Ark. Mat., **40** (2002), 27–45

Maximal plurisubharmonic functions and the polynomial hull of a completely circled fibration

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Abstract. Let $X\subseteq\partial\mathbf{B}^m\times\mathbf{C}^n$ be a compact set over the unit sphere $\partial\mathbf{B}^m$ such that for each $z\in\partial\mathbf{B}^m$ the fiber $X_z=\{w\in\mathbf{C}^n;(z,w)\in X\}$ is the closure of a completely circled pseudoconvex domain in $\mathbf{C}^n$. The polynomial hull $\widehat{X}$ of X is described in terms of the Perron–Bremermann function for the homogeneous defining function of X. Moreover, for each point $(z_0,w_0)\in\operatorname{Int}\widehat{X}$ there exists a smooth up to the boundary analytic disc $F:\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ with the boundary in X such that $F(0)=(z_0,w_0)$.

1. Introduction

Let m and n be positive integers. Let $\mathbf{B}^m=\{z\in\mathbf{C}^m;|z|<1\}$ be the open unit ball in $\mathbf{C}^m$ and let $\partial\mathbf{B}^m$ denote its boundary. Let $\varrho:\partial\mathbf{B}^m\times\mathbf{C}^n\to[0,\infty)$ be a nonnegative continuous function such that for each $z\in\partial\mathbf{B}^m$ the function $\varrho(z,\cdot):\mathbf{C}^n\to[0,\infty)$ is a homogeneous plurisubharmonic function on $\mathbf{C}^n$ with the only zero at the point $w=0$. We say that a function $u:\mathbf{C}^n\to[0,\infty)$ is *homogeneous* if $u(\lambda w)=|\lambda|u(w)$ for all $w\in\mathbf{C}^n$ and $\lambda\in\mathbf{C}$.

Let $X=\{(z,w)\in\partial\mathbf{B}^m\times\mathbf{C}^n;\varrho(z,w)\le1\}$. Then X is a compact subset of $\partial\mathbf{B}^m\times\mathbf{C}^n$ such that for each $z\in\partial\mathbf{B}^m$ the fiber $X_z=\{w\in\mathbf{C}^n;(z,w)\in X\}$ is the closure of a completely circled pseudoconvex domain $\Omega_z=\{w\in\mathbf{C}^n;\varrho(z,w)<1\}$ in $\mathbf{C}^n$.

The main result of the paper is the following theorem.

Theorem 1.1. *The polynomial hull $\widehat{X}$ of X is*

$$\widehat{X}=\{(z,w)\in\overline{\mathbf{B}}^m\times\mathbf{C}^n\,;\,\Psi_\varrho(z,w)\le1\},$$

(1) This work was supported in part by a grant from the Ministry of Science of the Republic of Slovenia.

where $\Psi_\varrho: \overline{\mathbf{B}}^m \times \mathbf{C}^n \to [0,\infty)$ *is the Perron–Bremermann function for* ϱ, *that is,* Ψ_ϱ *is the largest plurisubharmonic function on* $\mathbf{B}^m \times \mathbf{C}^n$ *whose boundary values are below* ϱ. *Moreover, for each point* $(z_0, w_0) \in \operatorname{Int} \widehat{X}$ *there exists a smooth up to the boundary analytic disc* $F: \Delta \to \mathbf{B}^m \times \mathbf{C}^n$ *with the boundary in* X *such that* $F(0) = (z_0, w_0)$.

Recall that the polynomial hull $\widehat{K}$ of a compact set $K \subseteq \mathbf{C}^n$ is defined as

$$\widehat{K} = \Big\{ z \in \mathbf{C}^n\,;\, |p(z)| \le \max_K |p| \text{ for every polynomial } p \text{ in } n \text{ variables} \Big\}$$

and that by the maximum principle the image $F(\Delta)$ of every H^∞ holomorphic mapping $F: \Delta \to \mathbf{C}^n$ with the boundary in K, that is, $F^*(e^{i\theta}) \in K$ for almost every θ, belongs to the polynomial hull $\widehat{K}$ of K.

The question of the description of the polynomial hull of a compact fibration X over the unit circle $\partial\Delta$ with analytic discs whose boundaries lie in X was considered in a series of papers [2], [9], [16], [17], [18], and quite recently by Whittlesey in [21], [22] and [23] (see also [6] and [7]). In the case $n=1$ the most general result was obtained by Slodkowski [17], where it was only assumed that each fiber is a simply connected continuum. In the case of higher dimensional fibers, results were obtained for convex fibers ([2], [16], [18]) and for the fibers which are smooth and strictly hypoconvex (lineally convex) ([22], [23]).

For higher dimensional base ($m>1$) and $n=1$ it is a classical result, [10, p. 99], that the polynomial hull of the set X, whose fibers are discs centered at the origin, is given by the Perron–Bremermann function for ϱ. Related results on the presence of analytic discs and even analytic balls in the hull of the set with the disc fibers are proved in [8] and [20]. Also, it was shown by an example in [8], that one can not, in general, expect to get a foliation of the whole $\widehat{X}$ with analytic discs even in such simple cases. Finally we remark that it was shown by H. Alexander [1] that in the case $m>1$ the polynomial hull of the graph of every continuous function φ on $\partial\mathbf{B}^m$ is nontrivial and it covers the whole $\mathbf{B}^m$.

Acknowledgement. The author is grateful to Z. Balogh and C. Leuenberger for very stimulating discussions.

2. Maximal plurisubharmonic functions

First we introduce some notation. Let D be an open subset of some complex space $\mathbf{C}^k$. By $\mathcal{PSH}(D)$ we will denote the set of all plurisubharmonic functions on D which are locally bounded from above near each point of $\overline{D}$. Also, for a function

$u: D \to [-\infty, \infty)$ which is locally bounded from above near each point of $\bar{D}$ we will denote by $u^\star: \bar{D} \to [-\infty, \infty)$ its upper semicontinuous regularization.

Let $\varrho: \partial\mathbf{B}^m \times \mathbf{C}^n \to [0, \infty)$ be a nonnegative continuous function and let $\mathcal{U}(\varrho)$ be the set of all plurisubharmonic functions on $\mathbf{B}^m \times \mathbf{C}^n$ whose boundary values are below ϱ:

$$\mathcal{U}(\varrho) = \{u\,;\, u \in \mathcal{PSH}(\mathbf{B}^m \times \mathbf{C}^n),\ u^\star(z,w) \le \varrho(z,w) \text{ on } \partial\mathbf{B}^m \times \mathbf{C}^n\}. \tag{2.1}$$

Since ϱ is a nonnegative function, the family $\mathcal{U}(\varrho)$ contains the function $u(z,w) \equiv 0$ and is thus nonempty. The *Perron–Bremermann function* $\Psi_\varrho: \mathbf{B}^m \times \mathbf{C}^n \to [0, \infty)$ for the function ϱ, [12, p. 89], is defined as

$$\Psi_\varrho(z,w) := \sup\{u(z,w)\,;\, u \in \mathcal{U}(\varrho)\}. \tag{2.2}$$

Let $H_\varrho: \bar{\mathbf{B}}^m \times \mathbf{C}^n \to [0, \infty)$ denote the function which for each fixed $w_0 \in \mathbf{C}^n$ is defined as the harmonic extension of the function $\varrho(\cdot, w_0): \partial\mathbf{B}^m \to [0, \infty)$ to $\mathbf{B}^m$. The function H_ϱ can be explicitly given as the Poisson integral

$$H_\varrho(z,w) = \frac{1}{\omega_{2m}} \int_{|\zeta|=1} \frac{1-|z|^2}{|\zeta - z|^{2m}} \varrho(\zeta, w)\, dS_\zeta, \tag{2.3}$$

where ω_{2m} is the measure of the unit sphere in $\mathbf{C}^m$. Obviously H_ϱ is a continuous function on $\bar{\mathbf{B}}^m \times \mathbf{C}^n$.

Since the restriction of a plurisubharmonic function to any complex subspace is also subharmonic, the values of the harmonic extension $H_\varrho(\cdot, w_0)$ have to be above the values $u(z, w_0)$ for every plurisubharmonic function $u \in \mathcal{U}(\varrho)$ and every fixed $w_0 \in \mathbf{C}^n$. Hence $\Psi_\varrho \le H_\varrho$ on $\mathbf{B}^m \times \mathbf{C}^n$ and then, by the continuity of the function H_ϱ, we also have $\Psi_\varrho^\star \le H_\varrho$ on $\bar{\mathbf{B}}^m \times \mathbf{C}^n$. The supremum of an arbitrary family of plurisubharmonic functions is not necessarily a plurisubharmonic function, but if it is locally bounded from above, then its upper semicontinuous regularization is plurisubharmonic, [12, p. 69]. Therefore $\Psi_\varrho^\star \in \mathcal{U}(\varrho)$. We conclude that $\Psi_\varrho = \Psi_\varrho^\star$ and hence $\Psi_\varrho \in \mathcal{U}(\varrho)$.

Proposition 2.1. *Let $\varrho: \partial\mathbf{B}^m \times \mathbf{C}^n \to [0, \infty)$ be a nonnegative continuous function such that for each $z \in \partial\mathbf{B}^m$ the function $\varrho(z, \cdot): \mathbf{C}^n \to [0, \infty)$ is a homogeneous plurisubharmonic function on $\mathbf{C}^n$ with the only zero at the point $w=0$.*

Then the Perron–Bremermann function Ψ_ϱ for the function ϱ is a nonnegative continuous function on $\bar{\mathbf{B}}^m \times \mathbf{C}^n$ such that

(1) *$\Psi_\varrho(z,w) = \varrho(z,w)$ for every $(z,w) \in \partial\mathbf{B}^m \times \mathbf{C}^n$;*

(2) *Ψ_ϱ is homogeneous in the w variable: $\Psi_\varrho(z, \lambda w) = |\lambda| \Psi_\varrho(z,w)$ for all $(z,w,\lambda) \in \mathbf{B}^m \times \mathbf{C}^n \times \mathbf{C}$;*

(3) *Ψ_ϱ is a maximal plurisubharmonic function on $\mathbf{B}^m \times \mathbf{C}^n$ for which $\Psi_\varrho(z,w) = 0$ if and only if $w = 0$.*

Proof. Clearly the function Ψ_ϱ is a maximal plurisubharmonic function on $\mathbf{B}^m\times\mathbf{C}^n$. Namely, if $D\subseteq\mathbf{B}^m\times\mathbf{C}^n$ is a relatively compact open set and $u:\overline{D}\to[-\infty,\infty)$ an upper semicontinuous function which is plurisubharmonic on D and such that $u\le\Psi_\varrho$ on ∂D, then the function

$$U(z,w):=\begin{cases}\max\{\Psi_\varrho(z,w),u(z,w)\}, & (z,w)\in D,\\ \Psi_\varrho(z,w), & (z,w)\notin D,\end{cases}$$

is in $\mathcal{U}(\varrho)$. Thus $U\le\Psi_\varrho$ on $\mathbf{B}^m\times\mathbf{C}^n$ and so also $u\le\Psi_\varrho$ on D.

The homogeneity of the function Ψ_ϱ follows immediately from the homogeneity of the function ϱ. Also, since ϱ is continuous and nonzero on $\partial\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ it follows that for small enough $m>0$, plurisubharmonic functions of the form $(z,w)\mapsto m|w|$ are in $\mathcal{U}(\varrho)$ and hence $\Psi_\varrho(z,w)=0$ if and only if $w=0$.

Now we will prove that Ψ_ϱ is continuous on $\partial\mathbf{B}^m\times\mathbf{C}^n$. Let $(z_0,w_0)\in\partial\mathbf{B}^m\times\mathbf{C}^n$ and let ε be a positive constant. Let $\lambda\in C(\partial\mathbf{B}^m)$ be a real continuous function on $\partial\mathbf{B}^m$ such that

(1) $\lambda(z_0)\ge-\varepsilon$;

(2) for every pair $(z,w)\in\partial\mathbf{B}^m\times\partial\mathbf{B}^n$ we have $\lambda(z)\le\log\varrho(z,w)-\log\varrho(z_0,w)$.

Such a function exists since the function $\sigma(z,w)=\log\varrho(z,w)-\log\varrho(z_0,w)$ is uniformly continuous on $\partial\mathbf{B}^m\times\partial\mathbf{B}^n$ and $\sigma(z_0,w)=0$.

Given $\lambda\in C(\partial\mathbf{B}^m)$, it is known, [12, p. 89], that there exists $\Lambda\in C(\overline{\mathbf{B}}^m)$ such that $\Lambda|_{\partial\mathbf{B}^m}=\lambda$ and $\Lambda|_{\mathbf{B}^m}$ is a maximal plurisubharmonic function on $\mathbf{B}^m$.

We consider the continuous function

$$u(z,w)=e^{\Lambda(z)}\varrho(z_0,w)$$

on $\mathbf{B}^m\times\mathbf{C}^n$. The assumptions on ϱ imply that $\log\varrho(z,w)$ is a plurisubharmonic function on $\mathbf{C}^n$ for each $z\in\partial\mathbf{B}^m$, [12, p. 84]. Therefore the function $\log u(z,w)=\Lambda(z)+\log\varrho(z_0,w)$ is plurisubharmonic on $\mathbf{B}^m\times\mathbf{C}^n$ and so $u\in\mathcal{PSH}(\mathbf{B}^m\times\mathbf{C}^n)$.

The conditions on the function Λ and the homogeneity of the function ϱ in the w variables imply that $u(z,w)\le\varrho(z,w)$ on $\partial\mathbf{B}^m\times\mathbf{C}^n$. Hence by the definition of the function Ψ_ϱ we have $u(z,w)\le\Psi_\varrho(z,w)$ on $\mathbf{B}^m\times\mathbf{C}^n$. Therefore

$$\begin{aligned}e^{-\varepsilon}\varrho(z_0,w_0)\le u(z_0,w_0)&\le\liminf_{(z,w)\to(z_0,w_0)}\Psi_\varrho(z,w)\\ &\le\limsup_{(z,w)\longrightarrow(z_0,w_0)}\Psi_\varrho(z,w)=\Psi^\star_\varrho(z_0,w_0)\le\varrho(z_0,w_0)\end{aligned}$$

and hence, letting $\varepsilon\searrow0$, we get that

$$\lim_{(z,w)\to(z_0,w_0)}\Psi_\varrho(z,w)=\Psi^\star_\varrho(z_0,w_0)=\varrho(z_0,w_0).$$

Thus the function Ψ_ϱ is continuous and equals ϱ at the points $(z,w)\in\partial\mathbf{B}^m\times\mathbf{C}^n$.

The continuity of Ψ_ϱ on $\mathbf{B}^m\times\mathbf{C}^n$ follows from an argument similar to the argument in the proof of Proposition 4 in [13] (see also [19]). Instead of the uniform continuity on the boundary, which we do not necessarily have, one uses the continuity of Ψ_ϱ on $\partial\mathbf{B}^m\times\mathbf{C}^n$ and its homogeneity in w variables to get that for every $\varepsilon>0$ there is a $\delta\in(0,\frac{1}{3})$ such that as soon as $|(z,w)-(z_0,w_0)|<3\delta$ for a pair of points $(z_0,w_0)\in\partial\mathbf{B}^m\times\mathbf{C}^n$ and $(z,w)\in\overline{\mathbf{B}}^m\times\mathbf{C}^n$, then

$$(1-\varepsilon)\Psi_\varrho(z,w)-\varepsilon\le\Psi_\varrho(z_0,w_0)\le(1+\varepsilon)\Psi_\varrho(z,w)+\varepsilon$$

and hence

$$\frac{1-\varepsilon}{1+\varepsilon}\Psi_\varrho(z',w')-\frac{2\varepsilon}{1+\varepsilon}\le\Psi_\varrho(z,w)$$

for any $(z,w),(z',w')\in\mathbf{B}^m\times\mathbf{C}^n$ with $\operatorname{dist}(z,\partial\mathbf{B}^m)<2\delta$ and $|(z',w')-(z,w)|<\delta$. $\square$

3. Polynomial hull and analytic discs

We are now prepared to formulate and prove our main results.

Theorem 3.1. *Let $\varrho:\partial\mathbf{B}^m\times\mathbf{C}^n\to[0,\infty)$ be as in Proposition 2.1 and let $X=\{(z,w)\in\partial\mathbf{B}^m\times\mathbf{C}^n;\varrho(z,w)\le1\}$. Then the polynomial hull $\widehat{X}$ of X is*

$$\widehat{X}=\{(z,w)\in\overline{\mathbf{B}}^m\times\mathbf{C}^n\,;\,\Psi_\varrho(z,w)\le1\},$$

where Ψ_ϱ is the Perron–Bremermann function for ϱ on $\mathbf{B}^m\times\mathbf{C}^n$.

Moreover, the polynomial hull $\widehat{X}$ contains a lot of analytic discs with boundaries in X.

Theorem 3.2. *For each point $(z_0,w_0)\in\operatorname{Int}\widehat{X}$ there exists a smooth up to the boundary analytic disc $F:\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ with the boundary in X such that $F(0)=(z_0,w_0)$.*

We will prove both theorems using Poletsky's characterization of the largest plurisubharmonic function below a given upper semicontinuous function ϕ on an open subset $D\subseteq\mathbf{C}^n$. It was proved in [14] that the function

$$u_\phi(z)=\inf_f\frac{1}{2\pi}\int_0^{2\pi}\phi(f^*(e^{i\theta}))\,d\theta,\qquad(3.1)$$

where the infimum is taken over all mappings $f:\bar{\Delta}\to D$ with $f(0)=z$ which are defined and holomorphic in some open neighbourhood V_f of the closed unit disc $\bar{\Delta}$, is a plurisubharmonic function on D and it equals the supremum of the plurisubharmonic functions v on D which are pointwise below ϕ. Moreover, it follows from results in [15, Lemma 8.3 and Theorem 8.1] that for a smoothly bounded strongly pseudoconvex domain $D\subseteq \mathbf{C}^n$ and a continuous function φ on ∂D the function

$$u_\varphi(z)=\inf_f \frac{1}{2\pi}\int_0^{2\pi}\varphi(f^*(e^{i\theta}))\,d\theta, \tag{3.2}$$

where the infimum is taken over all holomorphic mappings $f:\Delta\to D$ with $f(0)=z$ and whose boundary values satisfy $f^*(e^{i\theta})\in\partial D$ for almost every θ, is a continuous function on $\overline{D}$, a maximal plurisubharmonic function on D and is such that $u_\varphi|_{\partial D}=\varphi$. For a bounded holomorphic mapping F on Δ the notation F^* is used to denote its almost everywhere defined boundary values.

Remark 3.3. As already mentioned (3.2) follows from Lemma 8.3 and Theorem 8.1 in [15]. However, these two results are placed in a chain of other results in [15] as a part of a general theory of holomorphic currents developed by Poletsky and there is no explicit statement and proof of formula (3.2). To make our paper more self-contained a proof of (3.2) for the ball, which uses Poletsky's previous more direct result (3.1) from [14], is presented in the appendix.

Lemma 3.4. *Let ϱ be as in Proposition* 2.1. *Then the function*

$$\Phi_\varrho(z,w):=\inf_{(f,g)}\frac{1}{2\pi}\int_0^{2\pi}\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over all H^∞ holomorphic mappings of the unit disc $(f,g):\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ with $f(0)=z$ and $g(0)=w$ and whose boundary values satisfy $f^*(e^{i\theta})\in\partial\mathbf{B}^m$ almost everywhere on $\partial\Delta$, is a nonnegative upper semicontinuous function on $\mathbf{B}^m\times\mathbf{C}^n$ such that

(1) *$\Psi_\varrho(z,w)\le\Phi_\varrho(z,w)$ for every $(z,w)\in\mathbf{B}^m\times\mathbf{C}^n$;*

(2) *the function Φ_ϱ is locally bounded from above near each point of $\overline{\mathbf{B}}^m\times\mathbf{C}^n$ and $\Phi_\varrho^*=\varrho$ on $\partial\mathbf{B}^m\times\mathbf{C}^n$;*

(3) *the function Φ_ϱ is homogeneous in the w variable.*

Proof. The upper semicontinuity of the function Φ_ϱ follows directly from its definition with the help of the holomorphic automorphisms of the ball $\mathbf{B}^m$ and the fact that the function ϱ is continuous.

Recall that H_ϱ is the continuous function on $\overline{\mathbf{B}}^m\times\mathbf{C}^n$ which has the property that for each fixed $w_0\in\mathbf{C}^n$ the function $H_\varrho(z,w_0)$ solves the Dirichlet problem

$$\Delta u=0 \text{ on } \mathbf{B}^m \quad\text{and}\quad u|_{\partial\mathbf{B}^m}=\varrho(z,w_0).$$

We have already observed that $\Psi_\varrho \le H_\varrho$. On the other hand it is obvious from the submean value property that $\Psi_\varrho \le \Phi_\varrho$. We also compare the functions Φ_ϱ and H_ϱ. By the result of Poletsky $\Phi_\varrho(z,w_0)$ is pointwise below the maximal plurisubharmonic function $u_{w_0}(z)$ on $\mathbf{B}^m$ with $\varrho(z,w_0)$ as the boundary data. Thus $\Phi_\varrho(z,w_0)\le u_{w_0}(z)\le H_\varrho(z,w_0)$ and therefore

$$\Psi_\varrho \le \Phi_\varrho \le H_\varrho$$

on $\mathbf{B}^m \times \mathbf{C}^n$ (and hence also on $\overline{\mathbf{B}}^m \times \mathbf{C}^n$). This proves that Φ_ϱ can be continuously extended to the points $\partial\mathbf{B}^m \times \mathbf{C}^n$ and that $\Phi_\varrho^* = \varrho$ on $\partial\mathbf{B}^m \times \mathbf{C}^n$. The homogeneity of Φ_ϱ is clear. □

Proof of Theorem 3.1. Let $Y := \{(z,w)\in\overline{\mathbf{B}}^m \times \mathbf{C}^n ; \Psi_\varrho(z,w)\le 1\}$. We have to prove that $Y=\widehat{X}$. We also define the set $Z := \{(z,w)\in\overline{\mathbf{B}}^m \times \mathbf{C}^n ; \Phi_\varrho(z,w)\le 1\}$. The relation between the functions Ψ_ϱ and Φ_ϱ imply $Z\subseteq Y$.

First we will show that $Z\subseteq\widehat{X}\subseteq Y$. The inclusion $\widehat{X}\subseteq Y$ follows from the definition of the set Y with the plurisubharmonic function Ψ_ϱ and the fact, [12, p. 199, Corollary 5.3.5], that the polynomial hull $\widehat{X}$ of a compact set X in $\mathbf{C}^n$ equals the plurisubharmonic hull $\widehat{X}_{\mathcal{PSH}(D)}$ for any open neighbourhood D of $\widehat{X}$.

Let now (z_0,w_0) be a point from Z and let $\varepsilon>0$. If $w_0=0$, then it is obvious that $(z_0,w_0)\in\widehat{X}$. From now on we assume that $w_0\ne 0$ and so $\Phi_\varrho(z_0,w_0)\ne 0$.

By the definition of the function Φ_ϱ there exists an H^∞ analytic disc $(f,g)\colon\Delta\to \mathbf{B}^m\times\mathbf{C}^n$ $((f,g)(0)=(z_0,w_0))$ such that for its boundary values we have $f^*(e^{i\theta})\in \partial\mathbf{B}^m$ for almost every θ and such that

$$\Phi_\varrho(z_0,w_0)\le \frac{1}{2\pi}\int_0^{2\pi} \varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta \le \Phi_\varrho(z_0,w_0)+\varepsilon. \tag{3.3}$$

We let $\varphi(\xi)=\varrho(f^*(\xi),g^*(\xi))$, $\xi\in\partial\Delta$, and we observe the functional

$$p\longmapsto \frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^2\varphi(e^{i\theta})\,d\theta$$

over the space of holomorphic polynomials $p\in\mathcal{P}$ in one variable with $p(0)=1$. Recall a theorem of Szegő, [11, p. 144], which says that

$$\inf_{p\in\mathcal{P}} \frac{1}{2\pi}\int_0^{2\pi} |p(e^{i\theta})|^2\varphi(e^{i\theta})\,d\theta = \exp\biggl(\frac{1}{2\pi}\int_0^{2\pi} \log\varphi(e^{i\theta})\,d\theta\biggr). \tag{3.4}$$

Also, because $p(0)=1$ and the homogeneity of the function ϱ in w variable, we have

$$|p(e^{i\theta})|^2\varphi(e^{i\theta}) = \varrho(f^*(e^{i\theta}), p^2(e^{i\theta})g^*(e^{i\theta}))$$

and $(f,p^2g)(0)=(z_0,w_0)$ for every $p\in\mathcal{P}$. Hence by (3.3) we get
(3.5)

$$0<\Phi_\varrho(z_0,w_0)\le\exp\biggl(\frac{1}{2\pi}\int_0^{2\pi}\log\varphi(e^{i\theta})\,d\theta\biggr)\le\frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{i\theta})\,d\theta\le\Phi_\varrho(z_0,w_0)+\varepsilon.$$

Condition (3.5) implies that the function $\log\varphi$ is in $L^1(\partial\Delta)$ and hence there exists, [11, p. 103], a holomorphic function h on Δ which has nontangential limits almost everywhere on $\partial\Delta$ and is such that $\operatorname{Re}h^*=\log\varphi$ almost everywhere on $\partial\Delta$ and $\operatorname{Im}h(0)=0$. We define $F(\xi)=\Phi_\varrho(z_0,w_0)e^{-h(\xi)}$. Then

$$|F^*(\xi)|=\Phi_\varrho(z_0,w_0)e^{-\log\varphi(\xi)}=\frac{\Phi_\varrho(z_0,w_0)}{\varphi(\xi)}$$

almost everywhere on $\partial\Delta$. Also

$$F(0)=\Phi_\varrho(z_0,w_0)\exp\biggl(-\frac{1}{2\pi}\int_0^{2\pi}\log\varphi(e^{i\theta})\,d\theta\biggr)$$

and hence, using the inequalities (3.5), we get

$$1-\frac{\varepsilon}{\Phi_\varrho(z_0,w_0)}\le F(0)\le 1.$$

Since $|F^*(\xi)|\varphi(\xi)=\varrho(f^*(\xi),F^*(\xi)g^*(\xi))=\Phi_\varrho(z_0,w_0)\le 1$ and $|f^*(\xi)|=1$ for almost every $\xi\in\partial\Delta$, the analytic disc

$$\xi\longmapsto(f(\xi),F(\xi)g(\xi))$$

has the property that its boundary lies in X, that is, $(f^*,F^*g^*)(\xi)\in X$ for almost every $\xi\in\partial\Delta$. Also, the distance

$$|(z_0,w_0)-(f(0),F(0)g(0))|=|w_0-F(0)g(0)|\le|w_0|\frac{\varepsilon}{\Phi_\varrho(z_0,w_0)}$$

is arbitrarily small if only ε is chosen small enough. Since the polynomial hull of X is a closed subset of $\mathbf{C}^m\times\mathbf{C}^n$ and since an analytic disc with boundary in X belongs to $\widehat{X}$, we proved $(z_0,w_0)\in\widehat{X}$. Hence $Z\subseteq\widehat{X}$.

Finally we have to prove that $Y\subseteq\widehat{X}$. Let $(z_0,w_0)\in Y$. Since $\Psi_\varrho|_{\partial\mathbf{B}^m\times\mathbf{C}^n}=\varrho$, it is obvious that for any point $(z_0,w_0)\in Y$ such that $|z_0|=1$ we have $(z_0,w_0)\in X\subseteq\widehat{X}$. We assume from now on that $|z_0|<1$. Also, if $\Psi_\varrho(z_0,w_0)=0$, we know that $w_0=0$ and we obviously have $(z_0,0)\in\widehat{X}$. So from now on we also assume that $w_0\neq 0$ and hence $\Psi_\varrho(z_0,w_0)\neq 0$.

Let us define the function

$$\Psi^0(z,w)=\inf_{(f,g)}\frac{1}{2\pi}\int_0^{2\pi}\Phi_\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over all mappings $(f,g)\colon\bar\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ with $f(0)=z$ and $g(0)=w$ which are defined and holomorphic in some open neighbourhood of the closed unit disc $\bar\Delta$. By the result of Poletsky we have that Ψ^0 is plurisubharmonic on $\mathbf{B}^m\times\mathbf{C}^n$ and it equals the supremum of the plurisubharmonic functions on $\mathbf{B}^m\times\mathbf{C}^n$ which are pointwise below Φ_ϱ. Therefore $\Psi_\varrho\le\Psi^0\le\Phi_\varrho$. These inequalities together with Lemma 3.4 imply that the plurisubharmonic function Ψ^0 belongs to the space $\mathcal{U}(\varrho)$ and hence we must have $\Psi_\varrho=\Psi^0$.

Let $\varepsilon>0$. Then there exists a mapping $(f,g)\colon\bar\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ holomorphic on some open neighbourhood of Δ such that $(f,g)(0)=(z_0,w_0)$ and

$$\Psi_\varrho(z_0,w_0)\le\frac{1}{2\pi}\int_0^{2\pi}\Phi_\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta\le\Psi_\varrho(z_0,w_0)+\varepsilon. \tag{3.6}$$

Again using the theorem of Szegő and the homogeneity of the function Φ_ϱ we get that

$$0<\Psi_\varrho(z_0,w_0)\le\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\Phi_\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta\right)\le\Psi_\varrho(z_0,w_0)+\varepsilon. \tag{3.7}$$

A similar construction gives us a holomorphic function G on Δ such that

$$|G^*(\xi)|\Phi_\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))=\Phi_\varrho(f^*(\xi),G^*(\xi)g^*(\xi))=\Psi_\varrho(z_0,w_0)\le 1$$

and the distance

$$|(z_0,w_0)-(f(0),G(0)g(0))|=|w_0-G(0)g(0)|\le|w_0|\frac{\varepsilon}{\Psi_\varrho(z_0,w_0)}$$

is arbitrarily small if only ε is chosen small enough. Hence we have found an analytic disc $\xi\mapsto(f(\xi),G(\xi)g(\xi))$ with the property that its boundary lies in $Z\subseteq\widehat{X}$ and it passes arbitrarily close to the point (z_0,w_0). Hence $(z_0,w_0)\in\widehat{X}$. □

Before we prove Theorem 3.2 we state the following lemma whose proof is postponed and given in the appendix.

Lemma 3.5. *Let $0<a<1$ be a real number and let*

$$\Psi^a(z,w)=\inf_{(f,g)}\frac{1}{2\pi}\int_0^{2\pi}\Psi_\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over all smooth up to the boundary holomorphic mappings $(f,g)\colon\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ with $f(0)=z$, $g(0)=w$ and such that $a<|f^*(\xi)|<1$ for every $\xi\in\partial\Delta$. Then $\Psi^a=\Psi_\varrho$.

Proof of Theorem 3.2. Let $(z_0,w_0)\in\operatorname{Int}\widehat{X}$ and let $\varepsilon>0$ be so that $\Psi_\varrho(z_0,w_0)+\varepsilon<1$. By the continuity of the function Ψ_ϱ on $\overline{\mathbf{B}}^m\times\mathbf{C}^n$ there exists $\delta>0$ such that $|\Psi_\varrho(z,w)-\Psi_\varrho(\tilde z,\widetilde w)|<\varepsilon$ for any pair of points $(z,w)\in\overline{\mathbf{B}}^m\times\mathbf{C}^n$ and $(\tilde z,\widetilde w)\in\widehat{X}$ for which $|(z,w)-(\tilde z,\widetilde w)|<3\delta$.

The case $w_0=0$ is obvious. Let us now assume that $w_0\neq 0$. Let $a\in(|z_0|,1)$ be so close to 1 that $1/(1+\delta)<a$ and that for each $v\in(a,1)$ there exists a holomorphic automorphism A of the unit ball $\mathbf{B}^m$ which is δ uniformly on $\overline{\mathbf{B}}^m$ close to the identity map: $\|A-\mathrm{Id}\|<\delta$ and which takes $(1/v)z_0$ to z_0.

Using Lemma 3.5 and an argument similar to the argument in the proof of Theorem 3.1 we can show that there exists an H^∞ disc $F=(f,g)$ on Δ such that

(1) the mapping f is smooth up to the boundary $\partial\Delta$, $f(0)=z_0$ and $a<|f^*(\xi)|<1$ for every $\xi\in\partial\Delta$;

(2) $|w_0-g(0)|<\delta$;

(3) $\Psi_\varrho(F^*(\xi))=\Psi_\varrho(z_0,w_0)=t_0$ almost everywhere on $\partial\Delta$.

By Theorem 3.1 we know that the set $Y_{t_0}=\{(z,w)\in\overline{\mathbf{B}}^m\times\mathbf{C}^n;\Psi_\varrho(z,w)\le t_0\}$ is polynomially convex. Since F has the boundary in Y_{t_0}, we also have $F(\Delta)\subseteq Y_{t_0}\subseteq\widehat{X}$.

Let $v\in(a,\min_{\partial\Delta}|f|)$ be a regular value of the function $\xi\in\Delta\mapsto|f(\xi)|$ and let U_0 be the connected component of the set $\{\xi\in\Delta;|f(\xi)|<v\}$ which contains the point 0. Then U_0 is a smoothly bounded simply connected domain in $\mathbf{C}$ and so biholomorphic to Δ. Let A be a holomorphic automorphism of the unit ball $\mathbf{B}^m$ such that $\|A-\mathrm{Id}\|<\delta$ and $A((1/v)z_0)=z_0$.

We define

$$\widetilde{F}=\left(A\left(\frac{1}{v}f\right),g+(w_0-g(0))\right):U_0\longrightarrow\mathbf{B}^m\times\mathbf{C}^n.$$

Then obviously $\widetilde{F}(0)=(z_0,w_0)$ and $|A((1/v)f(\xi))|=1$ for every $\xi\in\partial U_0$. Also, since

$$|\widetilde{F}(\xi)-F(\xi)|<\|A-\mathrm{Id}\|+\left(\frac{1}{v}-1\right)+|w_0-g(0)|<3\delta$$

on $\overline{U}_0$, we get

$$|\Psi_\varrho(\widetilde{F}(\xi))-\Psi_\varrho(F(\xi))|<\varepsilon$$

and hence

$$\Psi_\varrho(\widetilde{F}(\xi))<\Psi_\varrho(F(\xi))+\varepsilon\le t_0+\varepsilon<1$$

for every $\xi\in\overline{U}_0$. Therefore the analytic disc $\widetilde{F}:U_0\to\mathbf{B}^m\times\mathbf{C}^n$ passes through the point (z_0,w_0) and it has the boundary contained in X. $\square$

For a bounded strongly pseudoconvex domain D in $\mathbf{C}^n$ the equality of the functions defined as Ψ_ϱ and Φ_ϱ was proved by Poletsky [15] and here the above proof shows the following result.

Corollary 3.6. *Under the assumptions of Proposition* 2.1 *the functions* Ψ_ϱ *and* Φ_ϱ *are equal, that is, for every point* $(z,w)\in\mathbf{B}^m\times\mathbf{C}^n$

$$\Psi_\varrho(z,w)=\inf_{(f,g)}\frac{1}{2\pi}\int_0^{2\pi}\varrho(f^*(e^{i\theta}),g^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over all H^∞ *holomorphic mappings* $(f,g)\colon\Delta\to\mathbf{B}^m\times\mathbf{C}^n$ *with* $f(0)=z$, $g(0)=w$ *and such that its boundary values* (f^*,g^*) *satisfy* $f^*(e^{i\theta})\in\partial\mathbf{B}^m$ *for almost every* θ.

The motivation for the next proposition comes from a result in [23] where the same conclusion was proved using nonelementary methods and under stronger assumptions. Also, we would like to show that the class of fibrations X over the unit circle considered in this paper and the class of fibrations considered in [22] and [23] are quite different.

Recall that a set $\Omega\subset\mathbf{C}^n$ is called *lineally convex* or *linearly convex* or also *hypoconvex* if its complement is the union of complex hyperplanes. Further, an open set $\Omega\subset\mathbf{C}^n$ is said to be *weakly lineally convex* if through every point of $\partial\Omega$ there passes a complex hyperplane which does not intersect Ω.

Proposition 3.7. *Let* Ω *be a completely circled weakly lineally convex domain in* $\mathbf{C}^n$. *Then* Ω *is convex.*

The homogeneous plurisubharmonic function on $\mathbf{C}^2$, $\varepsilon\in(0,1)$,

$$\varrho_\varepsilon(w_1,w_2)=\max\left\{|w_1|,|w_2|,\sqrt{\frac{|w_1w_2|}{\varepsilon}}\right\}$$

and the domain $\Omega_\varepsilon=\{(w_1,w_2)\in\mathbf{C}^2;\varrho_\varepsilon(w_1,w_2)<1\}$, [12, p. 224], then shows that there are completely circled pseudoconvex domains which are not convex and hence not lineally convex.

Proof. The conclusion is obvious for $n=1$. Let $n=2$ and let $w_0\in\partial\Omega$. Without loss of generality we may assume that $w_0=(1,0)$. Let $a,b\in\mathbf{C}$ be such that $\Lambda=\{(a\lambda+1,b\lambda);\lambda\in\mathbf{C}\}$ is a complex line through w_0 which does not intersect Ω. Let

$$H=\{(a\lambda+iy+1,b\lambda)\,;\lambda\in\mathbf{C},\ y\in\mathbf{R}\}$$

be the real hyperplane through w_0 spanned by Λ and the tangent line to the circle Δ at the point 1.

Let us assume that there is a point $(a\lambda_0+iy_0+1, b\lambda_0)\in H\cap\Omega$ for some $\lambda_0\in\mathbf{C}$ and $y_0\in\mathbf{R}$. Let $\mu=1/(1+iy_0)$. Then $|\mu|\le 1$ and, since Ω is a completely circled domain, we have $\mu(a\lambda_0+iy_0+1, b\lambda_0)\in\Omega$. Therefore

$$\left(a\frac{\lambda_0}{1+iy_0}+1, b\frac{\lambda_0}{1+iy_0}\right)\in\Omega\cap\Lambda,$$

which is a contradiction. Hence $H\cap\Omega=\emptyset$ and the proposition is proved for $n=2$.

For $n\ge 3$ the proposition follows by induction on n. □

4. The smooth case

It follows immediately from the maximum principle for subharmonic functions that if a holomorphic disc $F:\Delta\to\widehat{X}$ touches the boundary of $\widehat{X}$ over $\mathbf{B}^m$, that is $\Psi_\varrho(F(0))=1$, then the disc $F(\Delta)$ actually lies completely in the boundary of $\widehat{X}$. In this section we will show that under appropriate smoothness assumptions on the function Ψ_ϱ the boundary of $\widehat{X}$ over $\mathbf{B}^m$ is foliated by H^∞ holomorphic discs.

We recall that, [12, p. 99] (see also [3], [4], [5]), if a maximal plurisubharmonic function u on $D\subseteq\mathbf{C}^n$ is of class C^3 and the kernel of its Levi form is one-dimensional at each point of D, then there exists a foliation of D by Riemann surfaces $\{S_\alpha\}_{\alpha\in A}$ such that the restriction of u to any S_α is harmonic. The foliation is given by integrating the distribution of the kernels of the Levi form of the function u.

Proposition 4.1. *Let Ψ be a maximal plurisubharmonic function on $\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ of class C^3 such that*

(1) *Ψ is homogeneous in the w variable: $\Psi(z,\lambda w)=|\lambda|\Psi(z,w)$ for all $(z,w)\in\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ and $\lambda\in\mathbf{C}\setminus\{0\}$;*

(2) *the Levi form of Ψ has a one-dimensional kernel at each point $(z,w)\in\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$.*

Then the foliation of $\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ by Riemann surfaces $\{S_\alpha\}_{\alpha\in A}$ induced by Ψ is such that Ψ is constant on each leaf S_α.

Proof. For every $(z,w)\in\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ and $\lambda\in\mathbf{C}\setminus\{0\}$ we have

$$\Psi(z,\lambda w)=|\lambda|\Psi(z,w).$$

We differentiate this identity with respect to λ and get

$$\sum_{j=1}^{n}\frac{\partial\Psi}{\partial w_j}(z,\lambda w)w_j=\frac{1}{2}\frac{\bar\lambda}{|\lambda|}\Psi(z,w).$$

Set $\lambda=1$ to get

$$\sum_{j=1}^{n}\frac{\partial\Psi}{\partial w_j}(z,w)w_j=\frac{1}{2}\Psi(z,w)$$

for $(z,w)\in\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$.

Differentiation with respect to $\bar{z}_p$, $p=1,\dots,m$, and $\overline{w}_r$, $r=1,\dots,n$, gives us

$$\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial w_j\partial\bar{z}_p}(z,w)w_j=\frac{1}{2}\frac{\partial\Psi}{\partial\bar{z}_p}(z,w)\quad\text{and}\quad\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial w_j\partial\overline{w}_r}(z,w)w_j=\frac{1}{2}\frac{\partial\Psi}{\partial\overline{w}_r}(z,w).$$

Let $V(z,w)=(\mathcal{Z}(z,w),\mathcal{W}(z,w))$ be a vector field on $\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ which for each point $(z,w)\in\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$ spans the one-dimensional kernel of the Levi form of the function Ψ. This is also a vector field which is at each point tangent to the leaves of the foliation $\{S_\alpha\}_{\alpha\in A}$. By the above identities we get

$$\begin{aligned}\frac{1}{2}\biggl(\sum_{p=1}^{m}\overline{\mathcal{Z}_p(z,w)}\,\frac{\partial\Psi}{\partial\bar{z}_p}(z,w)+\sum_{r=1}^{n}\overline{\mathcal{W}_r(z,w)}\,\frac{\partial\Psi}{\partial\overline{w}_r}(z,w)\biggr)\\=\sum_{p=1}^{m}\overline{\mathcal{Z}_p(z,w)}\biggl(\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial w_j\partial\bar{z}_p}(z,w)w_j\biggr)+\sum_{r=1}^{n}\overline{\mathcal{W}_r(z,w)}\biggl(\sum_{j=1}^{n}\frac{\partial^2\Psi}{\partial w_j\partial\overline{w}_r}(z,w)w_j\biggr).\end{aligned}$$

Changing the order of summation and using the fact that the vector field $V(z,w)=(\mathcal{Z}(z,w),\mathcal{W}(z,w))$ spans the kernel of the Levi form of Ψ at the point (z,w) we get

$$\sum_{j=1}^{n}w_j\biggl(\sum_{p=1}^{m}\overline{\mathcal{Z}_p(z,w)}\,\frac{\partial^2\Psi}{\partial w_j\partial\bar{z}_p}(z,w)+\sum_{r=1}^{n}\overline{\mathcal{W}_r(z,w)}\,\frac{\partial^2\Psi}{\partial w_j\partial\overline{w}_r}(z,w)\biggr)=0.$$

Hence we have proved that at every point $(z,w)\in\mathbf{B}^m\times\mathbf{C}^n$ we have

$$\sum_{p=1}^{m}\mathcal{Z}_p(z,w)\,\frac{\partial\Psi}{\partial z_p}(z,w)+\sum_{r=1}^{n}\mathcal{W}_r(z,w)\,\frac{\partial\Psi}{\partial w_r}(z,w)=0,$$

and therefore the restriction of Ψ to any leaf S_α is constant. □

Remark 4.2. If the function Ψ has bounded level sets (this is the case for the function Ψ_ϱ from Proposition 2.1) each Riemann surface S_α is an image of a bounded holomorphic mapping $F_\alpha=(f_\alpha,g_\alpha)$ on Δ (a covering map). Since $\{S_\alpha\}_{\alpha\in A}$ form a foliation of $\mathbf{B}^m\times(\mathbf{C}^n\setminus\{0\})$, we must have $|f_\alpha^*(e^{i\theta})|=1$ almost everywhere on $\partial\Delta$.

Remark 4.3. There are examples of maximal plurisubharmonic functions Ψ on $\mathbf{B}^m$ $(m\geq 2)$ for which for certain points $z\in\mathbf{B}^m$ there is no germ V of an analytic variety containing z and such that $\Psi|_V$ is harmonic (Sibony's example [3, p. 73] and examples given by Poletsky). Therefore one can not in general expect to get a foliation of the whole $\widehat{X}$ with analytic discs, [8].

5. Appendix

Proposition 5.1. *Let φ be a continuous function on $\partial\mathbf{B}^m$ and let $u_0\in C(\overline{\mathbf{B}}^m)$ be the maximal plurisubharmonic function on $\mathbf{B}^m$ such that $u_0|_{\partial\mathbf{B}^m}=\varphi$. Then for every $z\in\mathbf{B}^m$,*

$$u_0(z)=\inf_f \frac{1}{2\pi}\int_0^{2\pi} \varphi(f^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over the family of all smooth up to the boundary mappings $f:\bar{\Delta}\to\overline{\mathbf{B}}^m$ which are holomorphic on Δ and such that $f(0)=z$ and $|f^*(\xi)|=1$ for every $\xi\in\partial\Delta$.

Proof. Let U be a continuous function on $\overline{\mathbf{B}}^m$, plurisuperharmonic on $\mathbf{B}^m$ and such that U equals φ on $\partial\mathbf{B}^m$. Then u_0 equals the supremum of the plurisubharmonic functions on $\mathbf{B}^m$ which are pointwise below U. Hence by [14] for every $z\in\mathbf{B}^m$ we have

$$u_0(z)=\inf_f \frac{1}{2\pi}\int_0^{2\pi} U(f^*(e^{i\theta}))\,d\theta,$$

where the infimum is taken over all mappings $f:\bar{\Delta}\to\mathbf{B}^m$ with $f(0)=z$ which are defined and holomorphic in some open neighbourhood V_f of $\bar{\Delta}$. Without loss of generality we may assume that the infimum is taken over the family $\mathcal{P}$ of polynomial mappings f for which $f(0)=z$ and $f(\bar{\Delta})\subseteq\mathbf{B}^m$.

Let $\varepsilon>0$ and let $f\in\mathcal{P}$ be such that

$$u_0(z)\le\frac{1}{2\pi}\int_0^{2\pi} U(f^*(e^{i\theta}))\,d\theta<u_0(z)+\varepsilon. \tag{5.1}$$

Let $\Gamma\subseteq f^{-1}(\mathbf{B}^m)$ be the connected component of $f^{-1}(\mathbf{B}^m)$ which contains $\bar{\Delta}$. The set Γ is a simply connected open set in $\mathbf{C}$ and we may also assume that it has a smooth (even real analytic) boundary.

The function $U\circ f\in C(\bar{\Gamma})$ is a superharmonic function on Γ. Let $w\in C(\bar{\Gamma})$ be the harmonic function on Γ such that $w|_{\partial\Gamma}=(U\circ f)|_{\partial\Gamma}$. Then w is the largest subharmonic function on Γ below $U\circ f$. Hence

$$w(0)=\inf_h \frac{1}{2\pi}\int_0^{2\pi} (U\circ f)(h^*(e^{i\theta}))\,d\theta, \tag{5.2}$$

where the infimum is taken over all mappings $h:\bar{\Delta}\to\Gamma$ with $h(0)=0$ which are defined and holomorphic in some open neighbourhood of $\bar{\Delta}$.

Let h_0 be a Riemann map from Δ to Γ, $h_0(0)=0$. Since $\partial\Gamma$ is smooth, h_0 is smooth up to the boundary and it takes $\partial\Delta$ into $\partial\Gamma$. Then

$$w(0)=(w\circ h_0)(0)=\frac{1}{2\pi}\int_0^{2\pi}(w\circ h_0)^*(e^{i\theta})\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}((U\circ f)\circ h_0)^*(e^{i\theta})\,d\theta.$$

By the submean property and because $w(0)$ is given as the infimum (5.2), we have

$$u_0(z) \le w(0) = \frac{1}{2\pi}\int_0^{2\pi} U((f\circ h_0)^*(e^{i\theta}))\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} U(f^*(e^{i\theta}))\,d\theta < u_0(z)+\varepsilon.$$

Hence the smooth up to the boundary holomorphic mapping $f\circ h_0\colon\Delta\to\mathbf{B}^m$ is such that $(f\circ h_0)(0)=z$, that it takes $\partial\Delta$ into $\partial\mathbf{B}^m$, and that it gives an ε-approximation of $u_0(z)$. □

Proof of Lemma 3.5. Obviously we have that Ψ^a is an upper semicontinuous function on $\mathbf{B}^m\times\mathbf{C}^n$ such that $\Psi_\varrho\le\Psi^a$. Using the continuity of Ψ_ϱ on $\overline{\mathbf{B}}^m\times\mathbf{C}^n$ and constant discs, we also have $(\Psi^a)^\star\le\varrho$ on $\partial\mathbf{B}^m\times\mathbf{C}^n$. Hence, to prove the lemma we have to show that Ψ^a is a plurisubharmonic function. The argument we use is a modification of the argument by Poletsky in [14] and we include it for the interested reader.

Let $\xi\in\Delta\mapsto L(\xi)=(z_0,w_0)+(c,d)\xi$ be a linear disc in $\mathbf{B}^m\times\mathbf{C}^n$. We would like to show that

$$\Psi^a(z_0,w_0)\le\frac{1}{2\pi}\int_0^{2\pi}\Psi^a(L(e^{i\theta}))\,d\theta.$$

Let $\varepsilon>0$. Then for each $\xi\in\partial\Delta$ there exists a smooth up to the boundary analytic disc $F(\xi,\cdot)=(f(\xi,\cdot),g(\xi,\cdot))$ such that $F(\xi,0)=L(\xi)$, $a<|f^*(\xi,e^{i\omega})|<1$ on $\partial\Delta$ and for which

$$\frac{1}{2\pi}\int_0^{2\pi}\Psi_\varrho(F^*(\xi,e^{i\omega}))\,d\omega<\Psi^a(L(\xi))+\varepsilon.$$

Since $\Psi^a(L(\xi))$ is an upper semicontinuous function on $\partial\Delta$, its integral can be arbitrarily well approximated by an integral of a continuous function $v\in C(\partial\Delta)$ such that $\Psi^a(L(\xi))\le v(\xi)$ on $\partial\Delta$. Hence, using the continuity of the function Ψ_ϱ, we may assume that $F(\xi,\cdot\,)$ is a piecewise continuous and uniformly bounded family of holomorphic discs. We will glue (find a homotopy between) the continuous pieces of $F(\xi,\cdot\,)$ on a set of arbitrarily small measure on $\partial\Delta$ to get a continuous family $F_1(\xi,\eta)=(f_1(\xi,\eta),g_1(\xi,\eta))$ of up to the boundary smooth holomorphic discs for which $F_1(\xi,0)=L(\xi)$, $a<|f_1^*(\xi,e^{i\omega})|<1$ on $\partial\Delta$ and

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\int_0^{2\pi}\Psi_\varrho(F_1^*(e^{i\theta},e^{i\omega}))\,d\omega\right)d\theta<\frac{1}{2\pi}\int_0^{2\pi}\Psi^a(L(e^{i\theta}))\,d\theta+\varepsilon.$$

The mappings $g(\xi,\cdot\,)$ are glued together by taking the convex combinations of nearby mappings, that is, for two nearby points $\xi_0,\xi_1\in\partial\Delta$ we set $g(\xi_t,\cdot\,)=(1-t)g(\xi_0,\cdot\,)+tg(\xi_1,\cdot\,)$, where ξ_t is some parametrization of the arc $(\xi_0,\xi_1)\subseteq\partial\Delta$ with the interval $[0,1]$. Then we define $\hat g(\xi_t,\cdot\,)=g(\xi_t,\cdot\,)-g(\xi_t,0)+w_0+d\xi_t$ to get

$\hat{g}(\xi_t,0)=w_0+d\xi_t$. We have to be more careful when gluing the mappings $f(\xi,\cdot)$. First we find a homotopy $\{\hat{f}(\xi_t,\cdot)\}_{t\in[0,1]}$ in $\mathbf{B}^m$ between $f(\xi_0,\cdot)$ and $f(\xi_1,\cdot)$ such that for each $t\in[0,1]$, the analytic disc $\hat{f}(\xi_t,\cdot)$ has no zeros on $\partial\Delta$. We distinguish two cases:

1. *The case* $m=1$. Each of the functions $f(\xi,\cdot)$ has a nonnegative winding number around 0 which is constant on each continuous piece of $f(\xi,\cdot)$. Multiplying continuous pieces of $f(\xi,\cdot)$ with functions of the form

$$\eta\longmapsto\frac{1}{r_0}\frac{\eta+r_0}{1+r_0\eta},$$

where $r_0\in(0,1)$ is a real number close to 1, we can arrange that the new family, which we still denote by $f(\xi,\cdot)$, has the same properties regarding approximation, boundary values and the position of the image of the point 0 as the original one, but all functions also have the same winding number k. Hence for each $\xi\in\partial\Delta$ the holomorphic function $f(\xi,\cdot)$ can be written in the form

$$f(\xi,\eta)=B(\xi,\eta)e^{\varphi(\xi,\eta)},$$

where $B(\xi,\eta)$ is a finite Blaschke product with k factors and $\varphi(\xi,\eta)$ a smooth up to the boundary holomorphic function on Δ with the property $\log a<\operatorname{Re}\varphi(\xi,\eta)<0$. Now a homotopy $\{\hat{f}(\xi_t,\cdot)\}_{t\in[0,1]}$ between functions $f(\xi_0,\cdot)$ and $f(\xi_1,\cdot)$ is obvious: the zeros of $B(\xi_0,\cdot)$ are moved to the zeros of $B(\xi_1,\cdot)$ and the convex combination of $\varphi(\xi_0,\cdot)$ and $\varphi(\xi_1,\cdot)$ is used.

2. *The case* $m>1$. Let $f(\xi_0,\cdot)$ and $f(\xi_1,\cdot)$ be two vector functions from the family $f(\xi,\cdot)$, $\xi\in\partial\Delta$. Since $m>1$, we can find a homotopy $\{\hat{f}(\xi_t,\cdot)\}_{t\in[0,1]}$ between $f(\xi_0,\cdot)$ and $f(\xi_1,\cdot)$ of smooth up to the boundary holomorphic discs in $\mathbf{B}^m$ such that $\hat{f}(\xi_t,\cdot)$ has no zeros on $\partial\Delta$ for each ξ_t. A small perturbation of the convex combination of $f(\xi_0,\cdot)$ and $f(\xi_1,\cdot)$ will be good enough.

Having a homotopy $\{\hat{f}(\xi_t,\cdot)\}_{t\in[0,1]}$ of smooth up to the boundary holomorphic discs in $\mathbf{B}^m$ with no zeros on $\partial\Delta$, we would like to modify it to satisfy the conditions $\hat{f}(\xi_t,0)=z_0+c\xi_t$ and $a<|\hat{f}^*(\xi_t,\eta)|<1$ for each $\eta\in\partial\Delta$ and $t\in[0,1]$. We may assume that $\hat{f}(\xi_t,\cdot)=f(\xi_t,\cdot)$ for $t\in[0,\delta]\cup[1-\delta,1]$ for some $0<\delta<\frac{1}{2}$. Let $r_0\in(0,1)$ be so close to 1 and $\varepsilon>0$ so small that $\|f(\xi_t,\cdot)\|_\infty<(1-\varepsilon)r_0$ and $\|\hat{f}(\xi_t,\cdot)\|_\infty<(1-\varepsilon)r_0$ for every $t\in[0,1]$ and that the family of functions

$$\eta\longmapsto\frac{1}{r_0}\frac{\eta+r_0}{1+r_0\eta}f(\xi,\eta)$$

has the same essential properties (approximation, boundary values, the position of the image of 0) as $f(\xi,\cdot)$.

Let $\varphi(\xi_t,\cdot)$ be a smooth up to the boundary holomorphic function on Δ such that $\operatorname{Re}\varphi(\xi_t,\eta)=\log|\hat{f}(\xi_t,\eta)|$ on $\partial\Delta$ and $\operatorname{Im}\varphi(\xi_t,0)=0$. Let $\chi(t)$ be a smooth function on $\mathbf{R}$ such that $\operatorname{supp}\chi\subset[0,1]$, $0\le\chi(t)\le1$ and $\chi(t)=1$ for $t\in[\delta,1-\delta]$.

We define a continuous family of analytic discs

$$\tilde{f}(\xi_t,\eta)=\frac{1}{r(t)}\frac{\eta+\alpha(t)}{1+\overline{\alpha(t)}\eta}e^{-\chi(t)\varphi(\xi_t,\eta)}\hat{f}(\xi_t,\eta),$$

where

$$r(t)=\max\left\{r_0,\frac{\|\hat{f}(\xi_t,\cdot)\|_\infty^{1-\chi(t)}}{1-\varepsilon}\right\}\quad\text{and}\quad\alpha(t)=r(t)e^{\chi(t)\varphi(\xi_t,0)}.$$

First we observe that

$$|r(t)e^{\chi(t)\varphi(\xi_t,0)}|\le r(t)\|\hat{f}(\xi_t,\cdot)\|_\infty^{\chi(t)}\le r_0<1$$

for every $t\in[0,1]$ and hence $\alpha(t)$ is well chosen. This shows that $\tilde{f}(\xi_t,\cdot)$, $t\in[0,1]$, is a well defined continuous family of analytic discs such that $\tilde{f}(\xi_t,0)=f(\xi_t,0)$ for every $t\in[0,\delta]\cup[1-\delta,1]$.

Also, for each $t\in[0,1]$ and $\eta\in\partial\Delta$ we have

$$|\tilde{f}(\xi_t,\eta)|=\frac{1}{r(t)}|\hat{f}(\xi_t,\eta)|^{1-\chi(t)}\le1-\varepsilon$$

and the equality holds for every $t\in[\delta,1-\delta]$. On the other hand for $t\in[0,\delta]\cup[1-\delta,1]$ and $\eta\in\partial\Delta$ we have

$$\frac{1}{r_0}|\hat{f}(\xi_t,\eta)|^{1-\chi(t)}\ge|\hat{f}(\xi_t,\eta)|^{1-\chi(t)}\ge|\hat{f}(\xi_t,\eta)|=|f(\xi_t,\eta)|>a$$

and

$$\frac{1-\varepsilon}{\|\hat{f}(\xi_t,\cdot)\|_\infty^{1-\chi(t)}}|\hat{f}(\xi_t,\eta)|^{1-\chi(t)}\ge(1-\varepsilon)\frac{a^{1-\chi(t)}}{((1-\varepsilon)r_0)^{1-\chi(t)}}>a$$

and hence $|\tilde{f}(\xi_t,\eta)|>a$ for every $t\in[0,1]$ and $\eta\in\partial\Delta$.

We finish the gluing by using an appropriate continuous family $\{A_t\}_{t\in[0,1]}$ of automorphisms of the ball $\mathbf{B}^m(0,1-\varepsilon)$ which are equal to the identity map on $[0,\delta]\cup[1-\delta,1]$ and are such that $A_t(\tilde{f}(\xi_t,0))=f(\xi_t,0)$ on $[\delta,1-\delta]$.

The rest is similar to [14], pp. 168–169, and we will only sketch it. First we approximate $F_1(\xi,\eta)$ uniformly on $\partial\Delta\times\bar{\Delta}$ by functions $F_2(\xi,\eta)$ which are holomorphic and smooth up to the boundary in $\eta\in\Delta$, rational in $\xi\in\Delta$, with a pole at $\xi=0$, and such that $F_2(\xi,0)=L(\xi)$. Then the pole at $\xi=0$ is erased using the change of variables $F_3(\xi,\eta)=F_2(\xi,\xi^N\eta)$. Finally the holomorphic mapping $\xi\in\Delta\mapsto(f_4(\xi),g_4(\xi))=F_3(\xi,e^{i\alpha}\xi)$ is for an appropriately chosen $\alpha\in\mathbf{R}$ such that $(f_4(0),g_4(0))=L(0)=(z_0,w_0)$, $a<|f_4^*(\xi)|<1$ on $\partial\Delta$ and

$$\begin{aligned}\Psi^a(z_0,w_0)&\leq\frac{1}{2\pi}\int_0^{2\pi}\Psi_\varrho(f_4^*(e^{i\theta}),g_4^*(e^{i\theta}))\,d\theta\\&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Psi_\varrho(F_3^*(e^{i\theta},e^{i\omega}))\,d\theta\,d\omega<\frac{1}{2\pi}\int_0^{2\pi}\Psi^a(L(e^{i\theta}))\,d\theta+\varepsilon.\qquad\square\end{aligned}$$

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Received April 25, 2000
in revised form June 22, 2001

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