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On a local uniqueness result for the inverse Sturm–Liouville problem

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Abstract. A new and fairly elementary proof is given of the result by B. Simon [S], that the potential in a Sturm–Liouville operator is determined by the asymptotics of the associated m-function near $-\infty$. The proof given is based on relations between the classical transformation operators and the m-function.

1. Introduction

In this paper we study the Sturm–Liouville operator

$$H=-\frac{d^2}{dx^2}+q$$

on $L^2([0,\infty))$ and the related Sturm–Liouville problem

$$Hu(x)=-\frac{d^2u}{dx^2}(x)+q(x)u(x)=\lambda u(x), \quad x\in[0,\infty), \tag{1}$$

$$u(0)=0. \tag{2}$$

We assume that $q\in L^1([0,\infty))$ is real valued. Under these assumptions it is well known (cf. [CL, p. 255, Problem 4]) that H is of limit-point type at infinity and selfadjoint on the domain

$$D(H)=\{u\in L^2([0,\infty))\mid u,u'\in AC_{\rm loc}([0,\infty)),\ u(0)=0,\ -u''+qu\in L^2([0,\infty))\}.$$

See [CL], [J] or [LS] for the theory of singular Sturm–Liouville problems. For a modern treatment see [Pe].

The special solution to (1) defined by the conditions (2) and $u'(0)=1$ is called the regular solution and denoted by $\phi(x,\lambda)$.

Since H is of limit-point type at infinity we can for $\lambda \notin \sigma(H)$ define $u(x, \lambda)$ to be the unique solution to (1) in $L^2([0,\infty))$ satisfying $u(0,\lambda)=1$, the so-called Weyl solution. Associated with (1) is the m-function defined by

$$m(\lambda; q) = u'(0,\lambda) \tag{3}$$

for λ not an eigenvalue of H. Since the spectrum of H is real and bounded from below ([LS, Theorem 3.1]) there is a constant $C>0$ such that the m-function is defined for $\lambda \in \mathbf{C} \setminus [-C,\infty)$.

The main result of this paper is a new proof of the following uniqueness result.

Theorem 1.1. ([S]) *Let $q_1, q_2 \in L^1([0,\infty))$ be real potentials for two Sturm–Liouville problems and let m_1 and m_2 be the associated m-functions. Assume there is an $a>0$ such that*

$$m(-k^2; q_1) - m(-k^2; q_2) = o(e^{-ak(1-\varepsilon)}), \quad \text{as } k \to \infty,$$

for every $\varepsilon>0$. Then $q_1(x)=q_2(x)$ for a.e. $x \in \left[0, \frac{1}{2}a\right]$.

As a corollary to Theorem 1.1 we recover the well-known result by [Bo], [GL] and [M] that the m-function (or equivalently the spectral measure associated with (1) and (2)) determines the potential q.

Note that since two potentials $q_1, q_2 \in L^1_{\text{loc}}([0,\infty))$ satisfying $q_1(x)=q_2(x)$ for a.e. $x \in \left[0, \frac{1}{2}a\right]$ have associated m-functions satisfying

$$m(-k^2; q_1) - m(-k^2; q_2) = o(^{-ak(1-\varepsilon)}), \quad \text{as } k \to \infty,$$

for all $\varepsilon>0$ (see [S, Theorem A.1.1]), Theorem 1.1 is valid under the less restrictive assumption that $q_1, q_2 \in L^1_{\text{loc}}([0,\infty))$.

In the paper [S] a new mathematical object is introduced and by this new formalism the result is proved. As observed in [GS2] this new object is closely related to a certain transformation operator relating the regular solutions to different Sturm–Liouville problems, and the main idea in the present paper is to give an elementary proof of Theorem 1.1 within the framework of these transformation operators.

Recently Gesztesy and Simon [GS1] and Bennewitz [Be] have given different new proofs, which are considerably shorter than the original proof.

The outline of the paper is the following: First we review the concept of transformation operators and prove several estimates concerning these operators. Next we derive an equation relating the Weyl solution to a particular transformation kernel. This relation then gives a relation between the m-function and the kernel through a kind of Laplace transform. At last we derive a relation between the different transformation kernels and by this relation and a uniqueness theorem for a related hyperbolic partial differential equation with Cauchy-data we prove Theorem 1.1.

2. Transformation operators

Let q_1 and q_2 be potentials and let ϕ_1 and ϕ_2 be the regular solutions to the associated Sturm–Liouville problems. Then there exists a unique transformation kernel $\widetilde{K}$ independent of λ such that

$$\phi_2(x,\lambda)=\phi_1(x,\lambda)+\int_0^x \widetilde{K}(x,t)\phi_1(t,\lambda)\,dt. \tag{4}$$

This is the Levitan–Povzner representation relating solutions to different Sturm–Liouville problems ([Po] and [L]).

In the special case $q_2=0$, the regular solution is $\phi_2(x,\lambda)=\sin(x\sqrt{\lambda})/\sqrt{\lambda}$, and the kernel denoted by $-L$ satisfies

$$\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}=\phi_1(x,\lambda)-\int_0^x L(x,t)\phi_1(t,\lambda)\,dt. \tag{5}$$

Similarly, when $q_1=0$, the kernel is denoted by K and satisfies

$$\phi_2(x,\lambda)=\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}+\int_0^x K(x,t)\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\,dt. \tag{6}$$

It is easily seen by inserting (4) in (1) and making use of the initial conditions of ϕ_1 and ϕ_2 that the kernel $\widetilde{K}$ must solve the Goursat problem

$$\begin{aligned} &\widetilde{K}_{xx}(x,t)-\widetilde{K}_{tt}(x,t)+(q_1(t)-q_2(x))\widetilde{K}(x,t)=0, && (x,t)\in D,\\ &2\frac{d}{dx}\widetilde{K}(x,x)=q_2(x)-q_1(x), && x\ge 0,\\ &\widetilde{K}(x,0)=0, && x\ge 0, \end{aligned} \tag{7}$$

where $D=\{(x,t)\in\mathbf{R}^2\,|\,0<t<x\}$.

The following lemma shows that the problem (7) is well posed.

Lemma 2.1. *Assume $q_1,q_2\in L^1([0,\infty))$. Then (7) has a unique solution $\widetilde{K}\in C^0(\overline{D})$. Moreover, if $q_1,q_2\in C^j([0,\infty))$, then $\widetilde{K}\in C^{1+j}(\overline{D})$ for $j\in\mathbf{Z}_+$.*

In any case we have the estimate

$$\begin{aligned} |\widetilde{K}(x,t)|\le &\int_0^{(x+t)/2}|q_2(y)-q_1(y)|\,dy\\ &\times\exp\left(\int_0^{(x-t)/2}\int_s^{(x+t)/2}|q_2(r+s)-q_1(r-s)|\,dr\,ds\right) \end{aligned} \tag{8}$$

for $0 \le t \le x$.

The solution operator $(q_1, q_2) \mapsto \widetilde{K}$ *is a continuous map in the sense, that if* $(q_1^{(n)}, q_2^{(n)}) \to (q_1, q_2)$, *as* $n \to \infty$, *in* $L^1([0,\infty)) \times L^1([0,\infty))$ *then* $\widetilde{K}^{(n)}(x,t) \to \widetilde{K}(x,t)$, *as* $n \to \infty$, *for* $(x,t) \in D$, *uniformly on compact subsets.*

Proof. The idea is to change coordinates and then formulate the problem as a Volterra integral equation of the second kind. This equation is then solved by iteration.

The change of variables $x = \xi + \eta$, $t = \xi - \eta$, defines the function

$$k(\xi, \eta) = \widetilde{K}(\xi + \eta, \xi - \eta), \quad 0 \le \eta \le \xi,$$

which solves

$$\begin{aligned} \frac{\partial^2 k}{\partial \xi \partial \eta}(\xi, \eta) - a(\xi + \eta, \xi - \eta) k(\xi, \eta) = 0, \qquad & 0 < \eta < \xi, \\ \frac{d}{d\xi} k(\xi, 0) = f(\xi), \quad & \xi \ge 0, \\ k(\xi, \xi) = 0, \qquad & \xi \ge 0, \end{aligned}$$

where $a(x,t) = q_2(x) - q_1(t)$ and $f(x) = \frac{1}{2}(q_2(x) - q_1(x))$. Integration with respect to η over the interval $[0, \eta]$ and then integration with respect to ξ over the interval $[\eta, \xi]$ yields the Volterra equation

$$k(\xi, \eta) = \int_\eta^\xi \int_0^\eta a(\xi' + \eta', \xi' - \eta') k(\xi', \eta') \, d\eta' \, d\xi' + \int_\eta^\xi f(u) \, du. \tag{9}$$

If we define the operator A on $C(D)$ by

$$Ak(\xi, \eta) = \int_\eta^\xi \int_0^\eta a(\xi' + \eta', \xi' - \eta') k(\xi', \eta') \, d\eta' \, d\xi',$$

then the equation (9) has the form

$$(I - A) k(\xi, \eta) = \int_\eta^\xi f(u) \, du = F(\xi, \eta). \tag{10}$$

Since for $c \in C(D)$ the inequality

$$|A^n c(\xi, \eta)| \le \Big(\sup_{0 \le \eta' \le \xi' \le \xi} |c(\xi', \eta')| \Big) \frac{1}{n!} \Big(\int_0^\eta \int_s^\xi |a(r+s, r-s)| \, dr \, ds \Big)^n \tag{11}$$

can be established by induction, the operator $(I-A)$ can be inverted by a convergent Neumann series. The unique solution k is thus obtained from (10). Moreover, the convergent Neumann series yields the estimate

$$|k(\xi,\eta)| \le 2\Big(\sup_{0\le\eta'\le\xi'\le\xi} |F(\xi',\eta')|\Big) \exp\Big(\int_0^\eta\int_s^\xi |a(r+s,r-s)|\,dr\,ds\Big), \tag{12}$$

from which (8) follows.

The regularity of $k(\xi,\eta)$ and of $\widetilde{K}(x,y)$ is obtained from the integral equation (9).

Since both the right- and left-hand side of (10) depend continuously on $q\in L^1([0,\infty))$, the solution operator is continuous in the specified sense. □

Next we study the special case of the transformation kernel L. In this case the partial differential equation is given by

$$\begin{aligned} L_{xx}(x,t)-L_{tt}(x,t)+q(t)L(x,t) &= 0, \qquad (x,t)\in D,\\ 2\frac{d}{dx}L(x,x) &= q(x), \quad x\ge 0,\\ L(x,0) &= 0, \qquad x\ge 0. \end{aligned} \tag{13}$$

In the following lemma we exploit (8).

Lemma 2.2. *Let $q\in L^1([0,\infty))$. Then*

$$|L(x,t)| \le \|q\|_{L^1}\exp(x\|q\|_{L^1}), \quad 0\le t\le x. \tag{14}$$

Moreover, $2L_t(2x,0)-q(x)$ is continuous and estimated by

$$|2L_t(2x,0)-q(x)| \le \|q\|_{L^1}^2\exp(x\|q\|_{L^1}) \tag{15}$$

If $q\in C_0^1([0,\infty))$, then

$$|L_t(x,t)| \le Ce^{x\|q\|_{L^1}}, \quad 0\le t\le x, \tag{16}$$

$$|L_{tt}(x,t)| \le Ce^{x\|q\|_{L^1}}, \quad 0\le t\le x, \tag{17}$$

where the constant C may depend on q.

Proof. The problem (13) is identical to (7) with $q_1=q$, $q_2=0$ and $\widetilde{K}=-L$.

Changing variables defines the function $l(\xi,\eta)=L(\xi+\eta,\xi-\eta)$ which because of (9) solves the equation

$$l(\xi,\eta) = -\int_\eta^\xi\int_0^\eta q(\xi'-\eta')l(\xi',\eta')\,d\eta'\,d\xi' + \frac{1}{2}\int_\eta^\xi q(u)\,du \tag{18}$$

and because of (8) is estimated by

$$|l(\xi,\eta)| \le 2\int_0^\xi |q(u)|\,du \exp\left(\int_0^\eta\int_s^\xi |q(r-s)|\,dr\,ds\right). \tag{19}$$

Since for $0\le\eta\le\xi$,

$$\int_0^\eta\int_s^\xi |q(r-s)|\,dr\,ds = \int_0^\eta\int_0^{\xi-s} |q(u)|\,du\,ds = \int_{\xi-\eta}^\xi \int_0^v |q(u)|\,du\,dv \le \eta\int_0^\xi |q(u)|\,du,$$

the estimate (19) yields

$$|l(\xi,\eta)| \le \int_0^\xi |q(u)|\,du \exp\left(\eta\int_0^\xi |q(u)|\,du\right),$$

which implies (14) for $q\in L^1([0,\infty))$.

Differentiating (18) yields a.e. that

$$l_\xi(\xi,\eta) = -\int_0^\eta q(\xi-\eta')l(\xi,\eta')\,d\eta' + \frac{1}{2}q(\xi) = -\int_{\xi-\eta}^\xi q(u)l(\xi,\xi-u)\,du + \frac{1}{2}q(\xi), \tag{20}$$

$$\begin{aligned} l_\eta(\xi,\eta) &= \int_0^\eta q(\eta-\eta')l(\eta,\eta')\,d\eta' - \int_\eta^\xi q(\xi'-\eta)l(\xi',\eta)\,d\xi' - \frac{1}{2}q(\eta) \\ &= \int_0^\eta q(u)l(\eta,\eta-u)\,du - \int_0^{\xi-\eta} q(u)l(\eta+u,\eta)\,du - \frac{1}{2}q(\eta). \end{aligned} \tag{21}$$

Hence $l_\xi(\xi,\eta)-\frac{1}{2}q(\xi)$ and $l_\eta(\xi,\eta)+\frac{1}{2}q(\eta)$ are continuous functions. Since

$$L_t(x,t) = \frac{1}{2}\left(l_\xi\left(\frac{x+t}{2},\frac{x-t}{2}\right) - l_\eta\left(\frac{x+t}{2},\frac{x-t}{2}\right)\right),$$

we find

$$2L_t(2x,0)-q(x) = -2\int_0^x q(u)l(x,x-u)\,du = -\int_0^{2x} q\left(\tfrac{1}{2}v\right)l\left(x,x-\tfrac{1}{2}v\right)dv,$$

from which the continuity and (15) follow.

When $q\in C_0^1([0,\infty))$ we also deduce

$$|l_\xi(\xi,\eta)| \le C\exp\left(\eta\int_0^\xi |q(u)|\,du\right),$$

$$|l_\eta(\xi,\eta)| \le C\exp\left(\eta\int_0^\xi |q(u)|\,du\right)$$

from which (16) follows. The estimate (17) follows similarly by differentiating (20) and (21) once again, since

$$L_{tt}(x,t)=\frac{1}{4}\left(l_{\xi\xi}\left(\frac{x+t}{2},\frac{x-t}{2}\right)+l_{\eta\eta}\left(\frac{x+t}{2},\frac{x-t}{2}\right)-2l_{\xi\eta}\left(\frac{x+t}{2},\frac{x-t}{2}\right)\right)$$

and since q and q' are compactly supported. □

We now give a result about the Cauchy problem for the partial differential equation in (7).

Lemma 2.3. *Let* $\Delta_b=\{(x,t)\in\mathbf{R}^2\,|\,0<t<x,\ t+x<b\}$ *for* $b\in[0,\infty)$. *Let* $q_1,q_2\in L^1([0,b])$, $f\in C([0,b])$, $g\in L^1([0,b])$. *Then*

$$\begin{aligned}\widetilde{K}_{xx}(x,t)-\widetilde{K}_{tt}(x,t)+(q_1(t)-q_2(x))\widetilde{K}(x,t)=0,\qquad &(x,t)\in\Delta_b,\\ \widetilde{K}(x,0)=f(x),\quad &x\in[0,b],\\ \widetilde{K}_t(x,0)=g(x),\quad &x\in[0,b],\end{aligned}$$

has a unique solution $\widetilde{K}\in C(\bar{\Delta}_b)$.

Proof. The proof follows the same lines as the proof of Lemma 2.1 (see [K] for more details). □

3. Relation between the m-function and a transformation kernel

In this section we prove a relation between the Weyl solution and the transformation kernel L. This result leads to the connection between L and the m-function given by

$$m(-k^2;q)=-k-\int_0^\infty L_t(x,0)e^{-xk}\,dx. \tag{22}$$

Note that (22) can be seen as a combination of [S, equation (2.3)] and [GS2, equation (9.11)]. We give a direct proof below.

We now establish a relation between u and L when $q\in C_0^1([0,\infty))$.

Lemma 3.1. *Let* $q\in C_0^1([0,\infty))$ *and let* L *be the transformation kernel* (5). *Then for* $k>\|q\|_{L^1}$,

$$u(t,-k^2)=e^{-tk}-\int_t^\infty L(x,t)e^{-xk}\,dx \tag{23}$$

is the Weyl solution to (1).

Proof. It will be shown that $u \in L^2([0,\infty))$ and that u solves (1) as well as $u(0,-k^2)=1$.

According to (14), $|L(x,t)| \le \|q\|_{L^1} \exp(x\|q\|_{L^1})$, which implies that $u(t,-k^2)$ is well defined by (23) for $k>\|q\|_{L^1}$.

Since $e^{-tk} \in L^2([0,\infty))$ and

$$\left| \int_t^\infty L(x,t)e^{-xk}\,dx \right| \le \frac{1}{k-\|q\|_{L^1}} e^{-t(k-\|q\|_{L^1})} \in L^2([0,\infty)),$$

it follows that $u(t,-k^2) \in L^2([0,\infty))$.

By Lemma 2.1 the assumption $q \in C_0^1([0,\infty))$ ensures that $L \in C^2(D)$. Differentiating (23) then yields

$$\begin{aligned} u'(t,-k^2) &= -ke^{-tk} + L(t,t)e^{-tk} - \int_t^\infty L_t(x,t)e^{-xk}\,dx, \\ u''(t,-k^2) &= k^2e^{-tk} + \frac{d}{dt}L(t,t)e^{-tk} - kL(t,t)e^{-tk} + L_t(t,t)e^{-tk} \\ &\quad - \int_t^\infty L_{tt}(x,t)e^{-xk}\,dx, \end{aligned} \tag{24}$$

since the estimates (16) and (17) justify differentiating under the integration sign.

Since L solves (13), we have

$$-\int_t^\infty L_{tt}(x,t)e^{-xk}\,dx = -\int_t^\infty (L_{xx}(x,t)+q(t)L(x,t))e^{-xk}\,dx, \tag{25}$$

and integration by parts gives

$$\begin{aligned} -\int_t^\infty L_{xx}(x,t)e^{-xk}\,dx &= L_x(t,t)e^{-xk} - k\int_t^\infty L_x(x,t)e^{-xk}\,dx \\ &= L_x(t,t)e^{-xk} - kL(t,t)e^{-xk} - k^2\int_t^\infty L(x,t)e^{-xk}\,dx. \end{aligned} \tag{26}$$

Inserting (25) and (26) in (24) gives

$$\begin{aligned} u''(t,-k^2) &= \left(k^2 + \frac{d}{dt}L(t,t) + L_t(t,t) + L_x(t,t)\right)e^{-xk} - (k^2+q(t))\int_t^\infty L(x,t)e^{-xk}\,dx \\ &= (k^2+q(t))u(t,-k^2), \end{aligned}$$

which is (1).

Moreover, $u(0,-k^2)=1$ since $L(x,0)=0$. □

The above result is the main ingredient in the proof of the relation (22), but to obtain the result for general $q \in L^1([0,\infty))$ we need the following continuity result.

Lemma 3.2. *Let $q \in L^1([0,\infty))$ and suppose $q_n \in L^1([0,\infty))$, $\|q-q_n\|_{L^1} \to 0$, as $n \to \infty$. Then $m(k^2;q_n) \to m(k^2;q)$, as $n \to \infty$, pointwise for every $k \in \mathbf{C}$ with $\operatorname{Im} k$ sufficiently large.*

Proof. For $k \in \mathbf{C}$ with $\operatorname{Im} k$ sufficiently large, the Weyl solution is given by

$$u(x,k^2) = \frac{f(x,k)}{F(k)},$$

where f is the Jost solution to (1) and F is the Jost function (cf. [CS]). The result follows since the map $q \mapsto f(x,k)$ is continuous on $L^1([0,\infty))$ for fixed (x,k). $\square$

We are now able to prove (22).

Lemma 3.3. *For $k > \|q\|_{L^1}$ the equation (22) is valid.*

Proof. Assume $q \in C_0^1([0,\infty))$. From (23) we have

$$u(t,-k^2) = e^{-tk} + \int_t^\infty L(x,t) e^{-xk}\, dx$$

which because of the estimate (15) gives

$$u'(t,-k^2) = -ke^{-tk} - L(t,t)e^{-tk} + \int_t^\infty L_t(x,t) e^{-xk}\, dx. \tag{27}$$

The result then follows by inserting $t=0$ in (27) since $L(0,0)=0$ and m is defined by (3).

For general $q \in L^1([0,\infty))$ let $(q_n)_{n \in \mathbf{Z}_+}$ be a sequence in $C_0^1([0,\infty))$ with $\|q_n\|_1 \le C < k$ such that $\lim_{n\to\infty} \|q-q_n\|_1 = 0$. Because of (14)

$$\int_0^\infty L_t(x,0;q_n) e^{-xk}\, dx \to \int_0^\infty L_t(x,0;q) e^{-xk}\, dx, \quad \text{as } n \to \infty,$$

by dominated convergence. The result then follows from Lemma 3.2. $\square$

4. Connection between different transformation kernels

In this section we give a result connecting the transformation kernels L_1 and L_2 associated with two Sturm–Liouville problems and the relative transformation kernel $\widetilde{K}$.

Lemma 4.1. *Assume $q_1, q_2 \in L^1([0,\infty))$. Let L_i be the transformation kernel given by (5) associated with the problem*

$$-u''(x)+q_i(x)u(x)=\lambda u(x),$$
$$u(0)=0,$$

$i=1,2$, and let $\widetilde{K}$ be the relative transformation kernel given by (4). If $(L_1)_t(x,0)=(L_2)_t(x,0)$ in $L^1([0,a])$ for some $a>0$, then $\widetilde{K}_t(x,0)=0$ in $L^1([0,a])$.

Proof. The kernels L_1, L_2 and $\widetilde{K}$ satisfy (5) and (4), respectively, that is

$$\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{\lambda}}=\phi_i(x,\lambda)-\int_0^x L_i(x,t)\phi_i(t,\lambda)\,dt, \quad i=1,2, \tag{28}$$

$$\phi_2(x,\lambda)=\phi_1(x,\lambda)+\int_0^x \widetilde{K}(x,t)\phi_1(t,\lambda)\,dt. \tag{29}$$

Denote by K_2 the kernel associated with q_2 given by (6), that is

$$\phi_2(x,\lambda)=\frac{\sin(\sqrt{\lambda}\,x)}{\sqrt{\lambda}}+\int_0^x K_2(x,t)\frac{\sin(\sqrt{\lambda}\,t)}{\sqrt{\lambda}}\,dt. \tag{30}$$

Combining (28) for $i=1$ with (30) and interchanging the order of integration yields

$$\begin{aligned}\phi_2(x,\lambda)&=\phi_1(x,\lambda)-\int_0^x L_1(x,t)\phi_1(t,\lambda)\,dt\\&\quad+\int_0^x K_2(x,t)\left(\phi_1(t,\lambda)-\int_0^t L_1(t,s)\phi_1(s,\lambda)\,ds\right)dt\\&=\phi_1(x,\lambda)+\int_0^x\left(K_2(x,t)-L_1(x,t)-\int_t^x K_2(x,s)L_1(s,t)\,ds\right)\phi_1(t,\lambda)\,dt.\end{aligned}$$

Since the kernel $\widetilde{K}$ is unique we find by (29) that

$$\widetilde{K}(x,t)=K_2(x,t)-L_1(x,t)-\int_t^x K_2(x,s)L_1(s,t)\,ds.$$

Hence

$$\widetilde{K}_t(x,t)=(K_2)_t(x,t)-(L_1)_t(x,t)+K_2(x,t)L_1(t,t)-\int_t^x K_2(x,s)(L_1)_t(s,t)\,ds$$

for almost all $(x,t)\in D$, and since $(L_1)_t(x,0)=(L_2)_t(x,0)$ in $L^1([0,a])$ and $L_1(0,0)=0$, we get

$$\widetilde{K}_t(x,0)=(K_2)_t(x,0)-(L_2)_t(x,0)-\int_0^x K_2(x,s)(L_2)_t(s,0)\,ds \tag{31}$$

for almost all $x\in[0,a]$.

On the other hand combining (28) for $i=2$ with (30) yields

$$\begin{aligned}\phi_2(x,\lambda)=\phi_2(x,\lambda)-&\int_0^x L_2(x,t)\phi_2(t,\lambda)\,dt\\ &+\int_0^x K_2(x,t)\left(\phi_2(t,\lambda)-\int_0^t L_2(t,s)\phi_2(s,\lambda)\,ds\right)dt,\end{aligned}$$

and interchanging the order of integration gives

$$\int_0^x\left(K_2(x,t)-L_2(x,t)-\int_t^x K_2(x,s)L_2(s,t)\,ds\right)\phi_2(t,\lambda)\,dt=0.$$

Using the fact that the generalized Fourier transform is unitary yields

$$K_2(x,t)-L_2(x,t)-\int_t^x K_2(x,s)L_2(s,t)\,ds=0,\quad 0\le t\le x.$$

The result is now obtained by combining this equation with (31). □

5. The uniqueness theorem

The last ingredient before we give the new proof of Theorem 1.1 is an inversion result for the Laplace transform.

Lemma 5.1. ([S]) *Let* $f\in L^1([0,a])$ *and assume that* $g(z)=\int_0^a f(y)e^{-zy}\,dy$ *satisfies the relation*

$$g(x)=o(e^{-ax(1-\varepsilon)}),\quad \textit{as } x\to\infty.$$

for all $\varepsilon>0$. *Then* $f\equiv 0$.

We are now able to prove the main theorem.

Proof of Theorem 1.1. By Lemma 3.3

$$\begin{aligned}m(-k^2;q_1)-m(-k^2;q_2)&=\int_0^\infty((L_1)_t(x,0)-(L_2)_t(x,0))e^{-xk}\,dx\\ &=o(e^{-ak(1-\varepsilon)}),\quad \text{as } k\to\infty,\end{aligned}\tag{32}$$

for all $\varepsilon>0$.

Since $q\in L^1([0,\infty))$ and $\left|2(L_i)_t(x,0)-q_i\left(\frac{1}{2}x\right)\right|\le c_1e^{c_2x}$ by (15), we have for $k>c_2$ that

$$\int_a^\infty |(L_1)_t(x,0)-(L_2)_t(x,0)|e^{-xk}\,dx=o(e^{-ak(1-\varepsilon)}),\quad \text{as } k\to\infty,$$

for all $\varepsilon>0$. Hence (32) gives

$$\int_0^a ((L_1)_t(x,0)-(L_2)_t(x,0))e^{-xk}\,dx=o(e^{-ak(1-\varepsilon)}),\quad \text{as } k\to\infty,$$

for all $\varepsilon>0$. By Lemma 5.1 we get

$$(L_1)_t(x,0)-(L_2)_t(x,0)=0\quad \text{for a.e. } x\in[0,a].$$

Lemma 4.1 now yields, that the relative transformation kernel $\widetilde{K}$ satisfies $\widetilde{K}_t(x,0)=0$ for a.e. $x\in[0,a]$. Since $\widetilde{K}$ is the unique solution to (7), the function $\widetilde{K}$, in particular, solves

$$\begin{aligned}\widetilde{K}_{xx}(x,t)-\widetilde{K}_{tt}(x,t)-(q_1(x)-q_2(t))\widetilde{K}(x,t)&=0, && (x,t)\in D,\\ \widetilde{K}(x,0)&=0, && x\in[0,a],\\ \widetilde{K}_t(x,0)&=0, && x\in[0,a].\end{aligned}$$

Since $\Delta_a\subset D$ this problem has according to Lemma 2.3 a unique solution $\widetilde{K}\in C(\bar{\Delta}_a)$. Hence $\widetilde{K}(x,t)\equiv 0$, $(x,t)\in\Delta_a$. Moreover, since $\widetilde{K}$ has a first order derivative almost everywhere and

$$0=\frac{d}{dx}\widetilde{K}(x,x)=q_1(x)-q_2(x)\quad \text{for a.e. } x\in\left[0,\tfrac{1}{2}a\right],$$

we have the result. □

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