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Best approximation in the supremum norm by analytic and harmonic functions

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1. Introduction

In this paper, we study the problem of finding, for a given bounded measurable function f on a domain Ω in $\mathbf{R}^n$, a harmonic function on Ω that best approximates f in the supremum norm, as well as (when $n=2$) the corresponding problem of approximating f by analytic functions. The analogous problem of approximating a bounded measurable function on the boundary of a plane domain (especially, the unit disk) by the *boundary values* of bounded analytic functions in the interior has been studied very extensively (see, e.g. [G]), but the present problem (which, as we shall see, is quite different in character) has received very little attention. There have been some studies, by Luecking [L1], [L2], Hintzman [H1], [H2] and Romanova [R1], [R2], pertaining to approximation by analytic functions.

Concerning the harmonic approximation problem in $\mathbf{R}^n$, its study seems to originate in a paper of Hayman, Kershaw and Lyons [HKL] from 1984. Our main motivation has been to refine and extend some of the results of that paper. This we have been able to do, in part by making greater use of functional analysis than they do. Those tools, *per se*, are well known. In the interest of a unified presentation, we have included proofs of some previously known results. The main novelty of the paper is Theorem 4.1 which, in the two-dimensional case, gives an affirmative answer to a question asked by Walter Hayman in 1984. Theorem 3.5 contains an answer to another question of Hayman. (These questions were contributed to the session on "New and Unsolved Problems" at the conference where [HKL] was presented, see p. 608 of the conference proceedings.) Also, several of our counterexamples are new, or sharper than previously known ones of similar character.

The present paper can be seen as complementary to [KMS], where the analogous problems were studied in L^p norm (with respect to volume measure) for $1\le p<\infty$.

We shall use the following notation. By Ω we denote a bounded, open connected set in $\mathbf{R}^n$ and by dx Lebesgue measure on $\mathbf{R}^n$. Then $L^p(\Omega; dx)$ (often abbreviated $L^p(\Omega)$) is the usual Lebesgue space, and $L^\infty(\Omega; dx) = L^\infty(\Omega)$ the (Banach) space of bounded measurable functions on Ω, endowed with supremum norm $\|\cdot\|_\infty$. When working with harmonic approximation we shall tacitly assume all functions are real-valued, and when working with analytic approximation complex-valued.

Let $C(\Omega)$ and $C(\bar{\Omega})$ denote the subspace of $L^\infty(\Omega)$ consisting of functions continuous on Ω, and of those elements of $C(\Omega)$ which extend continuously to the closure $\bar{\Omega}$ of Ω, respectively.

By $HL^p(\Omega)$ we denote the subspace of $L^p(\Omega)$ consisting of harmonic functions (and analogously with $HC(\bar{\Omega})$). For $\Omega \subset \mathbf{R}^2$, $AL^p(\Omega)$ denotes the subspace of (complex-valued) $L^p(\Omega)$ consisting of analytic functions. When working in the analytic context, we shall usually denote a generic point of $\mathbf{R}^2 \cong \mathbf{C}$ by $z = x + iy$ and Lebesgue area measure on $\mathbf{C}$ by dA.

We shall denote by $B(x^0, R)$ the open ball in $\mathbf{R}^n$ with center x^0 and radius R. When $n=2$ we shall usually denote $B(0,1)$ by $\mathbf{D}$, in the context of analytic functions. For a compact set $K \subset \mathbf{R}^n$, $M(K)$ denotes the space of real-valued Borel measures μ with support $\operatorname{supp}\mu$ in K. (Again, in the analytic function context in $\mathbf{R}^2$, $M(K)$ shall be a space of complex measures; this will always be made clear.) By $V(\mu) = \|\mu\|_M = \|\mu\|_{M(K)}$ we denote the total variation of the measure μ.

2. Existence of a best approximation

Given f in $L^\infty(\Omega)$, let

$$\lambda := \inf\{\|f-u\| : u \in HL^\infty(\Omega)\}. \tag{2.1}$$

It is elementary and well known that there exists at least one $u^\#$ in $HL^\infty(\Omega)$ such that $\|f-u^\#\| = \lambda$. (Such a function $u^\#$ is called a *best approximation* to f.) Indeed, if $\{u_j\} \subset HL^\infty(\Omega)$ with $\|f-u_j\| \to \lambda$, the u_j are uniformly bounded, hence there is a subsequence converging uniformly on compact subsets of Ω to an element $u^\#$ in $HL^\infty(\Omega)$ and it is easy to check that $\|f-u^\#\| = \lambda$. The corresponding result for approximation by elements of $AL^\infty(\Omega)$ ($\Omega \subset \mathbf{R}^2$) is proved similarly. As was noted in [HKL], even if Ω is a ball and f is very regular, there need not exist a best approximation which extends continuously to $\bar{\Omega}$. We shall show below that even for polynomial f, there need not exist a best approximation in $HC(\bar{\Omega})$.

3. Annihilating measures, duality

A measure μ in $M(\overline{\Omega})$ is said to annihilate a subset E of $C(\overline{\Omega})$ (which we write $\mu @ E$) if $\int f\,d\mu=0$ for all f in E. The set of all μ in $M(\overline{\Omega})$ annihilating E will be denoted $E^{@}$.

Theorem 3.1. *Given f in $C(\overline{B})$, where B is the open ball of $\mathbf{R}^n$,*

$$\min\{\|f-u\|, u\in HL^\infty(B)\}=\sup\left\{\left|\int f\,d\mu\right| : \mu\in HC(\overline{B})^{@},\ \|\mu\|_M=1\right\}. \tag{3.1}$$

Moreover, the supremum is attained for a suitable measure $\mu^{\#}$.

Remarks. (i) We could as well replace B by any domain with sufficiently regular boundary, the proof being essentially the same as what follows.

(ii) This kind of result is in principle familiar ([K1], [K2], [RS] and many others), but is not immediately contained in extant general theorems on "dual extremal problems" because f need not have a best approximation in $HC(\overline{B})$.

For the proof we require a lemma (essentially the same as one used in [L1]).

Lemma 3.2. *For any f in $C(\overline{B})$,*

$$\min\{\|f-u\| : u\in HL^\infty(B)\}=\inf\{\|f-v\| : v\in HC(\overline{B})\}. \tag{3.2}$$

Proof. It is clearly enough to show, for any u in $HL^\infty(B)$ and $\varepsilon>0$ that there exists v in $HC(\overline{B})$ with

$$\|f-v\|\le\|f-u\|+\varepsilon. \tag{3.3}$$

Now, for each t, $0<t<1$, and x in B,

$$|f(tx)-u(tx)|\le\|f-u\|,$$

hence

$$|f(x)-u(tx)|\le|f(x)-f(tx)|+\|f-u\|$$

so

$$\sup_{x\in B}|f(x)-u(tx)|\le\|f-u\|+\sup_{x\in B}|f(x)-f(tx)|.$$

The second summand on the right is less than ε if t is chosen sufficiently close to 1. Then defining $v(x):=u(tx)$, $v\in HC(\overline{B})$ and satisfies (3.3). □

Theorem 3.1 now follows easily. Indeed, a standard corollary to the Hahn–Banach theorem given in virtually all textbooks of functional analysis (e.g. [DS,

p. 64, Lemma 12] or [Si, p. 18, Theorem 1.1] states that the distance of an element f of a Banach space from a subspace equals the supremum of the numbers $|\varphi(f)|$ as φ ranges over all linear functionals on that Banach space having norm 1 and annihilating the given subspace, the supremum being moreover an attained maximum. In our case, with the underlying Banach space being $C(\overline{B})$, the dual space is $M(\overline{B})$. Thus the general theorem gives that $\inf\{\|f-v\|:v\in HC(\overline{B})\}$ equals the right-hand expression in (3.1). But, by Lemma 3.2 this infimum equals the left-hand term in (3.1), and that concludes the proof. □

The following is no doubt well known.

Lemma 3.3. *If μ in $M(\overline{B})$ annihilates $HC(\overline{B})$, its restriction to $S=\partial B$ is absolutely continuous with respect to hypersurface measure σ on the sphere S.*

Proof. Let ν be the measure on S defined as the restriction of μ to the Borel sets contained in S, and $\mu_i:=\mu-\nu$. It is enough to show that if G is a compact subset of S with $\sigma(G)=0$, then $\nu(G)=0$.

Suppose that φ belongs to $C(S)$ and satisfies

(i) $$\varphi(y)=1, \quad \text{if } y\in G,$$

(ii) $$0\le\varphi(y)<1, \quad \text{if } y\in S\setminus G.$$

(For example, if $\psi(y)$ is defined as the Euclidean distance from y to G, $\varphi(y)=1-\varepsilon\psi(y)$ for suitably small positive ε satisfies (i) and (ii).)

Let u_m denote the solution of Dirichlet's problem for B with boundary values φ^m, i.e. $u_m\in HC(\overline{B})$ and $u_m(y)=\varphi(y)^m$ on S, where m is a positive integer.

Since $\mu@HC(\overline{B})$ we have

$$0=\int u_m(x)\,d\mu(x)=\int u_m(x)\,d\mu_i(x)+\int \varphi(y)^m\,d\nu(y).$$

Suppose now that $m\to\infty$. By Lebesgue's bounded convergence theorem the second summand tends to $\nu(G)$. Hence the lemma will be proved if we show that the first summand tends to zero and, again by Lebesgue's theorem, it suffices (since $0\le u_m(x)\le 1$) to show that $\lim_{m\to\infty} u_m(x)=0$ for each x in B. But,

$$u_m(x)=\int \varphi(y)^m P(x;y)\,d\sigma(y),$$

where $P(x;y)$ is Poisson's kernel. For fixed x, $P(x;y)$ is a continuous function of y in S, so the limit of the integral on the right as $m\to\infty$ is $\int_G P(x,y)\,d\sigma(y)=0$. This completes the proof of Lemma 3.3. □

Corollary 3.4. *If μ in $M(\overline{B})$ annihilates $HC(\overline{B})$, then every u in $HL^\infty(B)$ is integrable with respect to μ, and moreover $\int u\,d\mu=0$.*

Here, the values of u on ∂B are understood as the radial limits of u from B, where they exist; observe that the subset E of S where the radial limits fail to exist satisfies $\sigma(E)=0$, and consequently (in view of Lemma 3.3) $|\mu|(E)=0$, where $|\mu|$ denotes the total variation measure associated to μ. Thus, u is well defined on $\overline{B}$ almost everywhere with respect to $|\mu|$ and $\int u\,d\mu$ makes sense.

Proof. Since $u(x)=\lim_{t\to 1} u(tx)$ holds almost everywhere with respect to $|\mu|$, we have by Lebesgue's theorem

$$\int u\,d\mu = \lim_{t\to 1} \int u(tx)\,d\mu(x)$$

and the integrals on the right vanish in view of our hypothesis, since $x\mapsto u(tx)$ is in $HC(\overline{B})$. □

Theorem 3.5. *Given that f belongs to $C(\overline{B})$, the necessary and sufficient condition that u in $HL^\infty(B)$ be a best approximation to f is: there exists μ in $M(\overline{B})$ of norm 1, annihilating $HC(\overline{B})$ such that*

(3.4) $\quad f(x)-u(x)=c\|f-u\|_\infty s(x)$ *almost everywhere with respect to* $|\mu|$,

where c is a unimodular constant (1 *or* -1) *and $s(x)$ is the signum function of μ, i.e. the Radon–Nikodym derivative $d\mu/d|\mu|$ (and hence $|s(x)|=1$ almost everywhere with respect to $|\mu|$).*

Proof. If $u^{\#}$ in $HL^\infty(B)$ is a best approximation of f and $\mu^{\#}$ is an extremal measure in (3.1), then

$$\left|\int f\,d\mu^{\#}\right| = \|f-u^{\#}\|_\infty \tag{3.5}$$

and, in view of Corollary 3.4 the left member of (3.5) equals $|\int (f-u^{\#})\,d\mu^{\#}|$. Inspection of the condition for equality in the inequality

$$\left|\int (f-u^{\#})\,d\mu^{\#}\right| \le \|f-u^{\#}\|_\infty \|\mu^{\#}\|_M$$

now yields the necessity of condition (3.4). For the sufficiency, suppose some $u\in HL^\infty(B)$, $\mu\in M(\overline{B})$, μ has norm 1 and (3.4) holds. Let v in $HL^\infty(B)$ be any best approximation to f. Then,

$$c\|f-u\|_\infty = c\|f-u\|_\infty \int d|\mu| = c\|f-u\|_\infty \int s(x)\,d\mu(x)$$

and from (3.4) the last term equals

$$\int (f-u)\,d\mu = \int (f-v)\,d\mu,$$

hence $\|f-u\|_\infty \le \|f-v\|_\infty \|\mu\| = \|f-v\|_\infty$ so that u is a best approximation. □

Remark. The proof shows slightly more than we claimed, namely any best approximation u, and any extremal measure μ in (3.1), satisfy (3.4).

With very slight changes, all the above analysis carries over to approximation of complex-valued functions in $L^\infty(\mathbf{D})$ by functions in $AL^\infty(\mathbf{D})$. The main change is that Lemma 3.3 requires a different proof. If μ, a complex measure on $\bar{\mathbf{D}}$, annihilates $AC(\bar{\mathbf{D}})$ we cannot conclude that it also annihilates $HC(\bar{\mathbf{D}})$, and apply Lemma 3.3. We reason instead as follows: let G be a closed subset of the unit circle with linear measure zero. It is well known (Rudin–Carleson theorem, [Ga, p. 58]) that there exists g in $AC(\bar{\mathbf{D}})$ with $\|g\|=1$, $g=1$ on G, and $|g|<1$ on $\bar{\mathbf{D}}\setminus G$. Then for positive integer m, $\int g^m\, d\mu=0$, and letting $m\to\infty$ and applying Lebesgue's convergence theorem, we get $\mu(G)=0$. This implies that $|\mu|$ restricted to $\partial\mathbf{D}$ is absolutely continuous with respect to linear measure.

The only other modification is in (3.4), where now the unimodular constant c is complex, as well as s which becomes the *conjugated* signum of μ. We state the result in the following theorem.

Theorem 3.6. *Given a complex-valued function $f\in C(\bar{\mathbf{D}})$, an element g of $AL^\infty(\bar{\mathbf{D}})$, and a complex measure μ of norm 1 on $\bar{\mathbf{D}}$ which annihilates $AC(\bar{\mathbf{D}})$, if*

$$f(z)-g(z)=c\|f-g\|\overline{s(z)} \quad (\textit{almost everywhere with respect to } |\mu|), \tag{3.6}$$

where c is a complex constant of modulus 1, and s is the Radon–Nikodym derivative $d\mu/d|\mu|$ (thus, $|s(z)|=1$ $|\mu|$-almost everywhere), then g is a best approximation to f from $AH^\infty(\bar{\mathbf{D}})$. Conversely, if g is any best approximation to f, there exists μ in $M(\bar{\mathbf{D}})$ of norm 1 annihilating $AC(\bar{\mathbf{D}})$ such that (3.6) holds, and indeed for μ we may take any extremal measure in (3.1) (adapted to the $AH^\infty(\bar{\mathbf{D}})$ scenario).

Remark. In place of the unit ball we could take any bounded domain Ω with fairly regular boundary (indeed, it suffices that the boundary be everywhere regular for Dirichlet's problem) and establish the corresponding results. There is only one point where a non-trivial change is required: in the proof of Lemma 3.2 we regularized $u(x)$ to $u(tx)$ which tacitly used that B is star-shaped with respect to 0. Let us indicate briefly the ideas needed for the general case. One requires the following lemma.

Lemma 3.7. *Let Ω be a bounded subdomain of $\mathbf{R}^n$, each boundary point of which is regular for Dirichlet's problem. Given that $f\in C(\bar{\Omega})$ we have*

$$\min\{\|f-u\| : u\in HL^\infty(\Omega)\}=\inf\{\|f-v\| : v\in HC(\bar{\Omega})\}. \tag{3.7}$$

Proof. If h in $HC(\bar{\Omega})$ solves the Dirichlet problem in Ω with $h=f$ on $\partial\Omega$, we can write $F:=f-h$ where now F vanishes on $\partial\Omega$. It is clear that it suffices to prove

(3.7) with F in place of f, i.e. there is no loss of generality to assume (as we shall henceforth)

$$f=0 \quad \text{on } \partial\Omega. \tag{3.8}$$

Suppose that $u^{\#}\in HL^{\infty}(\Omega)$, and that

$$\|f-u^{\#}\|=\lambda:=\min\{\|f-u\|:u\in HL^{\infty}(\Omega)\}.$$

We must show that for each positive ε, there is v in $HC(\bar{\Omega})$ such that

$$\|f-v\|\le\lambda+\varepsilon. \tag{3.9}$$

Note first that, because of (3.8) we have $\|u^{\#}\|\le\lambda$. We shall require the following lemma.

Lemma 3.8. *If $u\in HL^{\infty}(\Omega)$, where Ω is as in Lemma 3.7, there exists a sequence $\{v_j\}_{j=1}^{\infty}$ in $HC(\bar{\Omega})$ such that $\|v_j\|\le\|u\|$ and $v_j(x)\to u(x)$, for each $x\in\bar{\Omega}$.*

Assume this for the moment. Then, constructing the sequence $\{v_j\}_{j=1}^{\infty}$ for the function $u^{\#}$, we have that $\|v_j\|_{\infty}\le\lambda$ and $v_j(x)\to u^{\#}(x)$ for all x, and it is clear that the convergence is uniform on each compact subset of Ω. Let now $K=\{x\in\Omega: |f(x)|\ge\varepsilon\}$. Because of (3.8), K is compact. Hence for some sufficiently large j we have

$$|f(x)-v_j(x)|\le\lambda+\varepsilon, \quad x\in K.$$

For x not in K we have $|f(x)-v_j(x)|<\varepsilon+\|v_j\|<\varepsilon+\lambda$. Hence $\|f-v_j\|\le\lambda+\varepsilon$, verifying (3.9) and proving Lemma 3.7. □

Proof of Lemma 3.8. We only sketch the details. We suppose that Ω is regular enough so that u is the Poisson integral of a bounded measurable function φ on $\partial\Omega$. Then, there is a sequence $\{\varphi_j\}_{j=1}^{\infty}$, belonging to L^{∞} $(\partial\Omega;dm)$, where m denotes harmonic measure with respect to some arbitrary (but fixed) point of Ω, such that $\|\varphi_j\|_{\infty}\le\|\varphi\|_{\infty}$ and $\varphi_j(y)\to\varphi(y)$ m-almost everywhere for y in $\partial\Omega$. It is now easy to check that the Poisson integrals v_j of the φ_j satisfy the requirements of the lemma. □

One further remark: it is easy to show using the preceding ideas a harmonic analog of Sarason's "$H^{\infty}+C$ theorem" (see [G]).

Corollary 3.9. *If Ω satisfies the hypotheses of Proposition 3.7, $HL^{\infty}(\Omega)+C(\bar{\Omega})$ is a closed subspace of $L^{\infty}(\Omega)$.*

Outline of proof. It is sufficient to prove there is a constant M such that every f in $HL^{\infty}(\Omega)+C(\bar{\Omega})$ can be represented in the form $f=u+g$ where $u\in$

$HL^\infty(\Omega)$, $g\in C(\bar\Omega)$, and $\|u\|+\|g\|\le M\|f\|$, because then a Cauchy sequence $\{f_n\}_{n=1}^\infty$ from $HL^\infty(\Omega)+C(\bar\Omega)$ can always be represented as $f_n=u_n+g_n$ with $\{u_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ being Cauchy sequences in $HL^\infty(\Omega)$ and $C(\bar\Omega)$, respectively. Now, we can write $f=U+G$ with $U\in HL^\infty(\Omega)$ and $G\in C(\bar\Omega)$. Let h be the solution of Dirichlet's problem for Ω with boundary values G. Then

$$f=u+g,\quad u:=U+h,\quad g:=G-h. \tag{3.10}$$

Since g vanishes on $\partial\Omega$, the bounded harmonic function u satisfies

$$\limsup_{x\to y}|u(x)|\le\|f\|$$

for each $y\in\partial\Omega$, hence $\|u\|\le\|f\|$ by the maximum principle, and

$$\|g\|=\|f-u\|\le 2\|f\|.$$

Thus, (3.10) gives the desired representation of f, with $M=2$. □

4. Uniqueness of best approximations

Even in the unit ball B of $\mathbf{R}^n$, it is not known whether every f in $C(\bar B)$ has a unique best approximation from $HL^\infty(B)$. (It is easy, though, to show, as [HKL] did, that f has at most one best approximation that also is in $C(\bar B)$.) However, in two dimensions we can prove it, and this is one of our main results (in this section we work always in B, for simplicity but all the results extend readily to domains with sufficiently regular boundaries).

Theorem 4.1. *Every (real-valued) f in $C(\bar{\mathbf{D}})$, has a unique best approximation from $HL^\infty(\mathbf{D})$.*

Theorem 4.2. *Every (complex-valued) f in $C(\bar{\mathbf{D}})$ has a unique best approximation from $AL^\infty(\mathbf{D})$.*

The proofs are similar, but diverge at some points, and it seems slightly simpler to prove Theorem 4.2 first. We emphasize that the novelty in our results is that we prove uniqueness in the class of *bounded* approximants. Thus, uniqueness in the class $AC(\bar{\mathbf{D}})$ was already established in [H1] and some later work. As the referee pointed out to us, these results extend by conformal mapping to arbitrary Jordan domains.

Proof of Theorem 4.2. Let g_1 and g_2 be best approximations from $AL^\infty(\mathbf{D})$. By Theorem 3.6 there is a complex measure μ on $\bar{\mathbf{D}}$ of norm 1 satisfying

$$\mu @ AC(\bar{\mathbf{D}}). \tag{4.1}$$

$$f(z)-g_i(z)=\lambda\overline{s(z)} \quad |\mu|\text{-almost everywhere} \quad (i=1,2), \tag{4.2}$$

where $\lambda=\|f-g_1\|=\|f-g_2\|$ and s is the signum of μ. (We have dropped the unimodular constant c in the right member of (4.2): this c depends only on μ, and can always be taken equal to 1 upon replacing μ by $c\mu$.) Writing equation (4.2) for $i=1$ and $i=2$, and subtracting gives

$$g_1(z)-g_2(z)=0 \quad |\mu|\text{-almost everywhere.}$$

To complete the proof, therefore, it suffices to prove the following result.

Proposition 4.3. *Suppose that* $h\in AL^\infty(\mathbf{D})$ *and that* h *vanishes almost everywhere on* $\bar{\mathbf{D}}$ *with respect to* $|\mu|$, *where* μ *is a complex measure on* $\bar{\mathbf{D}}$ *of norm* 1 *satisfying* (4.1). *Then* $h=0$.

Indeed, applying this to $h=g_1-g_2$ finishes the proof of Theorem 4.2. It is essential to note that h is, as a bounded Borel function on $\mathbf{D}\cup E$, where E denotes the set of points ζ on $\partial\mathbf{D}$ where $\lim_{r\to 1} h(r\zeta)$ exists, $|\mu|$-measurable, since as we showed, (4.1) implies that $|\mu|$ vanishes on subsets of $\partial\mathbf{D}$ of length zero, and hence on $(\partial\mathbf{D})\setminus E$.

Proof. If $|\mu|$ restricted to $\partial\mathbf{D}$ is not the zero measure, then $|\mu|(F)>0$ for some $F\subset\partial\mathbf{D}$ of positive linear measure since, by virtue of a well-known theorem of F. and M. Riesz, cf. [Ga, Theorem 7.10], μ is absolutely continuous with respect to arc length measure on the unit circle. By hypothesis, $h(\zeta)=0$ almost everywhere on F and so by another theorem of F. and M. Riesz h vanishes identically. We are left with the case where $|\mu|(\partial\mathbf{D})=0$, i.e. the support (in the measure-theoretic sense) of μ lies in $\mathbf{D}$. By hypothesis, $\operatorname{supp}\mu$ is contained in the zero set of h, so unless h vanishes identically, μ is supported by a countable subset $\{z_j\}_{j=1}^\infty$ of $\mathbf{D}$ with $\sum_{j=1}^\infty(1-|z_j|)<\infty$. But then μ cannot satisfy (4.1)! For, (4.1) implies that for every $\varphi\in AL^\infty(\mathbf{D})$, μ annihilates $\varphi(tx)$ whenever $t<1$ and hence also φ (by the bounded convergence argument we have used several times). Thus there exist complex $\{c_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty|c_j|=1$ such that

$$0=\int\varphi\,d\mu=\sum_{j=1}^\infty c_j\varphi(z_j) \tag{4.3}$$

holds for every φ in $AL^\infty(\mathbf{D})$. And this is impossible: choose k so that $c_k \neq 0$, and take for φ in (4.3) a Blaschke product formed with the zero set $\{z_j : j \neq k\}$, and we obtain a contradiction.

Thus Proposition 4.3, and Theorem 4.2 are proven. $\square$

Proof of Theorem 4.1. In precisely the same way as above, the proof reduces to that of the following result.

Proposition 4.4. *Suppose that* $u \in HL^\infty(\mathbf{D})$ *and that* u *vanishes almost everywhere on* $\bar{\mathbf{D}}$ *with respect to* $|\mu|$, *where* μ *is a (real) measure on* $\bar{\mathbf{D}}$ *of norm* 1 *satisfying*

$$\mu @ HC(\bar{\mathbf{D}}). \tag{4.4}$$

Then $u=0$.

Remark. Again, it is in order to emphasize that u, defined on $\bar{\mathbf{D}}$ except in a boundary set of length 0, is $|\mu|$-measurable, so the hypothesis makes sense. Proposition 4.4 is rather deeper than Proposition 4.3 (for one thing, there is no easy way to rule out that $|\mu|$ charges some subset of $\partial\mathbf{D}$) and perhaps of independent interest. It makes good sense in $\mathbf{R}^n$, but we can only prove it when $n=2$.

Proof. There is no loss of generality in assuming (as we shall) that $|\mu|$ does not charge any set of positive area, for otherwise u vanishes on such a set and, being real-analytic in $\mathbf{D}$, identically. Write

$$M(\zeta) := \int \frac{d\mu(z)}{z-\zeta}, \quad \zeta \in \mathbf{C} \setminus \operatorname{supp} \mu.$$

For any polynomial P, since $(P(z)-P(\zeta))/(z-\zeta)$ is a polynomial in z (for any fixed ζ), we have by hypothesis (4.4)

$$\int \frac{P(z)-P(\zeta)}{z-\zeta}\, d\mu(z) = 0,$$

hence

$$P(\zeta)M(\zeta) = \int \frac{P(z)}{z-\zeta}\, d\mu(z), \quad \zeta \in \mathbf{C} \setminus \operatorname{supp} \mu. \tag{4.5}$$

Now, M is holomorphic on the open set $\mathbf{D} \setminus \operatorname{supp} \mu$. Moreover, it is not identically zero there. Indeed, M vanishes on $\{\zeta : |\zeta| > 1\}$ because of (4.4), so if it also were zero on all of $\mathbf{D} \setminus \operatorname{supp} \mu$ it would vanish almost everywhere on $\mathbf{C}$ (with respect to area measure). By a well-known theorem on Cauchy transforms (see [Ga, p. 46]) this would imply $\mu=0$, a contradiction. Hence, there is a non-empty disk D_0 contained in $\mathbf{D}$ and on which M is analytic and non-vanishing. From (4.5)

$$P(\zeta)=\frac{1}{M(\zeta)}\int\frac{P(z)}{z-\zeta}\,d\mu(z),\quad \zeta\in D_0. \tag{4.6}$$

Now, (4.6) can be interpreted so: to each ζ in D_0, there is a *complex* representing measure ν_ζ on $\bar{\mathbf{D}}$, where

$$d\nu_\zeta(z):=\frac{1}{M(\zeta)(z-\zeta)}\,d\mu(z)$$

for "point evaluation at ζ", i.e. $P(\zeta)=\int P\,d\nu_\zeta$ holds for every polynomial P and hence, by an approximation argument, when P is replaced by any element of the "disk algebra" $AC(\bar{\mathbf{D}})$. By virtue of a known result on function algebras ([Ga, p. 33]) there must also exist another representing measure m_ζ, i.e. a measure on $\bar{\mathbf{D}}$ satisfying

$$f(\zeta)=\int f\,dm_\zeta,\quad \text{all } f \text{ in } AC(\bar{\mathbf{D}}),\ \zeta \text{ in } D_0, \tag{4.7}$$

and with the additional properties

(4.8) $m_\zeta\geq 0$,

(4.9) m_ζ is absolutely continuous with respect to $|\nu_\zeta|$.

(And hence, m_ζ is absolutely continuous with respect to $|\mu|$.)

It is now easy to finish the proof. Since m_ζ is real, (4.7) continues to hold when f is replaced by its real part, so

$$u(\zeta)=\int u\,dm_\zeta,\quad \zeta\in D_0, \tag{4.10}$$

holds for all harmonic polynomials u, and hence all bounded harmonic functions on $\mathbf{D}$. Indeed, every such function, considered as defined on $\mathbf{D}\cup E$, where E is the subset of $\partial\mathbf{D}$ where u has radial limits, is the pointwise limit of a bounded sequence of harmonic polynomials, almost everywhere with respect to the measure which assigns to each Borel set $F\subset\bar{\mathbf{D}}$ the linear measure of $F\cap\partial\mathbf{D}$, and hence also $|\mu|$-almost everywhere, and so finally m_ζ-almost everywhere since m_ζ is absolutely continuous with respect to $|\mu|$.

Thus, in sum, (4.10) makes sense whenever $u\in HL^\infty(\mathbf{D})$. And (to return to the hypotheses of Proposition 4.4) if u vanishes $|\mu|$-almost everywhere on $\bar{\mathbf{D}}$ it also vanishes $|m_\zeta|$-almost everywhere so by (4.10), $u(\zeta)=0$ for all ζ in the disk D_0, and thus finally $u=0$. □

Remark. One of the main results of [HKL] is: *Every f in $C(\overline{B})$ admits at most one best approximation from $HC(\overline{B})$*, where B is the unit ball of $\mathbf{R}^n$, $n\geq 2$. (The analogous result for approximation from $AC(\overline{\mathbf{D}})$ had been proved earlier by Romanova [R1] and Luecking [L1].)

A proof in our context goes as follows: In view of the preceding, it is enough to prove that *if $g\in HC(\overline{B})$ and g vanishes almost everywhere with respect to $|\mu|$, where $\mu\in M(\overline{B})$ has norm 1 and annihilates $HC(\overline{B})$, then $g=0$.* To show this, let K denote the closure of $\operatorname{supp}\mu$. Then $\mathbf{C}\setminus K$ cannot be connected, otherwise (see [Br], [D]) $HC(\overline{B})|_K$ would be dense in $C(K)$ forcing $\mu=0$. Hence g (because it is continuous on $\overline{B}$!) vanishes on the boundary of some non-empty open subset of B. By the maximum principle it vanishes on this open set, and hence identically.

Additional remarks. It might not be amiss to draw attention to some interesting, purely potential-theoretic sidelights of the preceding discussion.

First of all, if μ is a (say, real) measure on $\overline{\Omega}$, where Ω is a smoothly bounded domain in $\mathbf{R}^n$ then, splitting $\mu=\mu_i+\mu_b$, where μ_i and μ_b denote the restrictions of μ to the interior, and boundary respectively of $\overline{\Omega}$, the condition $\mu @ HC(\overline{\Omega})$ can be restated in terms of the balayage concept (see, e.g. [La]) as: *μ_b is the balayage of $-\mu_i$ to $\partial\Omega$.* Thus, our Lemma 3.3 to the effect that μ_b is absolutely continuous with respect to hypersurface measure $d\sigma$ on $\partial\Omega$, for every annihilating measure μ, can be restated as: *the balayage onto $\partial\Omega$ of any bounded real measure on Ω is absolutely continuous with respect to $d\sigma$* (hence, if not identically zero, its support has positive area; consequently, a non-trivial measure supported on a subset of $\partial\Omega$ having area zero cannot be obtained as a balayage of any signed measure (charge distribution) on Ω).

This result is not deep, and doubtless known, but we have not found explicit mention of it. Also, it is in a sense best possible: using results of Bonsall [B] one can show that every measure on $\partial\Omega$, absolutely continuous with respect to $d\sigma$, is the balayage of a (real) measure on Ω, even one supported on a countable set clustering nowhere in Ω. (Of course, measures on $\partial\Omega$ arising by balayage of *positive* measures in Ω are more restricted; obviously their Radon–Nikodym derivatives with respect to harmonic measure are bounded away from zero.) The countable sets in question are those such that almost every boundary point of Ω is the non-tangential limit of some sequence chosen from the set. Such sets seem first to have occurred in the context of the unit disk in [BSZ]. It is easy to see that these sets always support non-trivial measures annihilating bounded harmonic functions.

In a similar vein, the proof of Proposition 4.4 contains an argument (essentially, the result quoted from [Ga] on representing measures) which, in an electrostatic interpretation, says the following is true for *planar* simply connected domains Ω:

If there is a real measure μ which (considered as a charge distribution on $\bar{\Omega}$) produces the same electric field outside Ω as does a point charge δ_ζ at some point ζ in $\Omega\setminus\operatorname{supp}\mu$, then there is also a *positive* charge distribution m absolutely continuous with respect to $|\mu|$, producing the same field outside Ω. (The absolute continuity implies, in particular, that m is not permitted to place charges anywhere off $\operatorname{supp}\mu$.)

This result seems not unreasonable, on "physical" grounds also in more than two dimensions. It is not known to us if it is true when Ω is e.g. the unit ball of $\mathbf{R}^3$.

5. Counterexamples, (a): best approximation may not be unique

In this and the next section, we construct examples to show (a) that uniqueness of the best approximation may fail if the function being approximated has even one single point of discontinuity, and (b) that a *continuous* best approximation need not exist, even if the function being approximated is very regular.

Theorem 5.1. *There is a function bounded in the unit disk* $\mathbf{D}$*, and continuous in* $\bar{\mathbf{D}}\setminus\{1\}$ *with more than one best approximation.*

Remark. A similar example was given in [HKL] but their function has a whole line of discontinuity points.

We precede the proof by a lemma that also will be needed in the next section.

Lemma 5.2. *Let* Ω *be a smoothly bounded domain in* $\mathbf{R}^n$*, and let* f *be a bounded real-valued continuous function on* Ω*. Suppose that* b *is an open ball with center* y *such that* $\bar{b}\subset\Omega$*. If further* $f(y)=1$*, and* $f(x)=-1$ *for all* x *in* ∂b*, then*

$$\operatorname{dist}(f, HL^\infty(\Omega))\geq 1. \tag{5.1}$$

Remark. In place of b we could take any other domain D with $\bar{D}$ homeomorphic to a ball, and $\bar{D}\subset\Omega$.

It is instructive to give two proofs of the lemma.

Proof #1 (by duality). In view of Theorem 3.1 it is enough to construct a measure μ in $M(\bar{\Omega})$ of norm 1, annihilating $HC(\bar{\Omega})$, and such that $|\int f\,d\mu|=1$. We can take $2\mu=\delta_y-\nu$, where ν is normalized Lebesgue measure on ∂b (so that $\int f\,d\nu$ is the mean value of f over the sphere ∂b). $\square$

Proof #2. We proceed by contradiction. Suppose $h\in HL^\infty(\Omega)$ and $\|f-h\|=d<1$. Then

$$-1-d\leq h(x)\leq -1+d,\quad x\in\partial b.$$

By the maximum principle, $h(y)\leq -1+d$. On the other hand $h(y)\geq f(y)-d=1-d$, and the last two inequalities contradict one another. □

Remarks. 1. Proof #2 shows that it is sufficient for (5.1) if $f(y)\geq 1$ and $f(x)\leq -1$ on ∂b.

2. For a later purpose, observe that if $d=1$ in the second proof, then $h\leq 0$ on ∂b while $h(y)\geq 0$, so, by the strong form of the maximum principle, h vanishes identically. Therefore we obtain the sharper conclusion:

Under the hypotheses of the lemma, if moreover $\|f\|=1$ then the function identically zero is the unique best approximation from $HL^\infty(\Omega)$.

Proof of Theorem 5.1. We begin by an auxiliary calculation: examine the set

$$E:=\{(x,y)\in\overline{\mathbf{D}}:|y|<1-x^2-y^2\}. \tag{5.2}$$

It is easy to see that E is the open lens-shaped set bounded by two circular arcs of radius $\frac{1}{2}\sqrt{5}$, centered at $(0,\frac{1}{2})$ and $(0,-\frac{1}{2})$.

Now, let $\{D_i\}_{i=1}^\infty$, be pairwise disjoint open disks with centers z_i and radii r_i, contained in E and such that $z_i\to 1$ as $i\to\infty$. One can construct a function φ on $\overline{E}$ such that

(i) $\varphi(z_i)=1$, $i=1,2,\dots$;

(ii) $\varphi(z)=-1$ for $z\in\partial D_i$, $i=1,2,\dots$;

(iii) $-1\leq\varphi\leq 1$ on $\overline{E}$;

(iv) $\varphi=0$ on ∂E;

(v) $\varphi\in C(\overline{E}\setminus\{1\})$.

Indeed, we define φ to be $1-2|z-z_i|/r_i$ in $\overline{D}_i$ and extend φ to $\overline{E}$ so that (iii), (iv) and (v) hold. Now, extend φ to $\overline{\mathbf{D}}$ as 0 on $\overline{\mathbf{D}}\setminus\overline{E}$ and call the resulting function $\widetilde{\varphi}$. Finally, define f by $f(x,y)=(x^2+y^2)\,\widetilde{\varphi}(x,y)$. Then $f\in C(\overline{\mathbf{D}}\setminus\{1\})$. Also, $\|f\|=1$ and $\operatorname{dist}(f,HL^\infty(\mathbf{D}))\geq 1$ since by Lemma 5.2 $\operatorname{dist}(\varphi,HL^\infty(D_i))\geq 1$, $i=1,2,\dots$. Hence, the function identically zero is a best approximation to f from $HL^\infty(\mathbf{D})$.

To complete the proof, we now show that y is also a best approximation to f. Indeed,

$$|f(x,y)-y|\leq(x^2+y^2)|\widetilde{\varphi}(x,y)|+|y|\leq\begin{cases}(x^2+y^2)+(1-x^2-y^2) & \text{on } E,\\ |y| & \text{on } \mathbf{D}\setminus E,\end{cases}$$

where in the first estimate we need (5.2). Hence $|f(x,y)-y|\leq 1$ on $\mathbf{D}$, and y is a best approximation. □

Remarks. 1. Since the set of all best approximations is always convex, in the present example all the functions cy, $-1\leq c\leq 1$, are best approximants.

2. As pointed out to us by the referee, this example also shows, e.g., that a function symmetric with respect to a line may admit non-symmetric best approximants (for the D_i can all be centered on the real axis).

6. Counterexamples, (b): smooth functions without continuous best approximation

A function continuous in, say, the closed ball of $\mathbf{R}^n$ may possess only discontinuous best approximations from $HL^\infty(B)$, i.e. such which do not extend continuously to all of $\bar{B}$, or are from $AL^\infty(\bar{\mathbf{D}})$ in the analytic case. An example of the latter was given by Hintzman [H2] in 1975. The same fact was remarked later in [HKL] in the $HL^\infty(B)$ context, and an example is presented there (Example 4.3) which, however, is faulty: the function approximated is in fact not continuous as claimed! Correct examples follow from our next theorem.

Theorem 6.1. *Let Ω be any bounded domain in $\mathbf{R}^n$. Every h in $HL^\infty(\Omega)$ appears as the unique best approximation to some function in $C_0^\infty(\Omega)$ (the class of infinitely differentiable functions with compact support).*

Proof. Without loss of generality, assume that Ω contains 0, and hence the closure of some ball $\beta:=\{x:|x|<\varrho\}$. We begin by constructing an auxiliary function φ. Fix $r<\varrho$ and construct $\varphi\in C_0^\infty(\beta)$ such that

(i) $\varphi(0)=-1$;

(ii) $\varphi=1$ on a neighborhood of $\{x:|x|=r\}$;

(iii) $0\le\varphi\le 1$ for $r\le|x|\le\varrho$;

(iv) $\|\varphi\|_\infty=1$.

It is clear that such φ exist, with $\varphi(x)$ radial, that is, of the form $p(|x|)$, where p is a suitably defined function on $[0,\varrho]$.

Let us now be given any $h\in HL^\infty(\Omega)$ with $\|h\|=1$ (clearly this implies no loss of generality). We claim that the function f defined by

$$f:=\begin{cases}\varphi+h, & |x|\le r,\\ \varphi+\varphi h, & r\le|x|\le\varrho,\end{cases}$$

and extended as 0 to $\bar{\Omega}\setminus\bar{\beta}$ fulfills the requirements of the theorem, i.e. h is its unique best approximation from $HL^\infty(\Omega)$.

Now,

$$f-h=\begin{cases}\varphi, & |x|\le r,\\ \varphi+(\varphi-1)h, & r\le|x|\le\varrho,\\ -h, & x\in\Omega\setminus\bar{\beta}.\end{cases}$$

In each case we see that $|f(x)-h(x)|\le 1$. For example, when $r\le|x|\le\varrho$ we have

$$|f(x)-h(x)|\le|\varphi(x)|+|\varphi(x)-1|\,|h(x)|\le\varphi(x)+(1-\varphi(x))=1$$

in view of (iii). Hence $\|f-h\|\le 1$.

On the other hand, in view of Lemma 5.2 the unique best approximation to φ on the ball $\{|x|\le r\}$ by a bounded harmonic function is identically zero. This is equivalent to saying that the only harmonic function that approximates f with error at most 1, even on the set $\{x:|x|\le r\}$, is that identically equal to h.

Hence, h is indeed the unique best approximation to f. □

Remarks. 1. Our construction is based on local behaviour: given h we can even construct f as in the theorem, with support in any given ball.

2. A similar result can be proved, in like manner, for approximation in $\mathbf{D}$, say, by bounded analytic functions. This shows an essential difference between the character of the approximation problem by bounded analytic functions when the norm is that of $L^\infty(\mathbf{D})$, and the (classical) one where the norm is that of $L^\infty(\partial\mathbf{D})$. In the latter case, it is known, e.g., that a Hölder continuous function has (a unique) best approximation which is moreover in $AC(\overline{\mathbf{D}})$, see [G] for details ("Carleson–Jacobs theorem") and references.

3. The *metric projection map* (see [Si]) is the set-valued map which assigns to each function its set of best approximations. In view of our results in Section 3, this map assigns to each real-valued f, continuous on the closed ball B of $\mathbf{R}^2$, a *unique* best approximation from $HL^\infty(B)$, so the restriction of the metric projection map to $C(\overline{B})$ is a map from $C(\overline{B})$ into (and, in view of Theorem 6.1 *onto*) $HL^\infty(B)$. A consequence of Theorem 6.1 is the following result.

Corollary 6.2. *The metric projection map from $C(\overline{B})$ to $HL^\infty(B)$ is discontinuous, with respect to the norm topologies of these spaces.*

Indeed, let h be any element with norm 1 of $HL^\infty(B)\setminus C(\overline{B})$ (i.e. not continuously extendable to $\overline{B}$). Let $\{t_j\}_{j=1}^\infty$ be a strictly increasing sequence on $(0,1)$ with $t_j\to 1$, and define $h_j(x)=h(t_jx)$. The construction used in the proof of Theorem 6.1 gives us a function f whose unique best approximation is h, and to each j a function f_j whose unique best approximation is $h_j/\|h_j\|$. Clearly $\|f_j-f\|\to 0$ since this only depends on the uniform convergence of h_j (and $h_j/\|h_j\|$) to h on each compact subset of B. But, the best approximant $h_j/\|h_j\|$ of course cannot tend *in norm* to h since that function is not continuous in $\overline{B}$. □

Our final example will show that even for such regular functions as polynomials, the best approximation need not be continuous!

Theorem 6.3. *Let g be defined in the closed unit disk $\overline{\mathbf{D}}$ by*

$$g(x,y):=y(x^2+y^2-1).$$

Its unique best approximation from $HL^\infty(\mathbf{D})$ is $-ch$, where h is the harmonic function in $\mathbf{D}$ with boundary values 1 and -1 on the upper and lower halves, respectively, of the unit circle and c is a uniquely determined positive constant.

Proof. Consider on the upper half $\mathbf{D}^+:=\mathbf{D}\cap\{x+iy:y>0\}$ of $\mathbf{D}$ the functions $F_t:=g+th$, where t is a positive parameter. Observe that F_t is subharmonic and its supremum on $\mathbf{D}^+$ is t. It is clear that, for sufficiently small t, $\inf F_t$ on $\mathbf{D}^+$ is negative (indeed, an attained minimum) at some point of $\mathbf{D}^+$ that can, in principle, be computed. For $t\to 0$, $\inf F_t$ tends to a negative limiting value, namely the minimum of g on $\mathbf{D}^+$. Thus, the continuous function $\inf F_t+\sup F_t$ (always relative to $\mathbf{D}^+$) is negative for small t, positive for large t and so takes the value zero for some t. Let c be the smallest such value of t. We then have that F_c has supremum c, and (attained) minimum $-c$ in $\mathbf{D}^+$, and moreover equals c on the upper half of the unit circle.

Since g and h have odd symmetry with respect to the x-axis, a similar reasoning applies on $\mathbf{D}^-:=\mathbf{D}\cap\{x+iy:y<0\}$. We conclude that $\|F_c\|=c$, moreover F_c equals c on the upper half, and $-c$ on the lower half of the unit circle, and for some $z_0\in\mathbf{D}^+$ we have $F_c(z_0)=-c$, $F_c(\bar{z}_0)=c$.

We now claim that *any function having these properties admits* 0 *as its unique best approximation from* $HL^\infty(\mathbf{D})$. Transferring this to g gives now the assertion of the theorem.

In proving this claim, the value of c is irrelevant, so we shall have proved the theorem if we show the following "Assertion". It will be convenient to denote by $X(\mathbf{D})$ the analog of Sarason's "$H^\infty+C$" space for the disk, namely

$$X(\mathbf{D}):=HL^\infty(\mathbf{D})+C(\bar{\mathbf{D}}).$$

[By earlier remarks, this space is *closed*, but that is not important here, only that each element of X has boundary values in $L^\infty(\partial\mathbf{D})$, whose essential supremum (with respect to Haar measure) cannot exceed its $L^\infty(\mathbf{D})$ norm.]

Assertion. *Suppose* $v\in X(\mathbf{D})$ *and that* v *satisfies*

(i) $\|v\|=1$;

(ii) $v=1$ *a.e. on the upper, and* $v=-1$ *a.e. on the lower half of the unit circle*;

(iii) *for some* $z_0\in\mathbf{D}^+$, $v(z_0)=-1$ *and* $v(\bar{z}_0)=+1$.

Then 0 *is the unique best approximation to* v *from* $HL^\infty(\mathbf{D})$.

Suppose that $u\in HL^\infty(\mathbf{D})$ and that $\|v-u\|\le 1$. We shall show that $u=0$. First of all, $|v(z)-u(z)|\le 1$ on the unit circle, so $u\ge 0$ on the upper half, and $u\le 0$ on the lower half of the unit circle. Moreover $u(z_0)\le 0\le u(\bar{z}_0)$. But, the only bounded harmonic function satisfying these properties is 0. Indeed, $u(z)-u(\bar{z})$ is harmonic in $\mathbf{D}^+$ and bounded, non-negative a.e. on $\partial\mathbf{D}^+$ and less than or equal to zero at z_0. By the strong maximum principle $u(z)-u(\bar{z})$ vanishes identically in $\mathbf{D}^+$, and hence also in $\mathbf{D}$. But then $u\ge 0$ on the upper half-circle, so that u is zero a.e. there, and likewise for the lower half-circle. Thus $u=0$, proving the assertion and with it Theorem 6.3. □

Remarks. 1. The last part of the proof could also be formulated in terms of annihilating measures. The gist of it is that for every $z_0 \in \mathbf{D}^+$ there is a measure μ annihilating $HC(\bar{\mathbf{D}})$, whose support is $\partial\mathbf{D} \cup \{z_0, \bar{z}_0\}$ and with the following further properties: $\mu|_{\partial\mathbf{D}}$ is absolutely continuous with respect to Haar measure, and the derivative of μ with respect to Haar measure is positive on the upper half, negative on the lower half of the unit circle. Moreover $\mu(\{z_0\})<0$ and $\mu(\{\bar{z}_0\})>0$.

Indeed, we have only to observe that $\delta_{z_0} - P_{z_0}\, d\theta$, where P_{z_0} denotes the Poisson kernel corresponding to z_0 and $d\theta$ is Haar measure, annihilates harmonic functions on $\mathbf{D}$, as does $\delta_{\bar{z}_0} - P_{\bar{z}_0}\, d\theta$. Hence so does their difference, which gives the required μ. It is easily verified that $P_{z_0} - P_{\bar{z}_0}$ is positive on the upper half-circle and negative on the lower half-circle.

2. Possibly one could prove a variant of Theorem 6.1 in which f is constructed to be, instead of an element of $C_0^\infty(\Omega)$, a function real-analytic on a neighborhood of $\bar{\Omega}$.

7. Concluding remarks

From the results of Section 6 it is clear that no amount of regularity of f can guarantee even continuity of its best approximation. However, other kinds of conditions can, as observed in [HKL].

Theorem 7.1. (Essentially from [HKL].) *Let Ω be a domain in $\mathbf{R}^n$ all of whose boundary points are regular for Dirichlet's problem, and suppose that $f \in C(\bar{\Omega})$. Let u denote the (unique) function in $HC(\bar{\Omega})$ equal on $\partial\Omega$ to f. If $f-u$ does not change sign on Ω, the unique best approximation of f from $HL^\infty(\Omega)$ is of the form $u+c$ for a suitable constant. More precisely: In case $f-u \geq 0$ on $\bar{\Omega}$ and $f-u$ has a positive maximum value M then $u+\frac{1}{2}M$ is the unique best approximation. Similarly, if $f-u \leq 0$ and $f-u$ has a negative minimum value $-M$, then $u-\frac{1}{2}M$ is the unique best approximation.*

Corollary 7.2. *If Ω satisfies the hypotheses of the theorem, and $f \in C(\bar{\Omega})$ and f is superharmonic in Ω, or subharmonic in Ω, its best approximation from $HL^\infty(\Omega)$ is unique, and continuous on $\bar{\Omega}$.*

This is proved in [HKL]. Let us give an alternative proof, since it is very short and illustrates well the usefulness of duality (Theorem 3.1).

Suppose e.g. $f-u \geq 0$ on $\bar{\Omega}$ and the maximum value $M>0$ of $f-u$ is assumed at a point y in Ω. Then, $\|f-(u+\frac{1}{2}M)\| = \frac{1}{2}M$, and $f-(u+\frac{1}{2}M)$ equals $-\frac{1}{2}M$ on all of $\partial\Omega$, and $\frac{1}{2}M$ at y. Therefore, in view of Theorem 3.1, it is sufficient to verify that there is a measure μ supported on $\partial\Omega\cup\{y\}$ of norm 1 annihilating $HC(\bar{\Omega})$, with $\mu(\{y\})>0$ such that the restriction of μ to $\partial\Omega$ is a non-positive measure. But, such μ is given by

$$\mu=\tfrac{1}{2}(\delta_y-\nu_y),$$

where ν_y is the "representing measure" on $\partial\Omega$ for evaluation at y, i.e. the harmonic measure on $\partial\Omega$, associated to the point y (which, in case $\partial\Omega$ is sufficiently smooth, is $P_y\,d\sigma$, where $d\sigma$ denotes Lebesgue measure on $\partial\Omega$ and P_y is Poisson's kernel for the point y). □

Despite its simplicity, Theorem 7.1 yields a large class of functions where best approximation can in principle be calculated (at any rate, reduced to the solution of Dirichlet's problem).

Further extension of this class can be given using the ideas we employed in proving Theorem 6.3. We shall merely illustrate the idea by an example. Let z_1 and z_2 be any two distinct points of $\mathbf{D}$, and let

$$P_j(\zeta)=\frac{1}{2\pi}\operatorname{Re}\left(\frac{\zeta+z_j}{\zeta-z_j}\right),\quad j=1,2;\ |\zeta|=1,$$

be the corresponding Poisson kernels. Writing $\zeta=e^{it}$, we have a measure μ on $\bar{\mathbf{D}}$ annihilating $HC(\bar{\mathbf{D}})$, given by

$$\mu=[\delta_{z_1}-P_1\,dt]-[\delta_{z_2}-P_2\,dt].$$

The signum of this measure on its support is $+1$ on $\{z_1\}$ and on the subset Γ_1 of the unit circle where $P_2(\zeta)>P_1(\zeta)$; it is -1 on $\{z_2\}$ and the subset Γ_2 of the unit circle where $P_1(\zeta)>P_2(\zeta)$. The sets Γ_1 and Γ_2 are easy to compute explicitly. It then follows that if $f\in C(\bar{\mathbf{D}})+HL^\infty(\mathbf{D})$ and $u\in HL^\infty(\mathbf{D})$, and f and u are such that $\|f-u\|=M$ and $f-u$ equals $+M$ at z_1 and a.e. on Γ_1, and $f-u=-M$ at z_2 and a.e. on Γ_2, then u is the unique best approximation to f from $HL^\infty(\mathbf{D})$. Theorem 6.3 exemplifies the special case where $z_1\in\mathbf{D}^+$, $z_2=\bar{z}_1$. In that case Γ_1 and Γ_2 are, respectively, the lower and upper halves of the unit circle.

This enables us to construct, backwards as it were, explicit smooth functions f in $C(\bar{\mathbf{D}})$ whose (unique) best approximation from $HL^\infty(\mathbf{D})$ is the harmonic function u equal to 1 on Γ_1 and -1 on Γ_2: we have only to construct f vanishing on $\partial\mathbf{D}$, and such that $\|f-u\|=1$, $f(z_1)-u(z_1)=1$, $f(z_2)-u(z_2)=-1$. Further details are left to the reader.

These are the simplest instances, in our context, of a well-known technique in the general theory of best uniform approximation: the signa of measures which annihilate the space of permissible approximants ("extremal signatures", cf. [RiS], [S]) play the role that alternating sequences of ± 1 play in the classical Chebyshev theory of polynomial approximation.

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