here $\mathbf{R}^{N^*}$ is the set of all non-zero linear functionals l on $\mathbf{R}^N$, i.e., $l\in\mathbf{R}^{N^*}$ is a real-linear mapping from $\mathbf{R}^N$ to $\mathbf{R}$. We can consider each $l\in\mathbf{R}^{N^*}$ as an element in $\mathbf{C}^{N^*}$ the (complex) vector space of complex-linear functionals on $\mathbf{C}^N$ via $l(x+iy)=l(x)+il(y)$.

Our original goal was to determine whether (1.3) is valid if the symmetry hypothesis is omitted. Using the example of the standard simplex $S_2\subset\mathbf{C}^2$, it is not too hard to see that the answer is no. However, equality in (1.3) does remain valid for real convex compact sets, symmetric or not, at every *real* point, i.e., for each $z=x\in\mathbf{R}^N\subset\mathbf{C}^N$. Indeed, more is true, see Corollary 3.2. The key idea is a geometric property of convex sets due to Kroo and Schmidt [KR]; we study this property in detail in the next section. This suggests a more general question: Let $N>1$ and suppose $K\subset\mathbf{C}^N$ is compact. Let

$$V(z):=\sup\{V_{l(K)}(l(z)):l\in\mathbf{C}^{N^*},\ l\neq 0\}, \tag{1.4}$$

i.e., l is permitted to vary over *all* non-zero complex-linear functionals on $\mathbf{C}^N$. When do we get equality in (1.3) if $\widetilde{V}$ is replaced by V? In Section 4, we discuss more general situations when the computation of V_K can be reduced to one-variable calculations; in particular, we show that if K is polynomially convex ($K=\widehat{K}$) and $V(z)=V_K(z)$ in $\mathbf{C}^N$, then K must be lineally convex, i.e. the complement of K is the union of complex hyperplanes. In Section 5 we show that for the simplex $S_2\subset\mathbf{C}^2$, $V^*\neq V_{S_2}$ (here, $V^*(z)=\limsup_{\zeta\to z}V(\zeta)$). This involves a detailed study of the Robin functions associated to V and V_{S_2}; these objects play a vital role in the study of functions in the class L if $N>1$. We conclude the paper in Section 6 by showing that among the regular, polynomially convex and lineally convex compact sets K in $\mathbf{C}^2$, the ones for which $V^*\neq V_K$ form a "large" class.

Remark. Note that if we replace l by a scalar multiple tl, then $V_{tl(K)}\circ tl=V_{l(K)}\circ l$. Thus considering upper envelopes over all linear functionals or simply, e.g., over all linear functionals normalized to have norm 1, yield the same functions V and $\widetilde{V}$. Similarly, if $l\in\mathbf{C}^{N^*}$ and $a\in\mathbf{C}$ is constant, then $V_{(l+a)(K)}((l+a)(z))=V_{l(K)}(l(z))$.

2. A geometric property of convex sets

In this section, K will be a *convex body* in $\mathbf{R}^N\subset\mathbf{R}^N+i\mathbf{R}^N=\mathbf{C}^N$, i.e., $K\subset\mathbf{R}^N$ is compact, convex, and has non-empty interior. Recall that a real hyperplane $H_a\subset\mathbf{R}^N$ is a *support hyperplane* for K at $a\in\partial K$ if $a\in H_a$ and K lies entirely in one of the two half-spaces determined by H_a, i.e., if H_a is given by

$$H_a=\{x\in\mathbf{R}^N:l(x)=l(a),\ l\in\mathbf{R}^{N^*}\},$$

then $l(x-a)$ is of constant sign for $x\in K$.

The geometric property of interest for us is given in the following theorem of Kroo and Schmidt.

Theorem 2.1. ([KS]) *Let K be a convex body in $\mathbf{R}^N$. Then for each $x\in \mathbf{R}^N\setminus K$, there exist two points $a,b\in\partial K$ with distinct support hyperplanes H_a and H_b such that*

(i) *a, b and x are collinear;*

(ii) *H_a and H_b are parallel.*

We write (a,b,H_a,H_b) for a 4-tuplet satisfying (i) and (ii). Note that for a given point x the points a and b are not, in general, unique (e.g., take K to be a square in $\mathbf{R}^2$). Even if the points a and b are unique, the support hyperplanes H_a and H_b need not be. We will obtain uniqueness if K is strictly convex (Corollary 2.2 below).

Analytically, Theorem 2.1 means that given a point $x\in\mathbf{R}^N\setminus K$, there exist points $a,b\in\partial K$, collinear with x, and a linear functional $l\in{\mathbf{R}^N}^*$ with

$$l(K)=[l(a),l(b)]. \tag{2.1}$$

We briefly indicate the idea of the proof. Without loss of generality, we may assume $x=0$ and consider the Minkowski-like functional f defined on the cone $c(K):=\{ax: a>0,\ x\in K\}$ by $f(y):=\inf\{\alpha>0: y/\alpha\in K\}$. Clearly f is homogeneous of order 1, i.e., $f(ty)=tf(y)$. Kroo and Schmidt show that, in addition, f is convex on $c(K)$. We discuss the function f in a bit more detail and show that f is continuous on the cone $c(K)$. Given $y\in c(K)$, the line determined by y and the origin 0 intersects K in a segment $[a(y),b(y)]$; we may assume $|a(y)|\le|b(y)|$. Then $f(y)=|y|/|b(y)|$. To show that f is continuous on $c(K)$ it suffices to verify continuity of the function $y\mapsto b(y)$. But clearly if $y\to y_0$ then the area of the triangle $\Delta(0,y,y_0)$ tends to 0 and hence the area of the quadrilateral $Q(a(y),b(y),b(y_0),a(y_0))$ also tends to 0 since $Q(a(y),b(y),b(y_0),a(y_0))\subset\Delta(0,y,y_0)$ (here we can assume $|y|>|b(y)|$ and $|y_0|>|b(y_0)|$). Thus $a(y)\to a(y_0)$ and $b(y)\to b(y_0)$. In particular, f is continuous on K and attains a positive minimum on K at a point $x_0\in\partial K$. This is used to show that there exist parallel support hyperplanes for K at $a=x_0$ and $b=x_0/f(x_0)$ and these points lie on a line through the origin 0.

Now suppose K is a *strictly* convex body, i.e., for every point $a\in\partial K$ there exists a unique support hyperplane H_a containing a and $K\cap H_a=\{a\}$. This means that each boundary point of K is an extreme point of K.

Corollary 2.2. *Let K be a strictly convex body in $\mathbf{R}^N$. Then for each $x\in\mathbf{R}^N\setminus K$, there exists a unique 4-tuple (a,b,H_a,H_b) satisfying* (i) *and* (ii) *of Theorem* 2.1.

Proof. It remains to prove the uniqueness. Fix $x \in \mathbf{R}^N \setminus K$ and suppose there are two 4-tuples (a, b, H_a, H_b) and $(a', b', H_{a'}, H_{b'})$ satisfying (i) and (ii). We may assume that a lies between b and x, and a' lies between b' and x. We show that $a=a'$ and $b=b'$, for then the strict convexity implies that $H_a=H_{a'}$ and $H_b=H_{b'}$. Suppose, for the sake of obtaining a contradiction, that $a \neq a'$. It follows that $b \neq b'$, or else the distinct points $a, a', b \in \partial K$ would be collinear, contradicting the strict convexity of K (the intermediate point would not be an extreme point).

Since the lines determined by the points a and b, and by the points a' and b' intersect at x, the four points a, b, a', b' lie in a two-dimensional plane P. We may assume, for simplicity, that, e.g., $a=(0,-1)$ and $b=(0,1)$ (in $P=\mathbf{R}^2$) and that the parallel lines l_a and l_b are given by $y=-1$ and $y=1$. The points a' and b' must lie in the strip $-1<y<1$ (as indeed must all points of K other than a and b). Now clearly both of a' and b' cannot lie on the same side of the y-axis, for the strip defined by $l_{a'}$ and $l_{b'}$ could not then contain both a and b. Hence one of a' or b' is to the left of the y-axis and the other to the right. It follows that a, a', b, b' form the vertices of a convex quadrilateral Q. The diagonals joining a to b and a' to b' intersect inside Q and hence in K. However, the lines determined by a and b, and by a' and b' intersect outside K, namely, at x. This implies that these lines coincide, i.e., the three distinct points $a, a', b \in \partial K$ are collinear, yielding a contradiction (as before) to strict convexity of K. □

In case K is a polytope in $\mathbf{R}^N$ we can say somewhat more. Suppose first that $K \subset \mathbf{R}^2$ is a (non-degenerate) polygon. In this case we are dealing with supporting lines.

Corollary 2.3. *Let $K \subset \mathbf{R}^2$ be a non-degenerate polygon. Then for each $x \in \mathbf{R}^2 \setminus K$, at least one of the two parallel lines H_a and H_b in* (ii) *of Theorem* 2.1 *can be taken to contain an edge of K.*

Proof. Fix $x \in \mathbf{R}^2 \setminus K$ and a 4-tuple (a, b, H_a, H_b) satisfying (i) and (ii) of Theorem 2.1. If at least one of a and b, say a, is in the interior of an edge, the support line H_a contains this edge. Otherwise, if both a and b are vertices of K, we can rotate H_a and H_b, keeping them parallel, until one of them contains an edge. □

Note that the linear map $l:=l_x$ in this situation is simply projection onto the normal to these edges, i.e., the support lines have equations $l(x)=l(a)$ and $l(x)=l(b)$. In general, if K is a polytope in $\mathbf{R}^N$ the situation is somewhat more complicated. It no longer suffices to take hyperplanes parallel to the $(N-1)$-dimensional faces. For example, in $\mathbf{R}^3$, suppose that A and B lie in the interiors of two non-parallel (1-dimensional) edges. The directions of these two edges completely determine the normal to the corresponding supporting hyperplanes, which clearly need not be parallel to any face.

However, the same ideas as in the two-dimensional case can be used here. Let K be a polytope in $\mathbf{R}^N$ and let F_1 and F_2 be two faces of K of dimensions k_1 and k_2 belonging to the parallel hyperplanes H_1 and H_2. Let $v(F_1)$ and $v(F_2)$ denote the k_1- and k_2-dimensional vector spaces associated to the affine subspaces generated by F_1 and F_2. Since these lie in parallel hyperplanes, $\dim[v(F_1)\oplus v(F_2)]\leq N-1$. We call the faces F_1 and F_2 *complementary* if $\dim[v(F_1)\oplus v(F_2)]=N-1$.

Corollary 2.4. *Let $K\subset\mathbf{R}^N$ be a polytope. Then for each $x\in\mathbf{R}^N\setminus K$, there exist complementary faces F_1 and F_2 in K such that a 4-tuple (a,b,H_a,H_b) satisfying* (i) *and* (ii) *of Theorem* 2.1 *can be chosen with $F_1\subset H_a$ and $F_2\subset H_b$.*

Proof. Fix $x\in\mathbf{R}^N\setminus K$ and a 4-tuple (a,b,H_a,H_b) satisfying (i) and (ii) of Theorem 2.1. Then H_a intersects K in a k_1-dimensional face F_1 containing a and H_b intersects K in a k_2-dimensional face F_2 containing b. If F_1 and F_2 are complementary, we are done. If not, the subspace $W:=v(F_1)\oplus v(F_2)$ has dimension at most $N-2$. By rotating the hyperplanes H_a and H_b until at least one of them hits a higher-dimensional face, we obtain two faces F_1' and F_2' in the new parallel support hyperplanes H_a' and H_b' with $\dim[v(F_1')\oplus v(F_2')]>\dim[v(F_1)\oplus v(F_2)]$. We continue this process until this dimension reaches $N-1$. □

3. The real restriction of V_K

Convex bodies are natural sets to study in pluripotential theory; for if $K\subset\mathbf{R}^N$ is a convex body, then

1. K is non-pluripolar as a subset of $\mathbf{C}^N$ (i.e., $V_K^*\not\equiv+\infty$);
2. K is regular.

The first statement follows since K has non-empty interior in $\mathbf{R}^N$ and hence contains a real ball. The second statement follows from the arc accessibility criterion of Pleśniak (cf. [P]). Indeed, the Siciak extremal function of a convex body is of Hölder class $\frac{1}{2}$ (cf., [B2]).

Our main result concerning these sets is the following.

Theorem 3.1. *Let $K\subset\mathbf{R}^N$ be a convex body. Fix $z_0\in\mathbf{C}^N\setminus K$. Suppose there exists $l=l_{z_0}:\mathbf{R}^N\to\mathbf{R}$ a real linear map such that, if we let $l(K):=[A,B]$, then there exist $a,b\in K$ with $l(a)=A$ and $l(b)=B$ and such that the points a, b and z_0 lie on a complex line, i.e., $z_0=ta+(1-t)b$ for some $t\in\mathbf{C}\setminus[0,1]$. Then $V_K(z_0)\leq V_{l(K)}(l(z_0))$.*

Proof. We use the following observation, which is essentially Lemma 1.1 of [B1]: *Let $K\subset\mathbf{C}^N$ be a regular compact set and fix $z_0\notin K$. If there exists a holomorphic* *map* $f\colon \mathbf{C}\setminus\overline{U}\to\mathbf{C}^N$ *(U is the unit disk) which is continuous up to* ∂U *satisfying*

1. $f(\zeta_0)=z_0$;
2. $f(\partial U)\subset K$;
3. $V_K(f(\zeta))-\log|\zeta|=O(1)$, $|\zeta|\to+\infty$,

then $V_K(z_0)\le\log|\zeta_0|$. This follows from the extended maximum principle applied to the subharmonic function $V_K(f(\zeta))-\log|\zeta|$ on $\mathbf{C}\setminus\overline{U}$.

To prove the lemma, if we let $h_{[A,B]}\colon \mathbf{C}\setminus[A,B]\to\mathbf{C}\setminus U$ denote the conformal map of $\mathbf{C}\setminus[A,B]$ onto $\mathbf{C}\setminus U$ with $h_{[A,B]}(\infty)=\infty$, and if $J(\zeta)=h^{-1}_{[-1,1]}(\zeta)=\frac{1}{2}(\zeta+1/\zeta)$, then upon setting $\zeta_0:=h_{[A,B]}(l(z_0))$ and taking $f(\zeta):=\frac{1}{2}(b-a)J(\zeta)+\frac{1}{2}(b+a)$, a computation shows that 1–3 are satisfied for the convex body K.

Corollary 3.2. *Let* $K\subset\mathbf{R}^N$ *be a convex body. If* $l=l_x$ *corresponds to the* l *in Theorem* 3.1 *associated to the point* $x\in\mathbf{R}^N\setminus K$, *then* $V_K(z)=V_{l(K)}(l(z))$ *for all* $z=ta+(1-t)b$ *with* $t\in\mathbf{C}\setminus[0,1]$. *In particular,* $\widetilde{V}(x)=V_K(x)$ *for* x *in* $\mathbf{R}^N$.

Proof. This follows from Theorems 2.1 and 3.1 using (2.1). □

For example, if $K\subset\mathbf{R}^2\subset\mathbf{C}^2$, we obtain a 3-real-dimensional set of points $z\in\mathbf{C}^2$ at which $\widetilde{V}(z)=V_K(z)$. From Corollary 2.3 and Theorem 3.1, we obtain a particularly simple geometric construction of the extremal function for a non-degenerate polygon in $\mathbf{R}^2$.

Corollary 3.3. *Let* $K\subset\mathbf{R}^2$ *be a non-degenerate polygon with* n *vertices. Then for* x *in* $\mathbf{R}^2$,

$$V_K(x)=\max\{V_{l_j(K)}(l_j(x)):l_j(x)=0 \text{ is parallel to the } j\text{-th edge of } K,\ j=1,\dots,n\}.$$

An analogous statement can be made concerning polytopes in $\mathbf{R}^N$, $N\ge3$.

4. Computing V_K using one-variable methods

Let $K\subset\mathbf{C}^N$ be compact. Recall that K is non-pluripolar as a subset of $\mathbf{C}^N$ if and only if $V_K^*\in L$ (equivalently, $V_K^*\not\equiv+\infty$) and that K is regular if and only if $V_K^*=0$ on K (equivalently, $V_K^*=V_K$, i.e., V_K is continuous on $\mathbf{C}^N$). We relate these notions in one and several variables for K and $p(K)$ when p is a non-constant polynomial.

Lemma 4.1. *Suppose that* $K\subset\mathbf{C}^N$ *is compact and that* $p\colon\mathbf{C}^N\to\mathbf{C}$ *is a non-constant polynomial. Then* (a) *if* K *is non-pluripolar,* $p(K)$ *is non-polar, and* (b) *if* K *is regular then* $p(K)$ *is regular.*

Proof. (a) Suppose that $p(K)$ is polar. Then there exists a subharmonic $u\colon\mathbf{C}\to\mathbf{R}\cup\{-\infty\}$, such that u is not identically $-\infty$ but u restricted to $p(K)$ is identically

$-\infty$. Then $U:=u\circ p$ is plurisubharmonic on $\mathbf{C}^N$, not identically $-\infty$, but $U|_K= u|_{p(K)}\equiv-\infty$, so that K is pluripolar, a contradiction. Note this proof is valid for any pluripolar *Borel* set K.

(b) We begin by noting that by Corollary 5.25 of Klimek [K1], $V^*_{E\cup Z}\equiv V^*_E$ for any bounded Borel $E\subset\mathbf{C}^N$, and $Z\subset\mathbf{C}^N$, pluripolar. Now suppose that $p(K)$ is not regular. Then there is a $\xi_0\in p(K)$ such that

$$V^*_{p(K)}(\xi_0)>0.$$

Let

$$Z:=\{z\in K:V^*_{p(K)}(p(z))>0\}=p^{-1}\{\xi\in p(K):V^*_{p(K)}(\xi)>0\}.$$

By the argument of (a) (for bounded Borel sets), Z is pluripolar. Let $z_0\in p^{-1}(\xi_0)\subset Z$. Then

$$V^*_K(z_0)=V^*_{K\setminus Z}(z_0)\geq\frac{1}{\deg(p)}V^*_{p(K)}(p(z_0))$$

(since the rightmost function is a competitor of the extremal function for $K\setminus Z$).

But

$$\frac{1}{\deg(p)}V^*_{p(K)}(p(z_0))=\frac{1}{\deg(p)}V^*_{p(K)}(\xi_0)>0$$

by assumption, and so we also have that

$$V^*_K(z_0)>0$$

implying that K is *not* regular, a contradiction. □

Now suppose K is regular and let p_d be a polynomial of degree $d\geq 1$. Then $p_d(K)$ is a regular compact set in $\mathbf{C}$. The function $V_{p_d(K)}(p_d(z))$ is plurisubharmonic in $\mathbf{C}^N$ with

1. $V_{p_d(K)}(p_d(z))=d\log|z|+O(1)$, $|z|\to+\infty$;
2. if $z\in K$, then $p_d(z)\in p_d(K)$ so that $V_{p_d(K)}(p_d(z))=0$.

Thus

$$\frac{1}{d}V_{p_d(K)}(p_d(z))\leq V_K(z).$$

Conversely, if $\|p_d\|_K\leq 1$, then $p_d(K)\subset U$, U being the unit disk in $\mathbf{C}$, so that $V_{p_d(K)}(w)\geq V_U(w)=\log^+|w|$ for all $w\in\mathbf{C}$. In particular, $V_{p_d(K)}(p_d(z))\geq\log^+|p_d(z)|$, from which it follows that

$$V_K(z)\leq\sup_{p_d}\frac{1}{d}V_{p_d(K)}(p_d(z))$$

and thus, in fact, we have

$$V_K(z) = \sup_{p_d} \frac{1}{d} V_{p_d(K)}(p_d(z)). \tag{4.1}$$

For example, if $d=1$, this implies that for any non-constant complex affine function $l(z)$, we have

$$V_{l(K)}(l(z)) \le V_K(z).$$

When are the functions of the form $V_{l(K)}(l(z))$ sufficient to determine V_K? More precisely, define

$$V(z) := \sup\{V_{l(K)}(l(z)) : l \in \mathbf{C}^{N^*},\ l \neq 0\} \tag{1.4}$$

as in the introduction. It is natural to ask for the most general situation under which we have the equality $V=V_K$.

We first show that should it be the case that $V=V_K$ then necessarily K must be lineally convex, i.e. the complement of K is the union of complex hyperplanes.

Proposition 4.2. *Let $N>1$. Suppose $K\subset\mathbf{C}^N$ is compact, regular, and polynomially convex ($K=\widehat{K}$). Define $V(z)$ using (1.4). If $V(z)=V_K(z)$ in $\mathbf{C}^N$, then K is lineally convex.*

Proof. It suffices to show that $K=\bigcap_l l^{-1}(l(K))$. For any K, the inclusion $K\subset\bigcap_l l^{-1}(l(K))$ is trivial. We prove the reverse inclusion by contradiction: if there exists $z_0\in\bigcap_l l^{-1}(l(K))$ but $z_0\notin K=\widehat{K}$, then, on the one hand, $V_K(z_0)>0$; on the other hand, for each l we have $V_{l(K)}(l(z_0))=0$ so that $V(z_0)=0$, contradicting $V=V_K$. □

Remarks. 1. For each positive integer n, we can define

$$V^{(n)}(z) := \sup\left\{ \frac{1}{\deg p} V_{p(K)}(p(z)) : 1 \le \deg p \le n \right\}.$$

The same proof shows that *if $K\subset\mathbf{C}^N$ is compact, regular, and polynomially convex, and if $V^{(n)}(z)=V_K(z)$ in $\mathbf{C}^N$, then K is "convex with respect to polynomials of degree n"*, i.e., the complement of K is the union of algebraic hypersurfaces of the form $\{p=0\}$ with $1\le\deg p\le n$.

2. Suppose $K\subset\mathbf{R}^N\subset\mathbf{R}^N+i\mathbf{R}^N=\mathbf{C}^N$ is compact (then K is automatically polynomially convex (cf., [K1], Lemma 5.4.1)) and connected. Let $\widetilde{V}(z)$ be defined as in (1.3), i.e.,

$$\widetilde{V}(z) := \sup\{V_{l(K)}(l(z)) : l \in \mathbf{R}^{N^*}\}.$$

Then clearly $\widetilde{V}(z)\le V(z)$; the same proof as above gives a partial converse to Corollary 3.2: *if* $\widetilde{V}(x)=V_K(x)$ *for* x *in* $\mathbf{R}^N$, *then* K *is convex.* For $\widetilde{V}(x)=V_K(x)$ for x in $\mathbf{R}^N$ implies that $\mathbf{R}^N\setminus K$ is the union of real hyperplanes; K being connected then yields that K is convex.

3. If K is non-pluripolar, then ${V^{(1)}}^*:=V^*$ (and hence ${V^{(n)}}^*$ for each $n=1,2,\ldots$) is in the class

$$L^+:=\{u\in L:\log^+|z|+C_1\le u(z)\le \log^+|z|+C_2 \text{ for some } C_1 \text{ and } C_2\}.$$

Indeed, it is well known that $V_K^*\in L^+$ if K is non-pluripolar; letting $l_j(z)=z_j$, $j=1,\ldots,N$, we have

$$V_K^*(z)\ge V_K(z)\ge V^{(1)}(z)\ge \max_{j=1,\ldots,N} V_{l_j(K)}(l_j(z)).$$

But $\max_{j=1,\ldots,N} V_{l_j(K)}(l_j(z))=V_{l_1(K)\times\ldots\times l_N(K)}(z)$ and $V^*_{l_1(K)\times\ldots\times l_N(K)}\in L^+$ since each $l_j(K)$ is non-polar by Lemma 4.1.

4. Note that if $N=1$, then $V=\widetilde{V}=V_K$ for all compact sets K.

5. The simplex $S_2\in\mathbf{C}^2$

The considerations of Section 3 might lead one to suspect that for a convex body K, one *always* has $\widetilde{V}=V_K$ in all of $\mathbf{C}^N$; or at least $V=V_K$ in $\mathbf{C}^N$. We next give an example to show that this is not the case.

Take $K=S_2=\{(x_1,x_2)\in\mathbf{R}^2:x_1,x_2\ge 0,\ x_1+x_2\le 1\}$, the standard simplex; then, as mentioned previously,

$$V_{S_2}(z_1,z_2)=\log h(|z_1|+|z_2|+|z_1+z_2-1|).$$

We show that $\{z\in\mathbf{C}^2:V^*(z)<V_K(z)\}\ne\emptyset$. We first recall the notion of the *Robin function* associated to a function $u\in L$. First of all, suppose that $K\subset\mathbf{C}^N$ is compact and regular. The Robin function of K is $\varrho_K:\mathbf{C}^N\to\mathbf{R}\cup\{-\infty\}$ defined by

$$\varrho_K(z):=\limsup_{|\lambda|\to\infty}[V_K(\lambda z)-\log|\lambda|].$$

More generally, for $V:\mathbf{C}^N\to\mathbf{R}$ in L we may define the Robin function of V to be, by abuse of notation,

$$\varrho_V(z):=\limsup_{|\lambda|\to\infty}[V(\lambda z)-\log|\lambda|]$$

(hence $\varrho_K = \varrho_{V_K}$). Note that for $\lambda \in \mathbf{C}$, $\varrho_V(\lambda z) = \log|\lambda| + \varrho_V(z)$ (logarithmic homogeneity).

Consider for $\alpha \in \mathbf{C}^N$ with $|\alpha| = 1$,

$$l(z) = l_\alpha(z) := \sum_{k=1}^{n} \alpha_k z_k,$$

the projection with normal vector α. From the remark at the end of the introduction, we need only consider $l \in \mathbf{C}^{N^*}$ with $\|l\| = 1$ in constructing the function $V(z)$ defined in (1.4), i.e.,

$$V(z) := \sup\{V_{l(K)}(l(z)) : l \in \mathbf{C}^{N^*},\ l \neq 0\} = \sup\{V_{l(K)}(l(z)) : l \in \mathbf{C}^{N^*},\ \|l\| = 1\}.$$

For simplicity in notation, we write

$$V(z) := \sup_l V_{l(K)}(l(z))$$

where we (implicitly) restrict the supremum to those $l \in \mathbf{C}^{N^*}$ with $\|l\| = 1$.

We note that $l(K)$ is regular by Lemma 4.1, and that also, for $l(z) \neq 0$ we may compute, by a change of variables,

$$\varrho_{l(K)}(l(z)) = \log|l(z)| + \varrho_{l(K)}.$$

Here $\varrho_{l(K)} = -\log \operatorname{cap}(l(K))$ is the Robin constant for $l(K) \subset \mathbf{C}$; $\operatorname{cap}(l(K))$ being the logarithmic capacity of $l(K)$.

We want to relate the functions $\varrho_{l(K)}(l(z))$ to $\varrho_V(z)$; we are able to do this only for K which satisfy an additional geometric regularity condition (Theorem 5.2 below).

Definition. We say that $K \subset \mathbf{C}^N$ is C-regular if there exists $b_0 \in (0,1)$ and $R \geq 1$ such that for all $|\eta| > R$,

$$\sup_l V_{l(K)}(b\eta) \leq \inf_l V_{l(K)}(\eta)$$

for all $0 < |b| \leq b_0$.

Note that, for example, the real standard simplex is C-regular since each $l(K)$ contains a line segment of a fixed minimal length having the origin as one endpoint, and also $l(K)$ is contained in the unit disk. Other examples include any compact set K containing a neighborhood of the origin.

For C-regular sets we may compare the extremal functions for different projections in the following manner.

Lemma 5.1. *Suppose that $K\subset \mathbf{C}^N$ is compact, regular, and C-regular. Then there is an $a\in(0,\frac{1}{2})$ and $R'\geq 1$ such that for all $|\lambda|\geq R'$ and $|z|=1$, if $l_1,l_2\in \mathbf{C}^{N^*}$, with $\|l_1\|=\|l_2\|=1$, satisfy $|l_1(z)|\leq a$ and $|l_2(z)|\geq 1-a$, then*

$$V_{l_1(K)}(\lambda l_1(z))\leq V_{l_2(K)}(\lambda l_2(z)).$$

Proof. Let $a:=b_0/(b_0+1)$ and $R':=R/(1-a)$, where b_0 and R are as in the definition of C-regularity. Further, let $\eta=\lambda l_2(z)$. Then

$$|\eta|=|\lambda|\,|l_2(z)|\geq R'|l_2(z)|\geq R'(1-a)=R.$$

Also,

$$\lambda l_1(z)=\frac{l_1(z)}{l_2(z)}\lambda l_2(z)$$

and

$$|b|:=\left|\frac{l_1(z)}{l_2(z)}\right|\leq \frac{a}{1-a}=b_0.$$

Consequently, from the definition of C-regularity,

$$V_{l_1(K)}(\lambda l_1(z))=V_{l_1(K)}(b\eta)\leq V_{l_2(K)}(\eta)=V_{l_2(K)}(\lambda l_2(z)). \quad \square$$

Theorem 5.2. *Suppose that K is compact, regular and C-regular. Then for all $z\in\mathbf{C}^N$*

$$\varrho_V(z)=\sup_l \varrho_{l(K)}(l(z)).$$

Proof. By logarithmic homogeneity of each side, it suffices to verify the equality for points z with $|z|=1$. We wish to show that

$$\limsup_{|\lambda|\to\infty}\Big[\sup_l V_{l(K)}(\lambda l(z))-\log|\lambda|\Big] \tag{5.1}$$

equals

$$\sup_l\Big[\log|l(z)|+\limsup_{|\lambda|\to\infty}\{V_{l(K)}(\lambda)-\log|\lambda|\}\Big]. \tag{5.2}$$

First of all (5.1) $\geq$ (5.2) since for each l, $V(z)\geq V_{l(K)}(l(z))$ and hence, for all l,

$$\limsup_{|\lambda|\to\infty}[V(\lambda z)-\log|\lambda|]\geq \limsup_{|\lambda|\to\infty}[V_{l(K)}(\lambda l(z))-\log|\lambda|];$$

the right-hand side is (5.2) (in disguised form) without the sup.

To show the reverse inequality, note that by Lemma 5.1,

$$(5.1) = \limsup_{|\lambda|\to\infty}\Big[\sup_{|l(z)|\geq a} V_{l(K)}(\lambda l(z)) - \log|\lambda|\Big],$$

i.e., we may take the supremum over this restricted class.

Now we claim that given $\varepsilon>0$, there exists $R>0$ such that for $|\eta|\geq R$,

$$|(V_{l(K)}(\eta)-\log|\eta|)-\varrho_{l(K)}|<\varepsilon \tag{5.3}$$

for *all* l satisfying $|l(z)|\geq a$. Supposing for the time that this is indeed the case, then given $\varepsilon>0$ choose such an R. Then for $|\lambda|\geq R/a$ we have, for each l with $|l(z)|\geq a$,

$$|\lambda l(z)|\geq|\lambda|a\geq R.$$

Hence,

$$V_{l(K)}(\lambda l(z))-\log|\lambda| = (V_{l(K)}(\lambda l(z))-\log|\lambda l(z)|)+\log|l(z)|$$

which implies that

$$|(V_{l(K)}(\lambda l(z))-\log|\lambda|)-(\varrho_{l(K)}+\log|l(z)|)|<\varepsilon$$

and thus that

$$V_{l(K)}(\lambda l(z))-\log|\lambda|\leq(\varrho_{l(K)}+\log|l(z)|)+\varepsilon.$$

Consequently,

$$\sup_{|l(z)|\geq a}(V_{l(K)}(\lambda l(z))-\log|\lambda|)\leq\sup_{l}(\varrho_{l(K)}+\log|l(z)|)+\varepsilon$$

for all $|\lambda|\geq R/a$. It follows then that (5.1) $\leq$ (5.2).

What remains is to prove our claim (5.3). To see this, let $\mu_{l(K)}$ be the equilibrium measure for $l(K)$. Then

$$P_{\mu_{l(K)}}(z) := \int_{l(K)}\log|z-\xi|\,d\mu_{l(K)}(\xi) = V_{l(K)}(z)-\varrho_{l(K)}$$

and hence

$$V_{l(K)}(z)-\log|z|-\varrho_{l(K)} = \int_{l(K)}\log\left|\frac{z-\xi}{z}\right|d\mu_{l(K)}(\xi) = \int_{l(K)}\log\left|1-\frac{\xi}{z}\right|d\mu_{l(K)}(\xi)$$

and the result follows since $l(K)$ is contained in a disk of fixed radius for all l (since K is bounded). $\square$

Returning to the case $K=S_2$, an easy calculation reveals that

$$\varrho_K(z_1,z_2)=\log(2(|z_1|+|z_2|+|z_1+z_2|)). \tag{5.4}$$

Moreover, for the simplex,

$$\sup_l \frac{|l(z_1,z_2)|}{\operatorname{cap}(l(K))}=e^{\varrho_{V^*}(z_1,z_2)} \quad \text{for all } (z_1,z_2)\in \mathbf{C}^2. \tag{5.5}$$

To verify (5.5), it suffices to show that $u(z_1,z_2):=\sup_l |l(z_1,z_2)|/\operatorname{cap}(l(K))$ is continuous in $\mathbf{C}^2\setminus\{(0,0)\}$ for then $e^{\varrho_{V^*}}$ and u are plurisubharmonic functions which agree q.e. in $\mathbf{C}^2$ and hence $e^{\varrho_{V^*}}\equiv u$. A further simplification follows by noting that u is homogeneous of order one: $u(tz_1,tz_2)=tu(z_1,z_2)$ for $t\in\mathbf{C}$; hence we need only verify continuity of u on the unit sphere in $\mathbf{C}^2$. We make the following observations:

(i) $u(z_1,z_2)=\max_{\|l\|=1}|l(z_1,z_2)|/\operatorname{cap}(l(K))$, for clearly for fixed (z_1,z_2), the mapping $l\mapsto|l(z_1,z_2)|/\operatorname{cap}(l(K))$ is continuous;

(ii) if $l(z_1,z_2)=az_1+bz_2$ with $|a|^2+|b|^2=1$ ($\|l\|=1$), then $l(0,0)=0$, $l(1,0)=a$, $l(0,1)=b$ so $\operatorname{cap}(l(K))\geq\frac{1}{4}\max\{|a|,|b|\}\geq 1/4\sqrt{2}=:1/c>0$.

Fix $z:=(z_1,z_2)$ and $z':=(z_1',z_2')$ with $|z|=|z'|=1$ and take l_z and $l_{z'}$ with $\|l_z\|=\|l_{z'}\|=1$ such that

$$u(z)=\frac{|l_z(z)|}{\operatorname{cap}(l_z(K))} \quad\text{and}\quad u(z')=\frac{|l_{z'}(z')|}{\operatorname{cap}(l_{z'}(K))}.$$

Then

$$u(z)\geq\frac{|l_{z'}(z)|}{\operatorname{cap}(l_{z'}(K))} \quad\text{and}\quad u(z')\geq\frac{|l_z(z')|}{\operatorname{cap}(l_z(K))}.$$

Hence

$$u(z)-u(z')\leq\frac{|l_z(z)|}{\operatorname{cap}(l_z(K))}-\frac{|l_z(z')|}{\operatorname{cap}(l_z(K))}\leq\left|\frac{l_z(z-z')}{\operatorname{cap}(l_z(K))}\right|$$

and

$$u(z')-u(z)\leq\frac{|l_{z'}(z')|}{\operatorname{cap}(l_{z'}(K))}-\frac{|l_{z'}(z)|}{\operatorname{cap}(l_{z'}(K))}\leq\left|\frac{l_{z'}(z'-z)}{\operatorname{cap}(l_{z'}(K))}\right|.$$

Thus

$$|u(z)-u(z')|\leq\max\left\{\left|\frac{l_z(z-z')}{\operatorname{cap}(l_z(K))}\right|,\left|\frac{l_{z'}(z'-z)}{\operatorname{cap}(l_{z'}(K))}\right|\right\}\leq c\max\{|l_z(z-z')|,|l_{z'}(z'-z)|\}.$$

Since $\|l_z\|=\|l_{z'}\|=1$, we have $|l_z(z-z')|\leq|z-z'|$ and $|l_{z'}(z'-z)|\leq|z-z'|$, and so

$$|u(z)-u(z')|\leq c|z-z'|.$$

Thus u is (uniformly) continuous on the unit sphere and (5.5) follows.

Our claim that $\varrho_{V^*}\neq\varrho_K$, and hence that $V^*\neq V_K$, now follows from (5.4), (5.5) and the following result.

Theorem 5.3. *Let $K=S_2$. There is a point $p=(z_1, z_2)$ in $\mathbf{C}^2$ such that*

$$\sup_l \frac{|l(z_1, z_2)|}{\operatorname{cap}(l(K))} < 2(|z_1|+|z_2|+|z_1+z_2|). \tag{5.6}$$

Proof. Take $p=(1, i)$. We will show that

$$\sup_l \frac{|l(1,i)|}{\operatorname{cap}(l(K))} < 2(1+1+|1+i|) = 2(2+\sqrt{2}). \tag{5.7}$$

To this end, write $l(z_1, z_2)=\alpha z_1+\beta z_2$. Since the left-hand side of our inequality is homogeneous in α we may divide by α or, in other words, assume that $l(z_1, z_2)= z_1+\lambda z_2$. Hence we need only show that

$$\sup_{\lambda\in\mathbf{C}} \frac{|1+\lambda i|}{\operatorname{cap}(l(K))} < 2(2+\sqrt{2}). \tag{5.7}$$

We shall bound the left-hand side by using three different lower estimates of the capacity of $l(K)$. Note that $l(K)$ is the (possibly degenerate) triangle in $\mathbf{C}$ with vertices 0, 1 and λ. Our first estimate is that the capacity of this triangle is at least the capacity of the edge $[0,1]$, i.e. $\operatorname{cap}(l(K))\ge\frac{1}{4}$. Hence

$$\frac{|1+\lambda i|}{\operatorname{cap}(l(K))} \le 4|1+\lambda i| = 4|\lambda - i|$$

and our claim is true for all $\lambda\in\mathbf{C}$ *inside* the circle centered at i with radius $r=a< 2(2+\sqrt{2})/4=\frac{1}{2}(2+\sqrt{2})$.

Next consider the edge $[0,\lambda]$. It follows that also $\operatorname{cap}(l(K))\ge\frac{1}{4}|\lambda|$, an estimate that is valid even if the triangle is degenerate. Hence,

$$\frac{|1+\lambda i|}{\operatorname{cap}(l(K))} \le 4\frac{|1+\lambda i|}{|\lambda|} = 4\left|\frac{\lambda-i}{\lambda}\right|.$$

Since $4(\lambda-i)/\lambda$ is a Möbius transformation that maps $\infty\mapsto 4<2(2+\sqrt{2})$, it follows that for $a>1$,

$$\left\{\lambda : 4\left|\frac{\lambda-i}{\lambda}\right| \le 4a\right\}$$

is the *exterior* of a circle whose centre and radius may easily be computed to be $-i/(a^2-1)$ and $r=a/(a^2-1)$, respectively.

Now, for a near $\frac{1}{2}(2+\sqrt{2})$, the interior of the first circle and the exterior of the second do not cover all of $\mathbf{C}$ (see the figure). Hence we require one further estimate on the capacity of a triangle (the third side does not suffice).

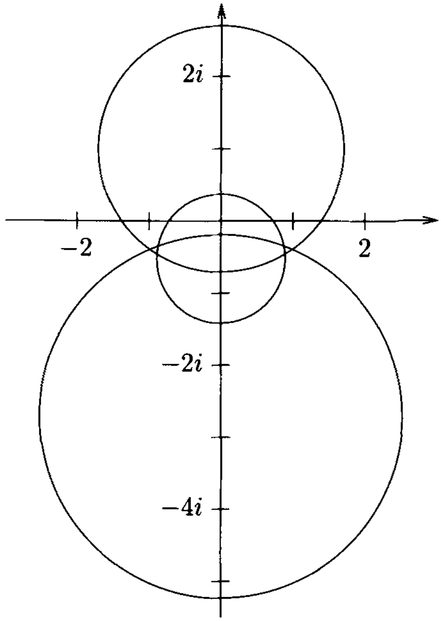

Figure 1. From top to bottom, circles 1, 2 and 3 with $a=\frac{1}{2}(2+\sqrt{2})$.

From [R, p. 146], the capacity of a triangle of area A is at least $\sqrt{A/\pi}$. Since the first two circle conditions do cover all of the upper half-plane, we consider $\lambda=\lambda_1+i\lambda_2$ with $\lambda_2<0$. Then in our case, we have $\mathrm{cap}(l(K))\geq\sqrt{-\lambda_2/2\pi}$, so that

$$\frac{|1+\lambda i|}{\mathrm{cap}(l(K))}\leq\frac{|\lambda-i|}{\sqrt{-\lambda_2/2\pi}}.$$

The set

$$\left\{\lambda:\frac{|\lambda-i|}{\sqrt{-\lambda_2/2\pi}}\leq 4a\right\}$$

is again the *interior* of a circle, this time with centre $i(1-4a^2)/\pi$ and radius $r=[(4a^2/\pi-1)^2-1]^{1/2}$.

From the figure it is clear that the interiors of the first and third circle together with the exterior of the second cover all of $\mathbf{C}$, for a near $\frac{1}{2}(2+\sqrt{2})$. Hence, for $a<\frac{1}{2}(2+\sqrt{2})$, but sufficiently close to $\frac{1}{2}(2+\sqrt{2})$ we see that, for all $\lambda\in\mathbf{C}$,

$$\frac{|1+\lambda i|}{\mathrm{cap}(l(K))}\leq 4a<2(2+\sqrt{2}),$$

and we have verified (5.7). $\square$

6. Final remarks

A natural question to ask is whether the example of the simplex, where $V \neq V_K$, is the "generic" case of a convex body in $\mathbf{R}^N$ which is not symmetric. To this end, we recall the definition of the *Klimek metric* Γ [K2]. Let $\mathcal{R}$ denote the set of all regular, polynomially convex compact sets $K \subset \mathbf{C}^N$. The set function $\Gamma: \mathcal{R} \times \mathcal{R} \to \mathbf{R}^+$ defined by

$$\Gamma(E,F) := \max\{\|V_E\|_F, \|V_F\|_E\} = \|V_E - V_F\|_{E\cup F} = \|V_E - V_F\|_{\mathbf{C}^N}$$

is easily seen to be a metric on $\mathcal{R}$ (indeed, Klimek showed that $(\mathcal{R},\Gamma)$ is a complete metric space). We can restrict Γ to the subset $\mathcal{K} \subset \mathcal{R}$ of lineally convex sets in $\mathcal{R}$; thus we obtain the metric space $(\mathcal{K},\Gamma)$. We modify the metric to suit our purposes. First we recall from Remark 3 in Section 4 that if $K \in \mathcal{R}$, then the function $V^* = V^{(1)^*} = V_K^{(1)^*}$ from (1.4) is a function in the class L^+. Thus for $E, F \in \mathcal{R}$, $\|V_E^{(1)^*} - V_F^{(1)^*}\|_{\mathbf{C}^N}$ is finite (although this is not necessarily the same as either of $\max\{\|V_E^{(1)^*}\|_F, \|V_F^{(1)^*}\|_E\}$ or $\|V_E^{(1)^*} - V_F^{(1)^*}\|_{E\cup F}$). It follows easily that

$$\widetilde{\Gamma}(E,F) := \max\{\|V_E - V_F\|_{\mathbf{C}^N}, \|V_E^{(1)^*} - V_F^{(1)^*}\|_{\mathbf{C}^N}\}$$

defines a metric on $\mathcal{R}$. Restrict this new metric to $\mathcal{K}$; we work in $\mathbf{C}^2$.

Proposition 6.1. *The set* $\mathcal{O} := \{K \in \mathcal{K} : V_K \neq V_K^{(1)^*}\}$ *is a non-empty open set in* $(\mathcal{K}, \widetilde{\Gamma})$.

Proof. From Theorem 5.3, $\mathcal{O} \neq \emptyset$. Fix $K \in \mathcal{O}$. Then there exists $p \in \mathbf{C}^2$ with $V_K(p) - V_K^{(1)^*}(p) = a > 0$. A simple calculation shows that for any δ with $0 < \delta < \frac{1}{2}a$,

$$B(K,\delta) := \{K' \in \mathcal{K} : \widetilde{\Gamma}(K', K) < \delta\} \subset \mathcal{O}.$$

Hence $\mathcal{O}$ is open. □

Let $\mathcal{K}_R$ denote the set of all convex bodies $K \subset \mathbf{R}^2 \subset \mathbf{C}^2$.

Corollary 6.2. *The set* $\mathcal{O}_R := \{K \in \mathcal{K}_R : V_K \neq V_K^{(1)^*}\}$ *is a non-empty open set in* $(\mathcal{K}_R, \widetilde{\Gamma})$.

We conjecture that $\mathcal{O}_R$ is dense in $\mathcal{K}_R$.

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Len Bos
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta T2N 1N4
Canada

Jean-Paul Calvi
Department of Mathematics
Université "Paul Sabatier"de Toulouse
FR-31062 Toulouse
France

Norman Levenberg
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland
New Zealand